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A COURSE
OF
PURE MATHEMATICS

BY

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PREFACE.

THIS book has been designed primarily for the use of first year students at the Universities whose abilities reach or approach something like what is usually described as 'scholarship standard'. I hope that it may be useful to other classes of readers, but it is this class whose wants I have considered first. It is in any case a book for mathematicians: I have nowhere made any attempt to meet the needs of students of engineering or indeed any class of students whose interests are not primarily mathematical.

A considerable space is occupied with the discussion and application of the fundamental ideas of the Infinitesimal Calculus, Differential and Integral. But the general range of the book is a good deal wider than is usual in English treatises on the Calculus. There is at present hardly room for a new *Calculus* of an orthodox pattern. It is indeed not many years since there was urgent need of such a book, but the want has been met by the excellent treatises of Professors Gibson, Lamb, and Osgood, to all of which, I need hardly say, I am greatly indebted. And so I have included in this volume a good deal of matter that would find a place in any *Traité d'Analyse*, though in English books it is usually separated from the Calculus and classed as 'Higher Algebra' or 'Trigonometry'.

In the first chapter I have discussed in some detail the various classes of numbers included in the arithmetical continuum. I have not attempted to include any account of any purely arithmetical theory of irrational number, since I believe all such theories to be entirely unsuitable for elementary teaching. My aim in this chapter is a more modest one: I take the 'linear continuum' for granted and assume the existence of a definite number corresponding to each of its points; and all that I attempt to do is to analyse and distinguish the various classes of numbers whose existence these assumptions involve.

Chapters II and III probably do not present many points of novelty. The account given in Chapter II of the most important classes of functions of x is more systematic and illustrated with much greater detail than is usual in English books. I have included, mainly for the sake of completeness, a certain amount of the elements of coordinate geometry of two and three dimensions: but I have, here and throughout the book, kept geometry in a strictly subordinate position and used it merely for purposes of illustration. I have also avoided any wealth of detail in connexion with the purely formal consequences of De Moivre's Theorem, and have devoted the space thus saved to the inclusion of a good deal of matter concerning vector analysis, bilinear transformation, and so on, which seemed to me likely to be more interesting and more useful as a preparation for Chapter X.

I have endeavoured to make Chapter IV one of the principal features of the book. The notion of a limit is one that has always presented grave difficulties to mathematical students even of great ability. It has been my good fortune during the last eight or nine years to have a share in the teaching of a good many of the ablest candidates for the Mathematical Tripos; and it is very rarely indeed that I have encountered a pupil who could face the simplest problem involving the ideas of infinity, limit, or continuity, with a vestige of the confidence with which he would deal with questions of a different character and of far greater intrinsic difficulty. I have indeed in an examination asked a dozen candidates, including several future Senior Wranglers, to sum the series $1 + x + x^2 + \dots$, and not received a single answer that was not practically worthless—and this from men quite capable of solving difficult problems connected with the curvature and torsion of twisted curves.

I cannot believe that this is due solely to the nature of the subject. There are difficulties in these ideas, no doubt: but they are not so great as many other difficulties inherent in mathematics that every young mathematician completely overcomes. The fault is not that of the subject or of the student, but of the text-book and the teacher. It is not enough for the latter, if he wishes to drive sound ideas on these points well into the mind of his pupils, to be careful and exact himself. He must be prepared not merely to tell the truth, but to tell it elaborately and ostentatiously. He must drill his pupils in 'infinity' and 'continuity', with an abundance of written exercises and examples, as he drills them at present in poles and polars or symmetric functions or the consequences of De Moivre's theorem. Then and only then he may hope that accurate thought in connexion with these matters will become an integral part of their ordinary mathematical habit of mind. It is this conviction that has led me to devote so much space to the most elementary ideas of all connected with limits, to be purposely diffuse about fundamental points, to illustrate them by so elaborate a system of examples, and to write a chapter of fifty pages without advancing beyond the ordinary geometrical series.

It is not necessary for me to say much about the general plan of the next four chapters. The two chapters on the Calculus are no doubt more difficult than the rest of the book. I have perhaps been inconsistent in the standards that I have adopted: but I have been influenced by the feeling that I shall have few readers who will not already have acquired some familiarity with the technique of the Calculus from other sources. I felt this particularly when I was writing the sections on integration. I also felt that the student is apt to carry away from the books in general use the quite mistaken impression that all methods of integration are essentially of a tentative and haphazard character. I have therefore deliberately given an account of the theory more systematic and general than would be suitable for a normal first course in the Calculus.

Chapters IX and X are devoted to the theory of the logarithm and exponential, starting from the definition of the logarithm as an integral. It was the desire to write an elementary account of this theory that originally led me to begin the book, and I have generally decided my choice of what was to be included in the earlier chapters by a consideration of what theorems would be wanted in the last two.

I regard the book as being really elementary. There are plenty of hard examples (mainly at the ends of the chapters): to these I have added, wherever space permitted, an outline of the solution. But I have done my best to avoid the inclusion of anything that involves really difficult ideas. For instance, I make no use of the 'principle of convergence': uniform convergence, double series, infinite products, are never alluded to: and I prove no general theorems whatever concerning the inversion of limit-operations—I never even define $\frac{d^2f}{dx\,dy}$ and $\frac{d^2f}{dy\,dx}$. In the last two chapters I have occasion once or twice to integrate a power-series, but I have confined myself to the very simplest cases and given a special discussion in each instance. Anyone who has read this book will be in a position to read with profit Mr Bromwich's *Infinite Series*, where a full and adequate discussion of all these points will be found.

It will be found that certain classes of theorems and examples that are prominent in many English books are here conspicuous by their absence. I may refer particularly to the standard theorems concerning the expression of the trigonometrical functions as infinite products or series of partial fractions, and to that familiar type of example the gist of which lies in the 'picking out of coefficients' from some combination of infinite series. The proofs of these results depend upon general theorems that seemed to me intrinsically too difficult to be included in a book professing to be at the same time rigorous and elementary: and I am on the whole of opinion that, if any proposition is too difficult to be proved properly, its statement and application had better be postponed. I am well aware that there is much to be said on the opposite side. A very plausible case can be made out for the habitual exercise of the student in the application of results whose proof is too difficult for his full comprehension. But I have found that I cannot myself write a book on those lines: nor am I fully convinced that such exercise is either necessary or desirable. After all there are plenty of theorems which are *not* too difficult to prove: and, if anyone believes that a sufficient variety of analytical training cannot be based upon them, I hope that my collections of Miscellaneous Examples may do something to convince him. I may say that it is only in these collections that examples of the character of 'problems' will be found. The sets of examples inside each chapter consist either of perfectly straightforward applications of the preceding 'book-work', or of summaries of parts of the theory for which there was no room in the text. They include many important theorems, some indeed to which reference is frequently made later in the book. No one can be more convinced than I am of the value of 'examples' designed merely to train the student's memory and powers of manipulation: but I see no reason why all examples should necessarily be trivial. I trust, however, that readers will not find it irritating to be referred back from the middle of a section in large type to an example in an earlier chapter. My decision as to whether a result should appear in the text or in the examples has always been based upon the relation that it bears to the general theorems in connexion with which it is first proved rather than upon the amount of use that is made of it later on.

I have throughout laid particular stress upon points that do not seem to me to be emphasized sufficiently in the text-books in general use, and passed rapidly over others that are of equal importance but stand in no such danger of neglect. Here again I have been influenced by the consideration that this book is likely to be used in conjunction with others rather than as a first text-book in any particular subject.

There are two respects in which I have diverged from the usually accepted notation and that seem of sufficient importance to be noticed here. I have entirely rejected the index notation for inverse functions ($\cos^{-1} x$, $\cosh^{-1} x$) in favour of the usual Continental notation ($\text{arc}\cos x$, $\text{arg}\cosh x$ or $\text{arg}\,\text{ch}\, x$). And I have followed Mr Leathem and Mr Bromwich in always writing

$$\lim_{n \to \infty}, \quad \lim_{x \to \infty}, \quad \lim_{x \to a}$$

and not $\lim\limits_{n=\infty}$, $\lim\limits_{x=\infty}$, $\lim\limits_{x=a}$. This last change seems to me one of considerable importance, especially when '∞' is the 'limiting value'. I believe that to write '$n = \infty$, $x = \infty$' (as if anything ever were 'equal to infinity'), however convenient it may be at a later stage, is in the early stages of mathematical training to go out of one's way to encourage incoherence and confusion of thought concerning the fundamental ideas of analysis.

The word 'quantity' occurs occasionally in the earlier chapters. It should be in each case altered to 'number'. Unfortunately I arrived at the decision never to use the term 'quantity' only after the earlier sheets had been passed for press.

The books to which I am most indebted (besides the treatises on the Calculus already mentioned) are Mr Bromwich's *Infinite*

Series and M. J. Tannery's *Leçons d'Algèbre et d'Analyse.* I must also acknowledge my obligations to a number of friends who have been kind enough to assist me in the preparation of the book. Mr Bromwich has read the whole of it (except Chapter III) either in manuscript or in proof, and a good deal of it twice; and I am indebted to him for corrections and suggestions on almost every page. Mr Berry read Chapters I, II, III, IX and X in manuscript, Professor J. E. Wright Chapters I, II, and III, and Dr Whitehead Chapters I and IV, and all gave me much valuable advice. In particular the earlier part of Chapter IV has been practically rewritten in consequence of Dr Whitehead's suggestions. I have also changed a good deal of Chapter VI in consequence of suggestions received from Dr Askwith. My thanks are also due to Messrs H. W. Turnbull and E. H. Neville, of Trinity College, who have between them read all the proofs and verified the examples: to the latter I am additionally indebted for the figures that appear in the Miscellaneous Examples to Chapter X. Finally I must express my gratitude to the readers and officials of the University Press for their close attention and unfailing courtesy.

G. H. HARDY.

TRINITY COLLEGE, CAMBRIDGE,
September 1908.

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CHAPTER I.

REAL VARIABLES.

1. The aggregate of rational numbers and their representation on a straight line. On a straight line L, produced indefinitely in both directions, we take a segment A_0A_1 of any length. We call A_0 *the origin*, or *the point* 0, and A_1 *the point* 1.

We now mark off a series of points

$$\dots A_{-m-1},\ A_{-m},\ \dots,\ A_{-1},\ A_0,\ A_1,\ \dots,\ A_n,\ \dots$$

along L, so that

$$\dots = A_{-m-1}A_{-m} = \dots = A_{-1}A_0 = A_0A_1 = \dots,$$

each segment being measured from left to right along L.

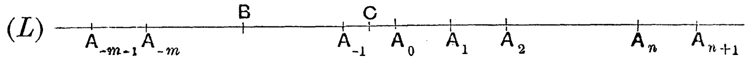

Fig. 1.

Then $$\frac{A_0A_n}{A_0A_1} = n \quad \dots\dots\dots\dots\dots\dots\dots\dots(1),$$

if n is any positive integer.

We will now agree that *length* is to be regarded as *a magnitude capable of sign*, positive if the length is measured in one direction along L (e.g. from B to C) and negative if measured in the other (from C to B), so that $CB = -BC$. We take the positive direction for the measurement of length to be from left to right.

Then $$\frac{A_0A_{-n}}{A_0A_1} = -\frac{A_{-n}A_0}{A_0A_1} = -n.$$

Hence the equation (1) is true for all integral values of n, positive or negative.

For the sake of uniformity we adopt the convention that (1) is also true when $n = 0$, in which case it reads

$$\frac{A_0A_0}{A_0A_1} = 0.$$

That is to say, we agree to regard BB, which is not, properly speaking, a segment at all, as a segment of no length.

Now let us take any positive proper fraction in its lowest terms, for example p/q, where p and q are positive integers without any common factor, and $p < q$. We divide A_0A_1 into q equal parts by points of division which it is natural to denote by

$$A_0,\ A_{1/q},\ A_{2/q},\ \ldots,\ A_{p/q},\ \ldots,\ A_{(q-1)/q},\ A_1.$$

It is evident that

$$\frac{A_0A_{p/q}}{A_0A_1} = \frac{p}{q} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

We thus obtain points on the line L corresponding to all such proper fractions p/q.

Any improper fraction may be expressed in the form $n + (p/q)$, where n is a positive integer and p/q a proper fraction. If we take a point $A_{n+(p/q)}$ such that $A_nA_{n+(p/q)} = A_0A_{p/q}$, it is evident that $\frac{A_0A_{n+(p/q)}}{A_0A_1} = n + \frac{p}{q}$: and if we thus find points $A_{n+(p/q)}$ corresponding to all possible positive values of n, p, and q, we shall have a point A_f corresponding to all possible positive integral or fractional values of f, and such that

$$\frac{A_0A_f}{A_0A_1} = f \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Finally, if $-f$ is a negative fraction, proper or improper, we take A_{-f} so that $A_{-f}A_0 = A_0A_f$, or

$$\frac{A_0A_{-f}}{A_0A_1} = -\frac{A_0A_f}{A_0A_1} = -f.$$

Thus we are able to determine a point A_r corresponding to *any integral or fractional value of* r, *positive or negative*, and such that

$$\frac{A_0A_r}{A_0A_1} = r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If we take, as is natural, the length A_0A_1 as our unit of length, so that $A_0A_1 = 1$, the equation (4) becomes

$$A_0A_r = r \quad \text{..........................} (5).$$

Definitions. *Any fraction $r = p/q$, where p and q are positive or negative integers, is called a* **rational number.**

The points A_r of the line L, which correspond to the rational numbers r in the manner explained above, are called the **rational points** *of the line.*

We can suppose (i) that p and q have no common factor, as if they have a common factor we can divide each of them by it, and (ii) that q is positive, since

$$p/(-q) = (-p)/q, \quad (-p)/(-q) = p/q.$$

The notion of a rational number obviously includes as a particular case that of an integer, since any integer may be expressed as a fraction whose denominator is unity.

Examples I. 1. If r and s are rational numbers, $r+s$, $r-s$, rs, and r/s are rational numbers, unless in the last case $s=0$ (when r/s is of course meaningless).

2. If P and Q are rational points, and PQ is divided into any number of equal parts, each of the points of division is a rational point.

3. If λ, m, and n are positive rational numbers, $\lambda(m^2 \sim n^2)$, $2\lambda mn$, and $\lambda(m^2+n^2)$ are positive rational numbers. Hence show how to determine any number of right-angled triangles the lengths of all of whose sides are rational.

4. Any terminated decimal represents a rational number whose denominator contains no factors other than 2 or 5. Conversely, any such rational number can be expressed, and in one way only, as a terminated decimal.

[The general theory of decimals will be considered in Chap. IV.]

5. The positive rational numbers may be arranged in the form of a simple series as follows:

$$\tfrac{1}{1}, \tfrac{2}{1}, \tfrac{1}{2}, \tfrac{3}{1}, \tfrac{2}{2}, \tfrac{1}{3}, \tfrac{4}{1}, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{1}{4}, \dots$$

Show that p/q is the $[\tfrac{1}{2}(p+q-1)(p+q-2)+q]$th term of the series.

[In this series every rational number is repeated indefinitely. Thus 1 occurs as $\tfrac{1}{1}, \tfrac{2}{2}, \tfrac{3}{3}, \dots$. We can of course avoid this by omitting every number which has already occurred in a simpler form, but then the problem of determining the precise position of p/q becomes more complicated.]

2. Irrational numbers. If the reader will mark off on the line all the points corresponding to the rational numbers whose denominators are 1, 2, 3, ... in succession, he will readily convince himself that he can cover the line with rational points as closely as he likes. We can state this more precisely as follows: *if we take any segment BC on L, we can find as many rational points as we please on BC.*

Suppose, for example, that BC falls within the segment A_1A_2. It is evident that if we choose a positive integer k so that

$$k \,.\, BC > A_0A_1 \qquad\text{.........................(1)}$$

and divide A_1A_2 into k equal parts, at least one of the points of division (say P) must fall *inside* BC, without coinciding with either B or C. For if this were not so BC would be entirely included in one of the k parts into which A_1A_2 has been divided, which contradicts the supposition (1). Thus at least one rational point P lies between B and C. But then we can find another such point Q between B and P, another between B and Q, and so on indefinitely; i.e., as we asserted above, we can find as many as we please. We may express this by saying that BC includes *infinitely many* rational points.

From these considerations the reader might be tempted to infer that these rational points account for all the points of the line, i.e. that *every* point on the line is a rational point. And it is certainly the case that if we imagine the line as being made up solely of the rational points, all other points (if any such there be) being imagined to be eliminated, the figure which remained would possess most of the properties which common sense attributes to the straight line, and would, to put the matter roughly, look and behave very much like a line.

There is, however, good reason for supposing that *there are on the line points which are not rational points.*

Let us look at the matter for a moment with the eye of common sense, and consider some of the properties which we may reasonably expect a straight line to possess if it is to satisfy the idea which we have formed of it in elementary geometry.

The straight line must be composed of points, and any segment of it by all the points which lie between its end points. With any such segment must be associated a certain entity called its *length*, which must be a *quantity* capable of *numerical measurement* in terms of any standard or unit length, and these lengths must be capable of combination with one another according to the ordinary rules of algebra by means of addition or multiplication. Again, it must be possible to construct a line whose length is the sum or product of any two given lengths. If the length PQ, along a given line, is a, and the length QR, along the same straight line, is b, the length PR must be $a+b$. Moreover, if the lengths OP, OQ, along one straight line, are 1 and a, and the length OR along another straight line is b, and if we determine the length OS by Euclid's construction (Euc. VI. 12) for a fourth proportional to the lines OP, OQ, OR, this length must be ab, the algebraical fourth proportional to 1, a, b. And it is hardly necessary to remark that the sums and products thus defined must obey the ordinary laws of algebra, such as

$$a+b=b+a, \quad a+(b+c)=(a+b)+c, \quad ab=ba,$$

and so on. The lengths of our lines must also obey a number of obvious laws concerning *inequalities* as well as equalities. Thus if A, B, C are three points lying along L from left to right, we must have $AB<AC$, and so on. Finally it must be possible, on our fundamental line L, to find a point P such that A_0P is equal to any segment whatever taken along L or along any other straight line.

Now it is very easy, by means of various elementary geometrical constructions, to construct a length x such that $x^2=2$. For example, we may construct an isosceles right-angled triangle

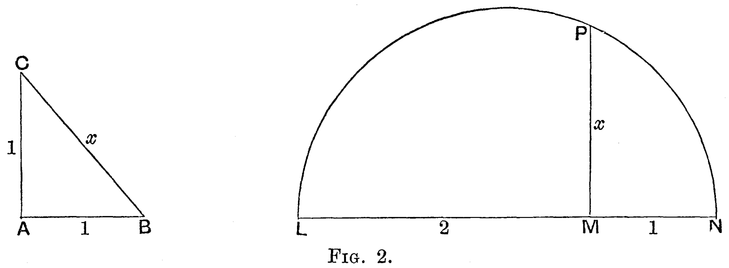

Fig. 2.

ABC such that $AB=AC=1$. Then if $BC=x$, $x^2=2$. Or we may determine the length x by means of Euclid's construction

(Euc. VI. 13) for a mean proportional to 1 and 2, as indicated in the figure.

It follows that there must be a point P on L such that

$$A_0P = x, \quad x^2 = 2.$$

But it is easy to see that *there is no rational number such that its square is* 2. In fact we may go further and say that there is no rational number whose square is m/n, where m/n is any positive fraction in its lowest terms, unless m and n are both perfect squares.

For suppose, if possible, that

$$\frac{p^2}{q^2} = \frac{m}{n},$$

p having no factor in common with q, and m no factor in common with n. Then

$$np^2 = mq^2.$$

Every factor of q^2 must divide np^2, and as p and q have no common factor, every factor of q^2 must divide n. Hence $n = \lambda q^2$, where λ is an integer. But this involves $m = \lambda p^2$: and as m and n have no common factor, λ must be unity. Thus $m = p^2$, $n = q^2$, as was to be proved.

We are thus led to believe in the existence of a point P, not one of the rational points already constructed, and such that $A_0P = x$, $x^2 = 2$; and (as the reader will remember from elementary algebra) we write $x = \sqrt{2}$. And if Q is the point such that $QA_0 = A_0P$, we write $A_0Q = -\sqrt{2}$.

The following alternative proof that $\sqrt{2}$ cannot be rational is interesting.

Suppose, if possible, that p/q is a positive fraction, in its lowest terms, such that $(p/q)^2 = 2$ or $p^2 = 2q^2$. It is easy to see that then we must have $(2q-p)^2 = 2(p-q)^2$, and so $(2q-p)/(p-q)$ is another fraction having the same property. But clearly $q < p < 2q$, and so $p - q < q$. Hence we obtain another fraction equal to p/q and having a *smaller* denominator, which contradicts the assumption that p/q is in its lowest terms.

DEFINITION. *Any point P on the line L which is not a rational point is called an* **irrational point.** *The length A_0P is called an* **irrational number.**

Examples II. 1. Show from first principles, without assuming the general theorem proved above, that $\sqrt{2}$ is not a rational number.

2. Give a similar proof for $\sqrt[3]{2}$.

3. Prove generally that, if p/q is a rational fraction in its lowest terms, $\sqrt[3]{(p/q)}$ cannot be rational unless p and q are both perfect cubes.

4. The square root of an integer must be either integral or irrational (i.e. it cannot be a *rational fraction*).

[For suppose, if possible, $\sqrt{n}=p/q$, where p, q are positive integers without a common factor. Then $nq^2=p^2$. Hence p^2 divides n, as p and q have no common factor; i.e. $n=\lambda p^2$, where λ is an integer, and so $\lambda q^2=1$, which shows that $\lambda=1$, $q=1$, $n=p^2$, and so $\sqrt{n}=p$.]

5. A more general proposition, due to Gauss, is the following: *if*

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\dots+p_n=0$$

is any algebraical equation with integral coefficients, it cannot have a rational but not integral root.

[For suppose that the equation has a root a/b, where a and b are integers without a common factor, and b is positive. Writing a/b for x, and multiplying by b^{n-1}, we obtain

$$-\frac{a^n}{b}=p_1a^{n-1}+p_2a^{n-2}b+\dots+p_nb^{n-1},$$

a fraction in its lowest terms equal to an integer, which is absurd. Thus $b=1$, and the root is a. It is evident that a must be a divisor of p_n.]

6. Show that if $p_n=1$ and $p_1+p_2+\dots+p_n\neq -1$, the equation cannot have a rational root.

7. Find the rational roots (if any) of

$$x^4-4x^3-8x^2+13x+10=0.$$

[The roots can only be integral, and so ± 1, ± 2, ± 5, ± 10 are the only possibilities: whether these are roots can be determined by trial. It is clear that we can in this way determine the rational roots of any such equation.]

3. Quadratic surds. If a is any rational number, the two numbers $\pm\sqrt{a}$ are either rational or irrational, and (as appears from what precedes) *generally* the latter. Numbers of this kind, when irrational, are called *pure quadratic surds.* A number $a\pm\sqrt{b}$, the sum of a rational number and a pure quadratic surd, is sometimes called a *mixed quadratic surd.*

The only kind of irrationals for whose existence the geometrical arguments of the preceding section have given us any warrant are these quadratic surds, pure and mixed, and the more complicated irrationals which may be expressed in a form involving the repeated extraction of square roots, such as

$$\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2+\sqrt{2+\sqrt{2}}}.$$

It is easy to construct geometrically a line whose length is equal to any number of this form, as the reader will easily see for himself. That *only* irrationals of these kinds can be constructed by Euclidean methods (i.e. by geometrical constructions with ruler and compasses) is a point the proof of which must be deferred for the present*. This particular property of quadratic surds naturally makes them peculiarly interesting.

Examples III. 1. Give geometrical constructions for

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}}.$$

2. The quadratic equation $ax^2+2bx+c=0$ has two real roots† if $b^2-ac>0$. Suppose a, b, c rational. Nothing is lost by taking all three to be integers, for we can multiply the equation by the L.C.M. of their denominators.

The reader will remember that the roots are $\{-b\pm\sqrt{(b^2-ac)}\}/a$. It is easy to construct these lengths geometrically, first constructing $\sqrt{(b^2-ac)}$. A much more elegant, though less straightforward, construction is the following.

Draw a circle of unit radius, a diameter PQ, and the tangents at the ends of the diameters.

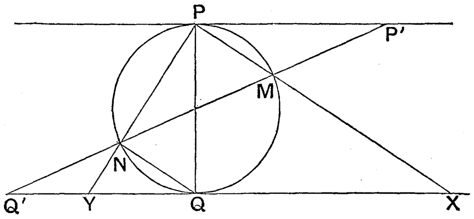

FIG. 3.

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* See Chap. II, Misc. Exs. 41.

† I.e. there are two values of x for which $ax^2+2bx+c=0$. If $b^2-ac<0$ there are no such values of x. The reader will remember that in books on elementary algebra the equation is said to have two 'imaginary' roots. The meaning to be attached to this statement will be explained in Chap. III.

When $b^2=ac$ the equation has only one root. For the sake of uniformity it is generally said in this case to have 'two equal' roots, but this is a mere convention.

*Take $PP' = -2a/b$ and $QQ' = -c/2b$, having regard to sign**. *Join $P'Q'$, cutting the circle in M and N. Draw PM and PN, cutting QQ' in X and Y. Then QX and QY are the roots of the equation with their proper signs*†.

The proof is simple and we leave it as an exercise to the reader. Another, perhaps even simpler, construction is the following. *Take a line AB of unit length. Draw $BC = -2b/a$ perpendicular to AB, and $CD = c/a$ perpendicular to BC and in the same direction as BA. On AD as diameter describe a circle cutting BC in X and Y. Then BX and BY are the roots.*

3. If ac is positive PP' and QQ' will be drawn in the same direction. Verify that if $b^2 < ac$ $P'Q'$ will not meet the circle, while if $b^2 = ac$ it will be a tangent. Verify also that if $b^2 = ac$ the circle in the second construction will touch BC.

4. **Some theorems concerning quadratic surds.** We shall assume that the reader is familiar with the ordinary rules for the manipulation of quadratic surds; such, e.g., as are expressed by the equations

$$\sqrt{(pq)} = \sqrt{p} \cdot \sqrt{q}, \quad \sqrt{(p^2 q)} = p\sqrt{q}.$$

He will find it a useful exercise at this stage to supply proofs of these equations.

Similar and dissimilar surds. Two pure quadratic surds are said to be *similar* if they can be expressed as rational multiples of the same surd, and otherwise *dissimilar*. Thus

$$\sqrt{8} = 2\sqrt{2}, \quad \sqrt{\tfrac{25}{2}} = \tfrac{5}{2}\sqrt{2},$$

and so $\sqrt{8}$, $\sqrt{\tfrac{25}{2}}$ are similar surds. On the other hand, if M and N are integers which have no common factor, and neither of which is a perfect square, $\sqrt{M}$ and $\sqrt{N}$ are dissimilar surds. For suppose, if possible,

$$\sqrt{M} = \frac{p}{q}\sqrt{\frac{\alpha}{\beta}}, \quad \sqrt{N} = \frac{r}{s}\sqrt{\frac{\alpha}{\beta}},$$

where all the letters denote integers.

Then $\sqrt{MN}$ is evidently rational, and therefore (Ex. II. 4) integral. Thus $MN = P^2$ where P is an integer. Let $a, b, c, \ldots$ be the prime factors of P, so that

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* The figure is drawn to suit the case in which b and c have the same and a the opposite sign. The reader should draw figures for other cases.

† I have taken this construction from Klein's *Leçons sur certaines questions de Géométrie Élémentaire* (French translation by J. Griess, Paris, 1896).

$$MN = a^2 b^2 c^2 \ldots .$$

Then MN is divisible by a^2, and therefore either (1) M is divisible by a^2, or (2) N is divisible by a^2, or (3) M and N are both divisible by a. The last case may be ruled out, since M and N have no common factor. This argument may be applied to each of the factors $a^2, b^2, c^2, \ldots$. Ultimately we see that M must be divisible by some of these factors and N by the rest. Thus

$$M = \lambda P_1^2, \quad N = \lambda P_2^2,$$

where P_1^2 denotes the product of some of the factors $a^2, b^2, c^2, \ldots$ and P_2^2 the product of the rest. Since M and N have no common factor we must have $\lambda = 1$, $M = P_1^2$, $N = P_2^2$; i.e. M and N are both perfect squares, which is contrary to our hypotheses.

THEOREM. *If A, B, C, D are rational and*

$$A + \sqrt{B} = C + \sqrt{D},$$

then either (i) $A = C$, $B = D$ *or* (ii) B *and* D *are both squares of rational numbers.*

If A is not equal to C, let $A = C + x$. Then, $\sqrt{B} = x + \sqrt{D}$,

or
$$B = x^2 + D + 2x\sqrt{D};$$

i.e.
$$\sqrt{D} = (B - D - x^2)/2x,$$

which is rational, and therefore D is the square of a rational number. In this case $\sqrt{B} = C - A + \sqrt{D}$ is also rational. On the other hand, if $A = C$ it is obvious that $B = D$.

Corollaries. (i) If $A + \sqrt{B} = C + \sqrt{D}$, then $A - \sqrt{B} = C - \sqrt{D}$ (unless $\sqrt{B}$ and $\sqrt{D}$ are both rational).

(ii) The equation $\sqrt{B} = C + \sqrt{D}$ is impossible unless $C = 0$, $B = D$, or both $\sqrt{B}$ and $\sqrt{D}$ are rational.

Examples IV. 1. Prove *ab initio* that $\sqrt{2}$ and $\sqrt{3}$ are not similar surds.

2. Prove that $\sqrt{x}$ and $\sqrt{(1/x)}$ are similar surds (unless both are rational).

3. If a and b are positive and rational $\sqrt{a} + \sqrt{b}$ cannot be rational unless $\sqrt{a}$ and $\sqrt{b}$ are rational. The same is true of $\sqrt{a} - \sqrt{b}$, unless $a = b$.

4. If $$\sqrt{A}+\sqrt{B}=\sqrt{C}+\sqrt{D},$$
then either (*a*) $A=C$ and $B=D$, or (*b*) $A=D$ and $B=C$, or (*c*) $\sqrt{A}$, $\sqrt{B}$, $\sqrt{C}$, $\sqrt{D}$ are all rational or all similar surds.

[Square the given equation and apply the theorem above.]

5. A quadratic surd cannot be the sum of two dissimilar quadratic surds.

6. Neither $(a+\sqrt{b})^3$ nor $(a-\sqrt{b})^3$ can be rational unless $\sqrt{b}$ is rational.

7. Prove that if $x=p+\sqrt{q}$, where p and q are rational, x^m, where m is any integer, can be expressed in the form $P+Q\sqrt{q}$, where P and Q are rational. For example,
$$(p+\sqrt{q})^2=p^2+q+2p\sqrt{q},\quad (p+\sqrt{q})^3=p^3+3pq+(3p^2+q)\sqrt{q}.$$
Deduce that any polynomial in x with rational coefficients (i.e. any expression of the form
$$a_0x^n+a_1x^{n-1}+\dots+a_n,$$
where $a_0, \dots a_n$ are rational numbers), can be expressed in the form $P+Q\sqrt{q}$.

8. Express $1/(p+\sqrt{q})$ in the same form.

$$\left[\text{We obtain } \frac{1}{p+\sqrt{q}}=\frac{p}{p^2-q}-\frac{\sqrt{q}}{p^2-q}.\right]$$

9. Deduce from Exs. 7 and 8 that any expression of the form $G(x)/H(x)$, where $G(x)$ and $H(x)$ are polynomials in x with rational coefficients, can be expressed in the form $P+Q\sqrt{q}$, where P and Q are rational.

10. If $a+\sqrt{b}$, where b is not a perfect square, is the root of an algebraical equation with rational coefficients, then $a-\sqrt{b}$ is another root of the same equation.

11. If p, q, and p^2-q are positive we can express
$$\sqrt{p+\sqrt{q}}$$
in the form $\sqrt{x}+\sqrt{y}$, where
$$x=\tfrac{1}{2}\{p+\sqrt{(p^2-q)}\},\quad y=\tfrac{1}{2}\{p-\sqrt{(p^2-q)}\}.$$

12. Determine the conditions that it may be possible to express $\sqrt{p+\sqrt{q}}$, where p and q are rational, in the form $\sqrt{x}+\sqrt{y}$, where x and y are rational.

13. If a^2-b is positive, the necessary and sufficient conditions that
$$\sqrt{(a+\sqrt{b})}+\sqrt{(a-\sqrt{b})}$$
should be rational are that a^2-b and $\frac{1}{2}(a+\sqrt{a^2-b})$ should both be squares of rational numbers.

5. Irrational numbers in general. The arguments which led us to believe in the existence of quadratic surds, and corresponding points on the line L, were based on considerations of elementary geometry. There is, however, another way of looking at the matter which is even more instructive and important, as it leads us to consider classes of irrational numbers far more general than quadratic surds.

Consider the equation $x^2 = 2$. We have already seen that there is no rational number x which satisfies this equation. The square of any rational number is either less than or greater than 2. We can therefore divide the rational numbers into two classes, those whose squares are less than 2, and those whose squares are greater than 2. We call these two classes *the class* T, or *the lower class*, and *the class* U, or *the upper class*. It is obvious that every member of U is greater than all the members of T. Moreover, we can find a member of the class T whose square, though less than 2, differs from 2 by as little as we please. In fact, if we carry out the ordinary arithmetical process for the extraction of the square root of 2 we obtain a series of rational numbers, viz.

$$1,\ 1{\cdot}4,\ 1{\cdot}41,\ 1{\cdot}414,\ 1{\cdot}4142,\ \dots$$

whose squares

$$1,\ 1{\cdot}96,\ 1{\cdot}9881,\ 1{\cdot}999396,\ 1{\cdot}99996164,\ \dots$$

are all less than 2, but approach nearer and nearer to it; and by taking a sufficient number of the figures given by the process, we can obtain as close an approximation as we want. Similarly we can find a member of the class U whose square, though greater than 2, differs from 2 by as little as we please. It is sufficient to increase the last figure, in the series of approximations given above, by unity: we obtain

$$2,\ 1{\cdot}5,\ 1{\cdot}42,\ 1{\cdot}415,\ 1{\cdot}4143,\ \dots.$$

Or again, we can find a member of T and a member of U which differ from one another by as little as we please.

This follows at once from the fact that *every* rational number belongs to one class or the other. A formal proof may be supplied as follows. Take any member x of T and any member y of U. Let n be any positive integer, and consider the numbers

$$x,\quad x+\frac{1}{n}(y-x),\quad x+\frac{2}{n}(y-x),\quad \dots\dots,\quad x+\frac{n-1}{n}(y-x),\quad y.$$

Each of these is rational and belongs either to T or to U. Let $x+(r/n)(y-x)$ be the *first* which belongs to T. Then $x+\{(r+1)/n\}(y-x)$ belongs to U. Thus we have found a member of T and a member of U which differ by $(y-x)/n$. And by taking a large enough value of n we can make this difference as small as we like.

We add a formal proof that an x can be found in T and a y in U such that $x^2<2$ and $y^2>2$, but both squares differ from 2 by as little as we please.

Suppose we want each difference to be less than ϵ (where ϵ may be, say, ·01 or ·0001 or ·00001). We can, in virtue of what precedes, choose x and y so that

$$y-x<\tfrac{1}{4}\epsilon.$$

We may obviously suppose both x and y less than 2, since $x^2<2$ and y is nearly equal to x. Then

$$y^2-x^2=(y-x)(y+x)<4(y-x)<\epsilon,$$

and since $x^2<2$ and $y^2>2$ it follows *à fortiori* that $2-x^2$ and y^2-2 are each less than ϵ.

We have thus divided all the positive rational points on L into two classes T and U such that (i) the class U lies entirely to the right of the class T, (ii) we can find a pair of points, one in T and one in U, whose distance from one another is as small as we please. And our common-sense notion of the attributes of a straight line demands *the existence of a number x and a corresponding point P such that P divides the class T from the class U.*

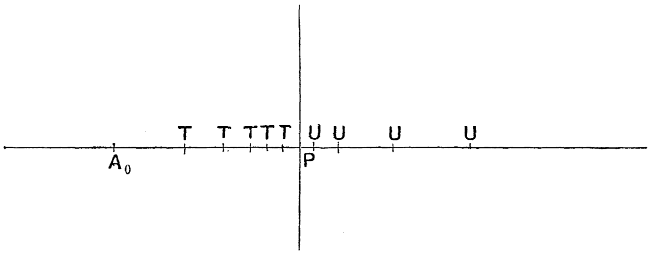

FIG. 4.

But (1) this number x cannot be rational. For if it were, P would belong either to the class T or the class U, let us say the former. Then $x^2<2$, or $x^2=2-\delta$, say, where δ is some positive number. But we can find a member of the class T whose square is as near to 2 as we like; and therefore we can find such a member of T whose square is greater than $2-\delta$, i.e. greater than x^2. That is to say, we can find a member of T which *lies to the right of P*: which is absurd. Hence P cannot belong to T. Similarly it cannot belong to U.

Again (2) x^2 cannot be either less than or greater than 2. If it were less than 2 we could, as above, find members of T to the right of P. This hypothesis is therefore untenable, and x^2 is not less than 2. Similarly it is not greater than 2.

Hence *there is a point P and a number x such that*

$$A_0P = x, \quad x^2 = 2.$$

This number x we denote by $\sqrt{2}$.

Examples V. 1. Find the difference between 2 and the squares of the decimals given in § 5 as approximations to $\sqrt{2}$.

2. Find the differences between 2 and the squares of

$$\tfrac{1}{1}, \tfrac{3}{2}, \tfrac{7}{5}, \tfrac{17}{12}, \tfrac{41}{29}, \tfrac{99}{70}.$$

3. Show that if m/n is a good approximation to $\sqrt{2}$, then $(m+2n)/(m+n)$ is a better one, and that the errors in the two cases are in opposite directions. Apply this result to continue the series of approximations in the last example.

4. If x and y are approximations to $\sqrt{2}$, by defect and by excess respectively, and $2-x^2<\epsilon$, $y^2-2<\epsilon$, then $y-x<\epsilon$.

5. The equation $x^2=4$ is satisfied by $x=2$. Examine how far the argument of the preceding sections applies to this equation (writing 4 for 2 throughout).

[We define the classes T, U as above. But in this case they do not include *all* rationals. The rational number 2 is an exception, since 2^2 is neither less than or greater than 4. And as before we are led to suppose the existence of a *dividing point.* But we cannot, of course, prove that this is not a rational point. It is, in fact, the point $x=2$.]

6. But the preceding argument may be applied to equations other than $x^2 = 2$, almost word for word; for example to $x^2 = N$, where N is any integer which is not a perfect square, or to

$$x^3 = 3, \quad x^3 = 7, \quad x^4 = 23,$$

or, as we shall see later on, to $x^3 = 3x + 8$. We are thus led to believe in the existence of points P on L such that $x = A_0P$ satisfies equations such as these, even when these lengths cannot be constructed by means of elementary geometrical methods.

The reader will no doubt remember that in treatises on elementary algebra the root of such an equation as $x^q = n$ is denoted by $\sqrt[q]{n}$ or $n^{1/q}$, and that a meaning is attached to such symbols as

$$n^{p/q}, \quad n^{-p/q}$$

by means of the equations

$$n^{p/q} = \sqrt[q]{n^p}, \quad n^{p/q}\, n^{-p/q} = 1.$$

And he will remember how, in virtue of these definitions, the 'laws of indices' such as

$$n^r \times n^s = n^{r+s}, \quad (n^r)^s = n^{rs}$$

are extended to cover the case in which r and s are any rational numbers whatever.

7. The continuum. The aggregate of points contained in a straight line L is called a **linear continuum.** It contains (1) the rational points, (2) the irrational points for whose existence we have the evidence summarized in the preceding sections, and the corresponding negative irrational points on the left of A_0, (3) all other points of the line, if any such there be. To each of these points corresponds a length measured from A_0, and capable of numerical measurement in terms of our unit-length A_0A_1. The measures of these lengths are the **real numbers,** positive or negative, integral, rational or irrational. The aggregate of all these numbers is called an **arithmetical continuum.** All the numbers contained in this arithmetical continuum may be operated with according to the ordinary rules of elementary algebra.

The substance of the preceding sections is not intended as a complete or rigorous analysis of the nature of either the linear or the arithmetical continuum. Such an analysis would be altogether beyond the scope of this book. What has been said is intended simply to remind the reader of some of the ideas on the subject which he no doubt already possesses, and to attempt to make them, and some of the obvious consequences which are involved in them, more explicitly present to his mind.

In order to show the incompleteness of the analysis of the numbers of the arithmetic continuum which has been given, we need only consider a few examples.

(i) Let us consider a more complicated surd expression, such as

$$z = \sqrt[3]{(4+\sqrt{15})} + \sqrt[3]{(4-\sqrt{15})}.$$

Our argument for supposing that the expression for z has a meaning, and that a point P exists on the line such that $A_0P = z$, might be as follows. We first show, as above, that there is a point P_1 such that if $y = A_0P_1$, $y^2 = 15$, and we can then determine points corresponding to the numbers $4+\sqrt{15}$, $4-\sqrt{15}$. Now consider the equation in z_1

$$z_1{}^3 = 4+\sqrt{15}.$$

The right-hand side of this equation is not rational: but exactly the same reasoning which leads us to suppose that the line contains a point x for which $x^3=2$ (or any other rational number) also leads us to the conclusion that it contains a point z_1 for which $z_1{}^3=4+\sqrt{15}$. Thus we find a point P' such that

$$z_1=A_0P'=\sqrt[3]{(4-\sqrt{15})}.$$

Similarly we find a point P'' such that

$$z_2=A_0P''=\sqrt[3]{(4-\sqrt{15})},$$

and taking $A_0P=A_0P'+A_0P''$ we have finally

$$A_0P=z=\sqrt[3]{(4+\sqrt{15})}+\sqrt[3]{(4-\sqrt{15})}.$$

Now it is easy to verify that

$$z^3=3z+8.$$

And we might have given a direct proof of the existence of a unique number z such that $z^3=3z+8$. It is easy to see that there cannot be two such numbers. For if $z_1{}^3=3z_1+8$ and $z_2{}^3=3z_2+8$, we find on subtracting and dividing by z_1-z_2 that $z_1{}^2+z_1z_2+z_2{}^2=3$. But if z_1 and z_2 are positive $z_1{}^3>8$, $z_2{}^3>8$ and therefore $z_1>2$, $z_2>2$, $z_1{}^2+z_1z_2+z_2{}^2>12$, and so the equation just found is impossible. And it is easy to see that neither z_1 nor z_2 can be negative. For if z_1 is negative and equal to $-\zeta$, ζ is positive and $\zeta^3-3\zeta+8=0$, or $3-\zeta^2=8/\zeta$. Hence $3-\zeta^2>0$, and so $\zeta<2$. But then $8/\zeta>4$ and cannot be equal to $3-\zeta^2$, which is less than 3.

Hence there is at most *one* z such that $z^3=3z+8$. And it cannot be rational. For any rational root of this equation must be integral and a factor of 8 (Ex. II. 5), and it is easy to verify that no one of ± 1, ± 2, ± 4, ± 8, is a root.

Thus $z^3=3z+8$ has at most one root and that root is not rational. We can now define the positive rational numbers x into two classes T, U according as $x^3<3x+8$, or $x^3>3x+8$. It is easy to see that if $x^3>3x+8$ and y is any number greater than x, then also $y^3>3y+8$. For suppose if possible $y^3\leqq 3y+8$. Then since $x^3>3x+8$ we obtain on subtracting $y^3-x^3<3(y-x)$, or $y^2+xy+x^2<3$, which is impossible, since y is positive and $x>2$ (since $x^3>8$). Similarly we can show that if $x^3<3x+8$ and $y<x$ then also $y^3<3y+8$.

Thus we have separated the rational numbers into two classes similar to the classes T, U of § 5. And we conclude, as there, that there is a number z which is greater than any number of T, and less than any number of U, and which satisfies the equation $z^3=3z+8$.

The reader who knows how to solve cubic equations by Cardan's method will be able to obtain directly from the equation the expression

$$z=\sqrt[3]{(4+\sqrt{15})}+\sqrt[3]{(4-\sqrt{15})}.$$

(ii) The direct argument applied above to the equation $x^3 = 3x + 8$ could be applied (though the application would be a little more difficult) to the equation

$$x^5 = x + 16,$$

and would lead us to the conclusion that a unique positive number exists which satisfies this equation. In this case, however, it is not possible to obtain a simple explicit expression for x composed of any combination of surds. It can in fact be proved (though the proof is difficult) that it is *generally* impossible to find such an expression for the root of an equation of higher degree than 4.

Thus, besides irrational numbers which can be expressed as pure or mixed quadratic or other surds, or combinations of such surds, there are others which cannot be so expressed. It is *only in very special cases* that such expressions can be found.

(iii) But even when we have added to our list of irrational numbers roots of equations (such as $x^5 = x + 16$) which cannot be explicitly expressed as surds, we have not exhausted the different kinds of irrational numbers contained in the continuum. Let us draw a circle whose diameter is equal to A_0A_1, i.e. to unity. It is natural to suppose that the circumference of such a circle has a length capable of numerical measurement as much as the diagonal of a square described on A_0A_1. This length is usually denoted by π. And it has been shown (though the proof is unfortunately long and difficult) that this number π is not the root of any algebraical equation with integral coefficients, so that we cannot have, for example, any such equation as

$$\pi^2 = n, \quad \pi^3 = n, \quad \pi^5 = \pi + n,$$

where n is an integer. If we take a point P such that $A_0P = \pi$, we have found a point which is not rational nor yet belongs to any of the classes of irrationals which we have so far considered. And this number π is no isolated or exceptional case. Any number of other examples can be constructed. In fact it is only special classes of irrational numbers which are roots of equations of this kind, just as it is only a still smaller class which can be expressed by means of surds.

Examples VI. 1. Show that $x=\sqrt[3]{(5+2\sqrt{6})}+\sqrt[3]{(5-2\sqrt{6})}$ satisfies the equation

$$x^3=3x+10,$$

and apply to this equation arguments similar to those used in § 7 (i). And, more generally, $x=\sqrt[3]{\{n+\sqrt{(n^2-1)}\}}+\sqrt[3]{\{n-\sqrt{(n^2-1)}\}}$ satisfies

$$x^3=3x+2n.$$

Consider this equation similarly, n being any positive integer.

2. Consider the equation

$$x^2-4x+3=0.$$

It is easy to see that $x=1$ and $x=3$ are roots of this equation. If we divide the rational numbers into two classes T and U according as

$$x^2-4x+3 \lesseqgtr 0,$$

we see that T contains all rational numbers between 1 and 3 and U all less than 1 or greater than 3. In this case we are not led to any irrational number, the numbers which divide the classes being 1 and 3. But if we consider instead the equation

$$x^2-4x+1=0,$$

(of which the roots are $2\pm\sqrt{3}$), we again have two points of division, in this case each irrational: and we might argue directly from the equation to the existence of two such numbers by dividing up the rational numbers into classes T, U as above.

8. **The continuous real variable.** The 'real numbers' may be regarded from two points of view. We may think of them *as an aggregate*, the 'arithmetical continuum' defined in the preceding section, or *individually*. And when we think of them individually, we may think either of a particular *specified* number (such as $1, -\frac{1}{2}, \sqrt{2}$, or π) or we may think of *any* number, *an unspecified* number, *the number* x. This last is our point of view when we make such assertions as 'x is a number,' 'x is the measure of a length,' 'x may be rational or irrational.' The x which occurs in propositions such as these is called *the continuous real variable*: and the individual numbers are called the *values* of the variable.

A 'variable,' however, need not necessarily be continuous. Instead of considering the aggregate of *all* real numbers, we might consider some partial aggregate contained in the former aggregate, such as the aggregate of rational numbers, or the aggregate of positive integers. Let us take the last case. Then in statements about *any* positive integer, or *an unspecified* positive integer, such as 'n is either odd or even,' n is called the variable, a *positive integral variable*, and the individual positive integers are its values.

In fact, this x and n are only examples of variables, the variable whose 'field of variation' is formed by all the real numbers, and that whose field is formed by the positive integers. These are the most important examples, but we have often to consider other cases. In the theory of decimals, for instance, we may denote by x any figure in the expression of any number as a decimal. Then x is a variable, but a variable which has only ten different values, viz. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The reader should think of other examples of variables with different fields of variation. He will find interesting examples in ordinary life. For instance—policeman x, the driver of cab x, star x in Herschel's catalogue, the year x, the xth day of the week.

MISCELLANEOUS EXAMPLES ON CHAPTER I.

1. If $a, b, c, \ldots k$ and $A, B, C, \ldots K$ are two sets of numbers, and all of the first set are positive, then

$$\frac{aA+bB+\ldots+kK}{a+b+\ldots+k}$$

lies between the algebraically least and greatest of $A, B, \ldots, K$.

2. What are the conditions that $ax+by+cz=0$, (1) for all values of x, y, z; (2) for all values of x, y, z subject to $\alpha x+\beta y+\gamma z=0$; (3) for all values of x, y, z subject to both $\alpha x+\beta y+\gamma z=0$ and $Ax+By+Cz=0$?

3. Any positive rational number can be expressed in one and only one way in the form

$$a_1+\frac{a_2}{1\,.\,2}+\frac{a_3}{1\,.\,2\,.\,3}+\ldots+\frac{a_k}{1\,.\,2\,.\,3\ldots k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \leqq a_1, \quad 0 \leqq a_2 < 2, \quad 0 \leqq a_3 < 3, \ \ldots \ 0 \leqq a_k < k.$$

[For example $\dfrac{47}{21}=\dfrac{11280}{7\,!}=\dfrac{7\cdot1611+3}{7\,!}=\dfrac{3}{7\,!}+\dfrac{6\cdot268+3}{6\,!}$

$$=\frac{3}{7\,!}+\frac{3}{6\,!}+\frac{5\cdot53+3}{5\,!}=\frac{3}{7\,!}+\frac{3}{6\,!}+\frac{3}{5\,!}+\frac{4\cdot13+1}{4\,!}$$

$$=\frac{3}{7\,!}+\frac{3}{6\,!}+\frac{3}{5\,!}+\frac{1}{4\,!}+\frac{1}{3\,!}+\frac{0}{2\,!}+2,$$

by continuing the same process. It is evident that k is *at most* equal to the largest prime factor of the denominator of the number given.]

4. Any positive rational number can be expressed in one and one way only as a simple continued fraction

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\dots\frac{1}{+a_n},$$

where a_1, a_2, ... are positive integers, of which the first only may be zero.

[Accounts of the theory of such continued fractions will be found in text-books of algebra.]

5. Find the rational roots (if any) of $9x^3-6x^2+15x-10=0$.

[Put $3x=y$ and apply the method of Ex. II. 7 to the resulting equation in y.]

6. A line AB is divided at C *in aurea sectione* (Euc. II. 11)—i.e. so that $AB\,.\,AC=BC^2$. Show that the ratio AC/AB is irrational.

[A direct geometrical proof will be found in Bromwich's *Infinite Series*, § 143, p. 363.]

7. A is irrational. In what circumstances can $\dfrac{aA+b}{cA+d}$, where a, b, c, d are rational, be rational?

8. Express $\sqrt{p}$, $\sqrt{q}$ in the form $ax+(b/x)$, where a, b are rational, and $x=\sqrt{p}+\sqrt{q}$.

9. If $\sqrt{p}$, $\sqrt{q}$ are dissimilar surds, and $a+b\sqrt{p}+c\sqrt{q}+d\sqrt{(pq)}=0$, where a, b, c, d are rational, then $a=0$, $b=0$, $c=0$, $d=0$.

[Express $\sqrt{p}$ in the form $M+N\sqrt{q}$, where M and N are rational, and apply the theorem of § 4.]

10. Show that if $a\sqrt{2}+b\sqrt{3}+c\sqrt{5}=0$, where a, b, c are rational numbers, then $a=0$, $b=0$, $c=0$.

11. Any polynomial in $\sqrt{p}$ and $\sqrt{q}$, with rational coefficients, (i.e. any sum of a finite number of terms of the form $A\,(\sqrt{p})^m(\sqrt{q})^n$, where m and n are integers, and A rational) can be expressed in the form

$$a+b\sqrt{p}+c\sqrt{q}+d\sqrt{pq},$$

where a, b, c, d are rational.

12. Express $\dfrac{a+b\sqrt{p}+c\sqrt{q}}{d+e\sqrt{p}+f\sqrt{q}}$, where a, b, etc. are rational, in the form

$$A+B\sqrt{p}+C\sqrt{q}+D\sqrt{pq},$$

where A, B, C, D are rational.

[Evidently

$$\frac{a+b\sqrt{p}+c\sqrt{q}}{d+e\sqrt{p}+f\sqrt{q}}=\frac{(a+b\sqrt{p}+c\sqrt{q})(d+e\sqrt{p}-f\sqrt{q})}{(d+e\sqrt{p})^2-f^2q}=\frac{\alpha+\beta\sqrt{p}+\gamma\sqrt{q}+\delta\sqrt{pq}}{\epsilon+\zeta\sqrt{p}},$$

where α, β, etc. are rational numbers which can easily be found. The required reduction may now be easily completed by multiplication of numerator and denominator by $\epsilon-\zeta\sqrt{p}$.]

For example, prove that

$$\frac{1}{1+\sqrt{2}+\sqrt{3}}=\frac{1}{2}+\frac{1}{4}\sqrt{2}-\frac{1}{4}\sqrt{6}.$$

13. If a, b, x, y are rational numbers such that

$$(ay-bx)^2+4(a-x)(b-y)=0,$$

prove that either (i) $x=a$, $y=b$, or (ii) $1-ab$ and $1-xy$ are squares of rational numbers. (*Math. Tripos*, 1903.)

[If we write $a-x=\xi$, $b-y=\eta$, we obtain

$$a^2\eta^2+b^2\xi^2+(4-2ab)\xi\eta=0.$$

Solving this equation for the ratio ξ/η we find that ξ/η (which we know to be rational) involves the quantity

$$\sqrt{\{(2-ab)^2-a^2b^2\}}=2\sqrt{(1-ab)}.$$

Hence $1-ab$ must be the square of a rational quantity. The only alternative is $\xi=\eta=0$.

But the equation given may also be written in the form

$$x^2\eta^2+y^2\xi^2+(4-2xy)\xi\eta=0.$$

Hence we deduce the same conclusion for $\sqrt{(1-xy)}$.]

14. If all the values of x and y given by

$$ax^2+2hxy+by^2=1, \quad a'x^2+2h'xy+b'y^2=1,$$

(where a, h, b, a', h', b' are rational) are rational, then

$$(h-h')^2-(a-a')(b-b'), \quad (ab'-a'b)^2+4(ah'-a'h)(bh'-b'h),$$

are both squares of rational quantities. (*Math. Tripos*, 1899.)

15. Show that $\sqrt{2}$ and $\sqrt{3}$ are cubic functions of $\sqrt{2}+\sqrt{3}$, with rational coefficients, and that $\sqrt{2}-\sqrt{6}+3$ is the ratio of two linear functions of $\sqrt{2}+\sqrt{3}$. (*Math. Tripos*, 1905.)

16. The expression

$$\sqrt{a+2m\sqrt{a-m^2}}+\sqrt{a-2m\sqrt{a-m^2}}$$

is equal to $2m$ if $2m^2>a>m^2$, and to $2\sqrt{(a-m^2)}$ if $a>2m^2$.

17. Show that any polynomial in $\sqrt[3]{2}$, with rational coefficients, can be expressed in the form

$$a+b\sqrt[3]{2}+c\sqrt[3]{4},$$

where a, b, c are rational.

More generally, any polynomial in $\sqrt[m]{p}$, with rational coefficients, can be expressed in the form

$$a_0+a_1x+a_2x^2+\dots+a_{m-1}x^{m-1},$$

where a_0, a_1, ... are rational and $x=\sqrt[m]{p}$. For any such polynomial is of the form

$$b_0+b_1x+b_2x^2+\dots+b_kx^k,$$

where the b's are rational. If $k \leqq m-1$ this is already of the form required. If $k > m-1$, let x^r be any power of x higher than the $(m-1)$th. Then $r = \lambda m + s$, where λ is an integer and $0 \leqq s \leqq m-1$: and $x^r = x^{\lambda m + s} = p^\lambda x^s$. Hence we can get rid of all powers of x higher than the $(m-1)$th.

18. Express $(\sqrt[3]{2}-1)^5$ in the form $a + b\sqrt[3]{2} + c\sqrt[3]{4}$, where a, b, c are rational.

19. Express $(\sqrt[3]{2}-1)/(\sqrt[3]{2}+1)$ in the same form.

[Multiply numerator and denominator by $\sqrt[3]{4} - \sqrt[3]{2} + 1$.]

20. If
$$a + b\sqrt[3]{2} + c\sqrt[3]{4} = 0,$$
where a, b, c are rational, then $a = 0$, $b = 0$, $c = 0$.

[Let $y = \sqrt[3]{2}$. Then $y^3 = 2$ and
$$cy^2 + by + a = 0.$$
Hence $2cy^2 + 2by + ay^3 = 0$ or
$$ay^2 + 2cy + 2b = 0.$$

Multiplying these two quadratic equations by a and c and subtracting we obtain $(ab - 2c^2)y + a^2 - 2bc = 0$, or $y = -(a^2 - 2bc)/(ab - 2c^2)$, a rational number, which is impossible. The only alternative is that $ab - 2c^2 = 0$, $a^2 - 2bc = 0$.

Hence $ab = 2c^2$, $a^4 = 4b^2c^2$. If $ab \neq 0$ we can divide the second equation by the first, which gives $a^3 = 2b^3$: and this is impossible, since $\sqrt[3]{2}$ cannot be equal to the rational quantity a/b. Hence $ab = 0$, $c = 0$, and it follows from the original equation that a, b, and c are all zero.

As a corollary, if $a + b\sqrt[3]{2} + c\sqrt[3]{4} = d + e\sqrt[3]{2} + f\sqrt[3]{4}$, then $a = d$, $b = e$, $c = f$.

It may be proved, more generally, that if
$$a_0 + a_1 p^{1/m} + \dots + a_{m-1} p^{(m-1)/m} = 0,$$
p not being a perfect mth power, then $a_0 = a_1 = \dots = a_{m-1} = 0$; but the proof is by no means so simple.]

21. Prove the theorem of § 4 by the method employed in Ex. 20.

22. If $A + \sqrt[3]{B} = C + \sqrt[3]{D}$, then either $A = C$, $B = D$, or B and D are both cubes of rational quantities.

[Assume $A = C + x$, cube, and apply the result of Ex. 20.]

23. If $\sqrt[3]{A} + \sqrt[3]{B} + \sqrt[3]{C} = 0$, then either one of A, B, C is zero, and the other two equal and opposite, or $\sqrt[3]{A}$, $\sqrt[3]{B}$, $\sqrt[3]{C}$ are rational multiples of the same surd $\sqrt[3]{X}$.

24. Find rational numbers α, β such that
$$\sqrt[3]{(7 + 5\sqrt{2})} = \alpha + \beta\sqrt{2}.$$

25. If
$$(a-b^3)\,b>0,$$
$$\sqrt[3]{a+\frac{9b^3+a}{3b}\sqrt{\left(\frac{a-b^3}{3b}\right)}}+\sqrt[3]{a-\frac{9b^3+a}{3b}\sqrt{\left(\frac{a-b^3}{3b}\right)}}$$
is rational.

[Each of the quantities under a cube root can be expressed in the form
$$\left\{\alpha+\beta\sqrt{\left(\frac{a-b^3}{3b}\right)}\right\}^3,$$
where α and β are rational.]

26. If $\alpha=\sqrt[n]{A}$, any polynomial in α is the root of an equation of degree n, with rational coefficients.

[We can express the polynomial (x say) in the form
$$x=l_1+m_1\alpha+\ldots+r_1\alpha^{(n-1)/n},$$
where l_1, m_1, ... are rational.

Similarly
$$x^2=l_2+m_2\alpha+\ldots+r_1\alpha^{(n-1)/n},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x^n=l_n+m_n\alpha+\ldots+r_n\alpha^{(n-1)/n}.$$

Hence
$$L_1x+L_2x^2+\ldots+L_nx^n=\Delta,$$
where Δ is the determinant
$$\begin{vmatrix} l_1 & m_1 & \ldots & r_1 \\ l_2 & m_2 & \ldots & r_2 \\ \ldots & \ldots & \ldots & \ldots \\ l_n & m_n & \ldots & r_n \end{vmatrix}$$
and L_1, L_2, ... the minors of l_1, l_2 ....]

27. Apply this process to $x=p+\sqrt{q}$. [The result is $x^2-2px+(p^2-q)=0$.]

28. Deduce from the result of the last example that if $p+\sqrt{q}=r+\sqrt{s}$, either $p=r$, $q=s$ or q, s are squares of rational quantities.

[It is easy to see that if $p+\sqrt{q}$ is not rational we must have
$$x^2-2px+(p^2-q)\equiv x^2-2rx+(r^2-s).]$$

29. Show that $y=a+bp^{1/3}+cp^{2/3}$ satisfies the equation
$$y^3-3ay^2+3y\,(a^2-bcp)-a^3-b^3p-c^3p^2+3abcp=0.$$

30. **Algebraical numbers**. We have seen that some irrational numbers (such as $\sqrt{2}$) are roots of equations of the type
$$a_0x^n+a_1x^{n-1}+\ldots+a_n=0,$$
where a_0, a_1, ..., a_n are integers. Such irrational numbers are called *algebraical* numbers: all other irrational numbers, such as π (§ 7), are called transcendental numbers. Show that if x is an algebraical number so are kx, where k is any rational number, and $x^{m/n}$, where m and n are any integers.

31. If x and y are algebraical numbers, so are $x+y$, $x-y$, xy and x/y.

[We have equations $\quad a_0x^m+a_1x^{m-1}+\dots+a_m=0,$

$$b_0y^n+b_1y^{n-1}+\dots+b_n=0,$$

where the a's and b's are integers. Write $x+y=z$, $y=z-x$ in the second, and eliminate x. We thus get an equation of similar form

$$c_0z^p+c_1z^{p-1}+\dots+c_p=0,$$

satisfied by z. Similarly for the other cases.]

32. If $$\mathrm{a}_0x^n+\mathrm{a}_1x^{n-1}+\dots+\mathrm{a}_n=0$$

where a_0, a_1, ..., a_n are any algebraical numbers, then x is an algebraical number.

[We have $n+1$ equations of the type

$$a_{0,r}\,\mathrm{a}_r^{\,m_r}+a_{1,r}\,\mathrm{a}_r^{\,m_r-1}+\dots+a_{m_r,r}=0,$$

($r=0$, 1, ..., n), in which the coefficients $a_{0,r}$, $a_{1,r}$, ... are integers. Eliminate a_0, a_1, ..., a_n between these and the original equation for x.]

33. Apply this process to the equation $x^2-2x\sqrt{2}+\sqrt{3}=0$.

[The result is $\quad x^8-16x^6+58x^4-48x^2+9=0.$]

34. Find equations, with rational coefficients, satisfied by

$$\sqrt{2}+\sqrt{3},\quad 1+\sqrt{2}+\sqrt{3},\quad \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}},\quad \sqrt{(2+\sqrt{2})}+\sqrt{(2-\sqrt{2})},$$

$$\sqrt{\{\sqrt{3}+\sqrt{2}\}}+\sqrt{\{\sqrt{3}-\sqrt{2}\}},\quad \sqrt[3]{2}+\sqrt[3]{3},\quad 1+\sqrt[3]{2}+\sqrt[3]{3},\quad \frac{\sqrt[3]{3}+\sqrt[3]{2}}{\sqrt[3]{3}-\sqrt[3]{2}}.$$

35. If $x^3=x+1$, then $x^{3n}=a_nx+b_n+c_n/x$, where

$$a_{n+1}=a_n+b_n,\quad b_{n+1}=a_n+b_n+c_n,\quad c_{n+1}=a_n+c_n.$$

36. If $x^6+x^5-2x^4-x^3+x^2+1=0$ and $y=x^4-x^2+x-1$, then y satisfies a quadratic equation with rational coefficients. (*Math. Tripos*, 1903.)

[It will be found that $y^2+y+1=0$.]

CHAPTER II.

FUNCTIONS OF REAL VARIABLES.

9. The idea of a function. Suppose that x and y are two continuous real variables, which we may suppose to be represented geometrically by distances $A_0P = x$, $B_0Q = y$ measured from fixed points A_0, B_0 along two straight lines L, M. And let us suppose that the positions of the points P and Q are not independent, but connected by a relation which we can imagine to be expressed as a relation between x and y: so that, when P and x are known, Q and y are also known. We might, for example, suppose that $y = x$, or $y = 2x$, or $\frac{1}{2}x$, or $x^2 + 1$. In all of these cases the value of x determines that of y. Or again, we might suppose that the relation between x and y is given, not by means of an explicit formula for y in terms of x, but by means of a geometrical construction which enables us to determine Q when P is known.

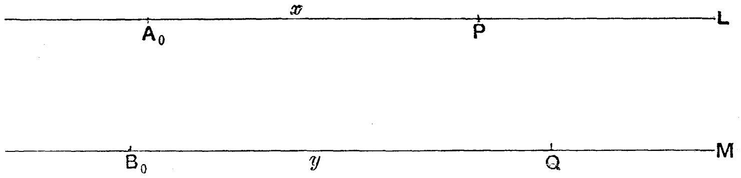

Fig. 5.

In these circumstances y is said to be a *function* of x. This notion of functional dependence of one variable upon another is perhaps the most important in the whole range of higher mathematics. In order to enable the reader to be certain that he understands it clearly we shall, in this chapter, illustrate it by means of a large number of examples.

But before we proceed to do this, we must point out that the simple examples of functions mentioned above possess three characteristics which are by no means involved in the general idea of a function, viz.:

(1) y is determined *for every value of* x;

(2) to each value of x for which y is given corresponds *one and only one value of* y;

(3) the relation between x and y is expressed by means of *an analytical formula.*

It is indeed the case that these particular characteristics are possessed by *many of the most important* functions. But the consideration of the following examples will make it clear that they are by no means essential to a function. All that is essential is that there should be some relation between x and y such that to some values of x at any rate correspond values of y.

Examples VII. 1. Let $y=x$ or $2x$ or $\frac{1}{2}x$ or x^2+1. Nothing further need be said at present about cases such as these.

2. Let $y=0$ *whatever be the value of* x. Then y is a function of x, for we can give x any value, and the corresponding value of y (viz. 0) is known. In this case the functional relation makes the same value of y correspond to all values of x. The same would be true were y equal to 1 or $-\frac{1}{2}$ or $\sqrt{2}$ instead of 0. Such a function of x is called *a constant.*

3. Let $y^2=x$. Then if x is positive this equation defines *two* values of y corresponding to each value of x, viz. $\pm\sqrt{x}$. If $x=0$, $y=0$. Hence to the particular value 0 of x corresponds *one* and only one value of y. But if x is negative there is *no* value of y which satisfies the equation. That is to say, *the function y is not defined for negative values of x.*

This function therefore possesses the characteristic (3), but not (1) or (2).

4. Consider a volume of gas maintained at a constant temperature and contained in a cylinder closed by a sliding piston*.

Let A be the area of the cross section of the piston and W its weight. The gas, held in a state of compression by the piston, exerts a certain pressure p_0 per unit of area on the piston, which balances the weight W, so that

$$W=Ap_0.$$

Let v_0 be the volume of the gas when the system is thus in equilibrium. If additional weight is placed upon the piston the latter is forced downwards. The volume (v) of the gas diminishes; the pressure (p) which it exerts upon unit area of the piston increases. Boyle's experimental law asserts that the product of p and v is very nearly constant, a correspondence which, if exact, would be represented by an equation of the type

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* I borrow this instructive example from Prof. H. S. Carslaw's *Introduction to the Calculus.*

$$pv = a \quad \text{.......................................(i)},$$

where a is a number which can be determined approximately by experiment.

Boyle's law, however, only gives a reasonable approximation to the facts provided the gas is not compressed too much. When v is decreased and p increased beyond a certain point the relation between them is no longer expressed with tolerable exactness by the equation (i). It is known that a much better approximation to the true relation can then be found by means of what is known as 'van der Waals' law,' expressed by the equation

$$\left(p + \frac{\alpha}{v^2}\right)(v - \beta) = \gamma \quad \text{..............................(ii)},$$

where α, β, γ are numbers which can also be determined approximately by experiment.

Of course the two equations, even taken together, do not give anything like a complete account of the relation between p and v. This relation is no doubt in reality much more complicated, and its form changes, as v varies, from a form nearly equivalent to (i) to a form nearly equivalent to (ii). But, from a mathematical point of view, there is nothing to prevent us from contemplating an ideal state of things in which, for all values of v above a certain limit, V say, (i) would be exactly true, and (ii) exactly true for all values of v less than V. And then we might regard the two equations as together defining p as a function of v. It is an example of a function which *for some values of v is defined by one formula and for other values of v is defined by another.*

This function possesses the characteristic (2): to any value of v only one value of p corresponds: but it does not possess (1). For p is at any rate not defined as a function of v for negative values of v; a *negative volume* means nothing, and so negative values of v do not present themselves for consideration at all.

5. Suppose that a perfectly elastic ball is dropped (without rotation) from a height $\frac{1}{2}g\tau^2$ on to a fixed horizontal plane, and rebounds continually.

The ordinary formulae of elementary dynamics, with which the reader is probably familiar, show that

$$h = \tfrac{1}{2}gt^2$$

if $0 \leqq t \leqq \tau$,

$$h = \tfrac{1}{2}g(2\tau - t)^2$$

if $\tau \leqq t \leqq 3\tau$, and generally

$$h = \tfrac{1}{2}g(2n\tau - t)^2$$

if $(2n-1)\tau \leqq t \leqq (2n+1)\tau$, h being the depth of the ball, at time t, below its original position. Obviously h is a function of t which is only defined for positive values of t.

The reader should construct other examples of functions which occur in physical problems.

6. Suppose that y is defined as being *the largest prime factor of* x. This is an instance of a definition which only applies to a particular class of values of x, viz., *integral* values. 'The largest prime factor of $\frac{11}{3}$ or of $\sqrt{2}$ or of π' means nothing, and so our defining relation fails to define for such values of x as these. Thus this function does not possess the characteristic (1). It does possess (2), but not (3), as there is no simple formula which expresses y in terms of x.

7. Let y be defined as *the denominator of* x *when* x *is expressed in its lowest terms*. This is an example of a function which is defined if and only if x is *rational*. Thus $y=7$ if $x=-11/7$: but y is not defined for $x=\sqrt{2}$, 'the denominator of $\sqrt{2}$' being a meaningless expression.

8. Let y be defined as *the height in inches of policeman* $C.x$, *in the Metropolitan Police, at* 5.30 *p.m. on* 8 *Aug.* 1907. Then y is defined for a certain number of integral values of x, viz., $1, 2, \dots, N$, where N is the total number of policemen in division C at that particular moment of time.

10. The graphical representation of functions. Coordinate geometry of two dimensions. Suppose that the variable y is a function of the variable x. It will generally be open to us also to regard x as a function of y, in virtue of the functional relation between x and y. But for the present we shall look at this relation from the first point of view. We shall then call x the *independent variable* and y the *dependent variable*; and, when the particular form of the functional relation is not specified, we shall express it by the general form of equation

$$y=f(x)$$

(or $F(x)$, $\phi(x)$, $\psi(x)$, ... as the case may be).

The nature of particular functions may, in very many cases, be illustrated and made easily intelligible as follows: draw two lines OX, OY at right angles to one another and produced indefinitely in both directions. We can represent values of x and y by distances measured from O along the lines OX, OY respectively, regard being paid, of course, to sign, and the positive directions of measurement being those indicated by arrows in Fig. 6.

Let a be any value of x for which y is defined and has (let us suppose) the single value b. Take $OA=a$, $OB=b$, and complete the rectangle $OAPB$. Imagine the point P marked on the diagram. This marking of the point P may be regarded as showing that the value of y for $x=a$ is b.

If to the value a of x corresponded *several* values of y (say b, b', b'') we should have, instead of the single point P, a number of points P, P', P''.

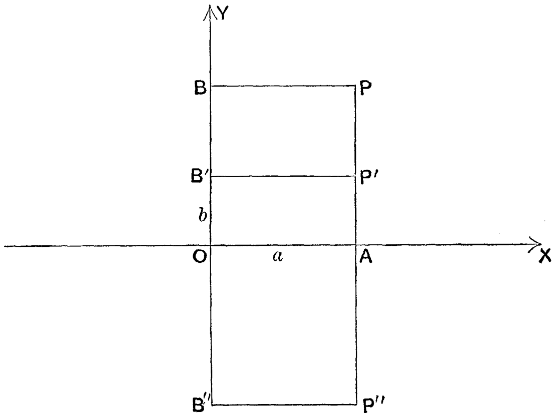

Fig. 6.

We shall call P the *point* (a, b); a and b the *coordinates of* P *referred to the axes* OX, OY; a the *abscissa*, b the *ordinate of* P; OX and OY the *axis of* x and the *axis of* y, or together the *axes of coordinates*, and O the *origin of coordinates*, or simply the *origin*.

Examples VIII. 1. Let P be the point (a, b), Q the point (α, β). Complete the parallelogram $OPRQ$. Show that R is the point $(a+\alpha, b+\beta)$.

2. The middle point of PQ is the point $\frac{1}{2}(a+\alpha)$, $\frac{1}{2}(b+\beta)$.

3. More generally, the line which divides PQ in the ratio $\mu : \lambda$ is the point $(\lambda a+\mu\alpha)/(\lambda+\mu)$, $(\lambda b+\mu\beta)/(\lambda+\mu)$. These expressions give, if the ratio $\mu : \lambda$ is properly chosen, the coordinates of any point on the line PQ.

4. The centre of mass of equal particles at the points (a_1, b_1), (a_2, b_2), ... (a_n, b_n) is the point $(a_1+a_2+\ldots+a_n)/n$, $(b_1+b_2+\ldots+b_n)/n$.

5. **Change of axes.** Draw through O lines OX', OY' making angles θ with OX, OY (Fig. 7). Draw PA', PB' perpendicular to OX', OY'. It is clear that P is determined if OA' and OB' are given just as much as if OA and OB are given. Let $OA=x$, $OB=y$, $OA'=x'$, $OB'=y'$. Then x' and y' are *the coordinates of* P *referred to the new axes* OX', OY'.

Prove that $x'=x\cos\theta+y\sin\theta$, $y'=-x\sin\theta+y\cos\theta$, and express x and y in terms of x' and y'.

6. In Ex. 5, the origin was left unchanged, but the new axes were inclined to the old ones. We might have taken new axes parallel to the old ones, but passing through a new origin O'. Let the coordinates of O' referred to the old axes be α, β. Express x' and y' in terms of x and y, and conversely.

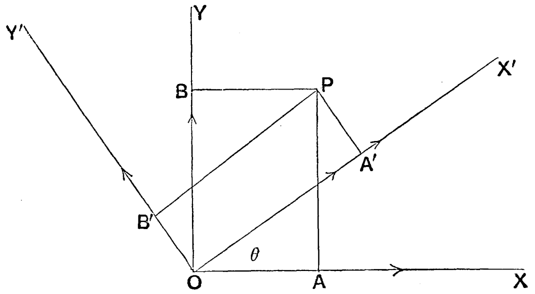

Fig. 7.

7. A new origin O' is taken, and new axes $O'X'$, $O'Y'$ inclined at any angle to the old ones. Show, by means of the results of Exs. 5 and 6, that x' and y' may be expressed in terms of x and y by formulae of the type $x'=ax+by+c$, $y'=dx+ey+f$, where $a, b, \ldots$ are numbers independent of x and y.

11. **The equation of a straight line.** Let us now suppose that for all values a of x for which y is defined, the value b (or values $b, b', b'', \ldots$) of y, and the corresponding point P (or points $P, P', P'', \ldots$) have been determined. We call the aggregate of all these points the **graph** of the function y.

To take a very simple example, suppose that y is defined as a function of x by the equation

$$ax + by + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where a, b, c are any fixed numbers. Then y is a function of x which possesses all the characteristics (1), (2), (3) of § 9. It is easy to show that *the graph of y is a straight line.*

First suppose $a = 0$. Then y has the constant value $-c/b$, and the graph is obviously a straight line parallel to OX.

Next suppose a different from zero, and suppose that (x_1, y_1) (x_2, y_2) are any two points on the graph, so that

$$ax_1 + by_1 + c = 0, \quad ax_2 + by_2 + c = 0 \quad \ldots\ldots\ldots\ldots(2).$$

The coordinates of any point P on the line joining (x_1, y_1) and (x_2, y_2) may (Ex. VIII. 3) be expressed in the form

$$\xi = (\lambda x_1 + \mu x_2)/(\lambda + \mu), \quad \eta = (\lambda y_1 + \mu y_2)/(\lambda + \mu).$$

But if we multiply the two equations (2) by λ and μ respectively, add the results, and divide by $\lambda + \mu$, we obtain

$$a\xi + b\eta + c = 0,$$

which shows that P lies on the graph. Hence the graph includes all the points of the line. And it cannot include any other points. For the line is not parallel to OX, since if it were y would be constant for all points on it, which is not the case. Hence there is one point on the line for which y has any value we like to assign. And so, if the graph contained a point (x', y') which did not lie on the line, there would be *two* values of x given by the equation $ax + by + c = 0$ when y had the value y': and this is obviously untrue. Thus the graph includes all the points of the line and no others.

We shall sometimes use another mode of expression. We shall say that when x and y vary in such a way that equation (1) is always true, *the locus of the point* (x, y) *is a straight line*, and we shall call (1) *the equation of the locus*, and say that the equation *represents* the locus. This use of the terms 'locus,' 'equation of the locus' is quite general, and may be applied whenever the graph of y is, in the ordinary sense of the word, a curve*, and the relation between x and y is capable of being represented by an analytical formula.

The preceding work does not apply when $b = 0$. The equation then reduces to $x = -c/a$, so that the distance of P from OY is constant—i.e. P lies on a line parallel to OY. In this case y does not occur in the equation at all, and so the latter cannot be regarded as defining y as a function of x. But it may be regarded as defining x as a function of y, viz. the constant $-c/a$.

The equation $ax + by + c = 0$ is *the general equation of the first degree*, for $ax + by + c$ is the most general polynomial in x and y which does not involve any terms of degree higher than the first in x and y. Hence *the general equation of the first degree represents a straight line.* It is equally easy to prove the converse proposition, *the equation of any straight line is of the first degree.* After the discussion which precedes we may leave this as an exercise for the reader.

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* 'Curve' of course includes straight line as a particular case. Some examples in which the 'graph' is *not*, in the ordinary sense of the word, a curve, will be found in Exs. XVI.

Examples IX. 1. The angles which the line $ax+by+c=0$ makes with OX are $\text{arc tan}\,(-a/b)$ and $\pi - \text{arc tan}\,(-a/b)$, where $\text{arc tan}\,\lambda$ denotes the numerically least angle whose tangent is λ.

2. If P is a point on the line, and x', y' are defined as in Ex. VIII. (5), show that

$$Ax'+By'+C=0,$$

where $$A=a\cos\theta+b\sin\theta,\quad B=b\cos\theta-a\sin\theta.$$

We call this equation *the equation of the line referred to the new axes OX', OY'*—it is the relation which connects the new coordinates x', y'. It will be observed that this equation also is of the first degree, as it obviously should be, since the proof that the equation of a straight line is of the first degree in no way depends upon what particular axes are chosen.

3. The coordinates of the point of intersection of $ax+by+c=0$ and $a'x+b'y+c'=0$ are

$$\frac{bc'-b'c}{ab'-a'b},\quad -\frac{ac'-a'c}{ab'-a'b};$$

unless $a/b=a'/b'$, in which case the lines are parallel.

4. The tangents of the angles between the lines in Ex. 3 are

$$\pm(ab'-a'b)/(aa'+bb'),$$

and the lines are perpendicular if $aa'+bb'=0$.

5. The length of the perpendicular from (ξ, η) on to $ax+by+c=0$ is

$$\frac{a\xi+b\eta+c}{\sqrt{(a^2+b^2)}},$$

the perpendicular being regarded as positive or negative according to the side of the line on which the point lies. [Positive when it is on the same side as 0, if $c>0$: negative in the same circumstances if $c<0$.]

6. The equation $(ax+by+c)+\lambda(\alpha x+\beta y+\gamma)=0$ represents a line through the intersection of $ax+by+c=0$ and $\alpha x+\beta y+\gamma=0$, and, by proper choice of λ, may be made to represent any such line. Discuss the particular case in which $\alpha/a=\beta/b$.

7. Hence show how to find the equation of a line through the intersection of two given lines and parallel or perpendicular to a third.

8. The equation of the circle whose centre is (a, b) and radius r is

$$(x-a)^2+(y-b)^2=r^2.$$

Conversely, any equation of this form represents a circle.

9. The most general equation of the *second* degree in x and y, in which there is no term in xy, and x^2 and y^2 have equal coefficients, viz.

$$a(x^2+y^2)+2gx+2fy+c=0$$

represents a circle if $f^2+g^2>ac$. Discuss the cases in which $f^2+g^2\leqq ac$.

10. Verify that the characteristic form of the equation of a circle (Ex. 9) is not altered by change of axes.

11. The general equation of a circle which passes through the points of intersection of two intersecting circles

$$(x-a)^2+(y-b)^2=r^2, \quad (x-\alpha)^2+(y-\beta)^2=\rho^2$$

is

$$(x-a)^2+(y-b)^2-r^2+\lambda\{(x-\alpha)^2+(y-\beta)^2-\rho^2\}=0.$$

12. If $\lambda=-1$ this last equation is of the first degree only, and represents the *common chord* of the two circles.

13. The two circles

$$x^2+y^2+2dx+2ey+k^2=0, \quad x^2+y^2+2\delta x+2\epsilon y+\kappa^2=0,$$

will represent a pair of intersecting circles if

$$d^2+e^2-k^2>0, \quad \delta^2+\epsilon^2-\kappa^2>0$$

and

$$4(d^2+e^2-k^2)(\delta^2+\epsilon^2-\kappa^2)>(2d\delta+2e\epsilon-k^2-\kappa^2)^2.$$

14. Show that the two circles in Ex. 13 will cut at right angles if

$$2d\delta+2e\epsilon=k^2+\kappa^2.$$

15. The area of the triangle formed by the points (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

taken positively. Hence deduce the result of § 11.

Examples X. 1. A point moves (a) so that its distance from a given line is constant, (b) so that its distances from two given lines are equal. Show that in each case the locus of the point is two straight lines (α) by geometrical reasoning, (β) by means of the results of § 11 and Ex. IX. 5.

2. The distances of a variable point P from a number of lines are p, p', p'', ..., and P moves so that

$$ap+bp'+cp''+\dots=0$$

where a, b, c, ... are constants. Show that the locus of P consists of a number of straight lines.

3. A, B are fixed points, and P a variable point which moves so that (a) $\lambda \,.\, AP^2+\mu \,.\, BP^2=$const., ($b$) $AP/BP=$const. Show that the locus of P is in either case a circle.

4. A line of fixed length moves with its ends upon OX and OY. Find the equation of the locus of the point which divides the line in the ratio $\mu : \lambda$.

[Let the line be AB, meeting OX, OY in A, B, and let $OA=a$, $OB=b$. The coordinates of the point P in question (Ex. VIII. 3) are $\lambda a/(\lambda+\mu)$ and $\mu b/(\lambda+\mu)$. Also $a^2+b^2=\text{const.}=c^2$, say. Thus if $x=\lambda a/(\lambda+\mu)$, $y=\mu b/(\lambda+\mu)$, we deduce $\dfrac{x^2}{\lambda^2}+\dfrac{y^2}{\mu^2}=\dfrac{c^2}{(\lambda+\mu)^2}$.

If $\lambda=\mu$, i.e. if P is the middle point of AB, this is the equation of a circle.]

5. A line of constant length moves with its ends on a fixed circle. Prove that the locus of the point which divides the line in a fixed ratio is a concentric circle.

12. Polar coordinates. In what precedes we have determined the position of P by the lengths of $OM=x$, $MP=y$. If $OP=r$ and $MOP=\theta$, θ being an angle between 0 and 2π (measured in the positive direction), it is evident that

$$x=r\cos\theta,\qquad y=r\sin\theta,$$
$$r=\sqrt{(x^2+y^2)},\qquad \cos\theta:\sin\theta:1::x:y:r,$$

and that the position of P is equally well determined by a knowledge of r and θ. We call r and θ the *polar coordinates* of P. The former, it should be observed, is essentially positive.

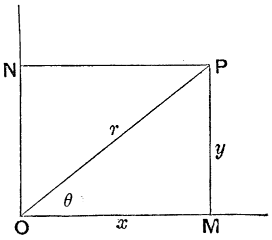

FIG. 8.

If P moves on a curve there will be some relation between r and θ, say $r=f(\theta)$ or $\theta=F(r)$. This we call the *polar equation* of the locus. The polar equation may be deduced from the (x, y) equation (or *vice versa*) by the formulae above.

It should be observed that (x, y) and (r, θ) are only two out of an infinite variety of 'systems of coordinates' which may be used to fix the position of P.

Examples XI. 1. The polar equation of a straight line is of the form

$$r\cos(\theta-\alpha)=p,$$

where p and α are constants.

2. The equation $r=2a\cos\theta$ represents a circle passing through the origin. So do $r=2a\sin\theta$ or $r=\lambda\cos\theta+\mu\sin\theta$. Find the radius of each circle.

3. The general equation of a circle is of the form

$$r^2+c^2-2rc\cos(\theta-\alpha)=a^2,$$

where a, c, and α are constants.

13. Further examples of functions and their graphical representation. In all of Exs. IX. and X. we were concerned with two very simple functions of x, viz. the functions y defined by the equations $\alpha x + \beta y + \gamma = 0$ or $(x-a)^2 + (y-b)^2 = r^2$. Only in Ex. X. 4 did we meet for a moment a slightly more general type of functional relation. The examples which follow will give the reader a better notion of the infinite variety of possible types of functions.

A. **Polynomials.** The meaning of the term *polynomial in x* was explained in Ch. I. It denotes a function of the form

$$a_0 x^m + a_1 x^{m-1} + \dots + a_m$$

where $a_0, a_1, \dots a_m$ are constants. The simplest polynomials are the *simple powers*

$$y = x,\ x^2,\ x^3,\ \dots,\ x^m,\ \dots.$$

The graph of the function x^m is of two distinct types, according as m is even or odd.

First let $m = 2$. Then three points on the graph are

$$(0, 0),\quad (1, 1),\quad (-1, 1).$$

Any number of additional points on the graph may be found by assigning other special values to x: thus the values

$$x = \tfrac{1}{2},\ 2,\ 3,\ -\tfrac{1}{2},\ -2,\ 3$$

give

$$y = \tfrac{1}{4},\ 4,\ 9,\ \ \tfrac{1}{4},\ \ 4,\ 9.$$

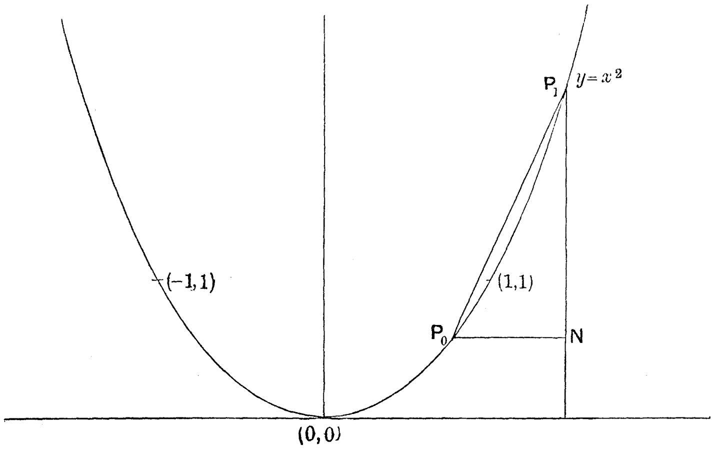

FIG. 9.

If the reader will plot off a fair number of points on the graph he will be led to conjecture that the form of the graph is something like that shown in Fig. 9. If he draws a curve through the special points which he has proved to lie on the graph and then tests its accuracy by giving x new values, and calculating the corresponding values of y, he will find that they lie as near to the curve as it is reasonable to expect, when the inevitable inaccuracies of drawing are considered.

There is, however, one fundamental question which we cannot answer adequately at present. The reader has no doubt some notions as to what is meant by a *continuous* curve, a curve without breaks or jumps—such a curve, in fact, as is roughly represented in Fig. 9. The question is whether the graph of the function $y=x^2$ is in fact such a curve. This cannot be *proved* by merely constructing any number of isolated points on the curve, although the more such points we construct the more probable it will appear.

This question cannot be discussed properly until Ch. IV. In that chapter we shall consider in detail what our common sense idea of continuity really means, and how we can prove that such graphs as the one now considered, and others which we shall consider later on in this chapter, are really continuous curves. For the present the reader may be content to draw his curves as common sense dictates.

It is easy to see that the curve $y=x^2$ is everywhere *convex* to the axis of x. Let P_0, P_1 (Fig. 9) be the points (x_0, x_0^2), (x_1, x_1^2). Then

$$\tan NP_0P_1 = \frac{NP_1}{P_0N} = \frac{x_1^2 - x_0^2}{x_1 - x_0} = x_0 + x_1,$$

and, if P_0 is kept fixed, this increases as x_1 increases—i.e. the slope of P_0P_1 becomes steeper and steeper.

The curve $y=x^4$ is similar to $y=x^2$ in general appearance, but flatter near O, and steeper beyond the points A, A' (Fig. 10). And $y=x^m$, where m is even and greater than 4, is still more so. And as m gets larger and larger the flatness and steepness grow more and more pronounced, until the curve is practically indistinguishable from the thick broken line in the figure.

The reader should next consider the curves given by $y=x^m$, when m is odd. The fundamental difference between the two cases is that whereas when m is even $(-x)^m = x^m$, so that the curve is symmetrical about OY, when m is odd $(-x)^m = -x^m$, so that y is negative when x is negative. Fig. 11 shows the curves $y = x$, $y = x^3$, and the form to which $y = x^m$ approximates for larger odd values of m.

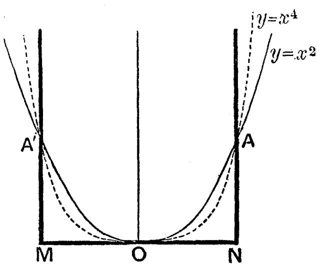

Fig. 10.

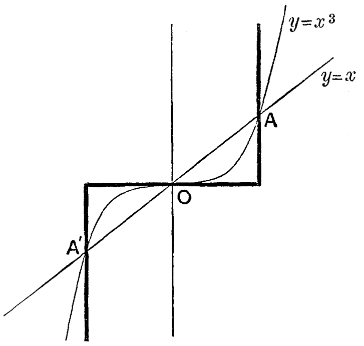

Fig. 11.

It is now easy to see how (theoretically at any rate) the graph of any polynomial may be constructed. In the first place, from the graph of $y = x^m$ we can at once derive that of Cx^m, where C is a constant, by multiplying the ordinate of every point of the curve by C. And if we know the graphs of $f(x)$ and $F(x)$ we can find that of $f(x) + F(x)$ by taking the ordinate of every point to be the sum of the ordinates of the corresponding points on the two original curves.

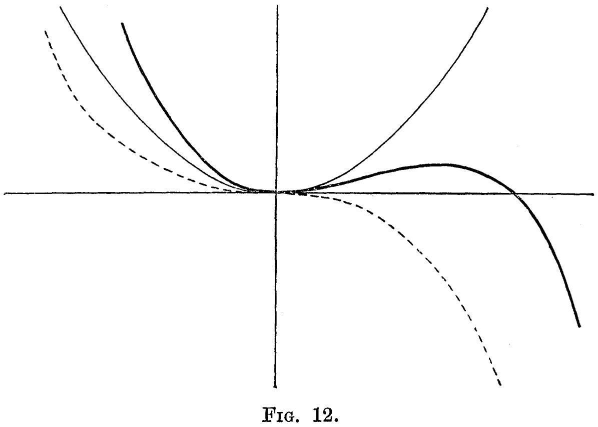

Fig. 12.

Fig. 12 shows the graph of $y=2x^2-x^3$, constructed in this way. The thin line is $y=2x^2$; the dotted line $y=-x^3$. In order to prevent the figure becoming of an awkward size, the scale for measurements along the axis of y has been taken to be one-quarter of that for measurements along the axis of x. This is often convenient: of course any ratio of the scales may be chosen.

The drawing of graphs of polynomials is however so much facilitated by the use of more advanced methods, which will be explained later on, that we shall not pursue the subject further here.

Examples XII. 1. Trace the curves $y=7x^4$, $y=3x^5$, $y=x^{10}$.

[The reader should draw the curves carefully, choosing the scales of measurement along OX and OY so as to get a convenient figure: but all three curves should be drawn *in one figure*. The reader will then realise how rapidly the higher powers of x increase, as x gets larger and larger, and will see that, in such a polynomial as

$$x^{10}+3x^5+7x^4,$$

(or even $x^{10}+30x^5+700x^4$) it is the *first* term which is of really preponderant importance when x is fairly large. Thus even when $x=4$, $x^{10}>1{,}000{,}000$, while $30x^5<35{,}000$ and $700x^4<180{,}000$; while if $x=10$ the preponderance of the first term is still more marked.]

2. Compare the relative magnitudes of x^{12}, $1{,}000{,}000x^6$, $1{,}000{,}000{,}000{,}000x$ when $x=1$, 10, 100, etc.

[The reader should make up a number of examples of this type for himself. This idea of the *relative rate of growth* of different functions of x is one with which we shall often be concerned in the following chapters.]

3. Draw the graph of $ax^2+2bx+c$.

[Here $y-\{(ac-b^2)/a\}=a\{x+(b/a)\}^2$. If we take new axes parallel to the old and passing through the point $-b/a$, $(ac-b^2)/a$, the new equation is $y'=ax'^2$. The reader should consider a few different cases in which a, b, c have numerical values, sometimes positive and sometimes negative.]

4. Trace the curves $y=x^3-3x+1$, $y=x^2(x-1)$, $y=x(x-1)^2$.

14. B. **Rational Functions.** The class of functions which ranks next to that of polynomials in simplicity and importance is that of rational functions. In Ch. I. we defined a rational function as the quotient of one polynomial by another: thus if $P(x)$, $Q(x)$ are polynomials we may denote the general rational function by

$$R(x)=\frac{P(x)}{Q(x)}.$$

In the particular case when $Q(x)$ reduces to unity or any other constant (i.e. does not involve x), $R(x)$ reduces to a polynomial: thus the class of rational functions includes that of polynomials as a sub-class. The following points concerning the definition should be noticed.

(1) We usually suppose that $P(x)$ and $Q(x)$ have no common factor $x+a$ or $x^p+ax^{p-1}+bx^{p-2}+\dots+k$, all such factors being removed by division.

(2) It should however be observed that this removal of common factors *does as a rule change the function.* Consider for example the function x/x, which is a rational function. On removing the common factor x we obtain $1/1=1$. But the original function is not *always* equal to 1: it is equal to 1 only so long as $x \neq 0$. If $x=0$ it takes the form $0/0$, which is meaningless. Thus the function x/x is equal to 1 if $x \neq 0$ and is undefined when $x=0$. It therefore differs from the function 1 which is *always* equal to 1.

(3) Such a function as

$$\left(\frac{1}{x+1}+\frac{1}{x-1}\right)\Big/\left(\frac{1}{x}+\frac{1}{x-2}\right)$$

may be reduced, by the ordinary rules of algebra, to the form

$$\frac{x^2(x-2)}{(x-1)^2(x+1)},$$

which is a rational function of the standard form. But here again it must be noticed that the reduction is not *always* legitimate. In order to calculate the value of a function for a given value of x we must substitute the value for x in the function *in the form in which it is given.* In the case of this function the values $x=-1, 1, 0, 2$ all lead to a meaningless expression, and so the function is not defined for these values. The same is true of the reduced form, so far as the values ± 1 are concerned. But $x=0$ or 2 gives the value 0. Thus once more the two functions are not strictly equivalent.

(4) But, as appears from the particular example considered under (3), even when the function has been reduced to a rational function of the standard form there will generally be a certain number of values of x for which it is not defined. These are the values of x (if any) for which the denominator vanishes. Thus $(x^2-7)/(x^2-3x+2)$ is not defined when $x=1$ or 2.

(5) Generally we agree, in dealing with expressions such as those considered in (2) and (3), to disregard the exceptional values of x for which such processes of simplification as were used there are illegitimate, and to reduce our function to the standard form of rational function. The reader will easily verify that (on this understanding) the sum, product, or quotient of two rational functions may themselves be reduced to rational functions of the standard type. And generally *a rational function of a rational function is itself a rational function*: i.e. if in $z=P(y)/Q(y)$, where P and Q are

polynomials, we substitute $y=P_1(x)/Q_1(x)$, we obtain on simplification an equation of the form $z=P_2(x)/Q_2(x)$.

(6) It is in no way presupposed in the definition of a rational *function* that the *constants* which occur as coefficients should be rational *numbers*. The word rational has reference solely to the way in which the variable x appears in the function. Thus

$$\frac{x^2+x+\sqrt{3}}{x\sqrt[3]{2}-\pi}$$

is a rational function.

The use of the word rational arises as follows. The rational function $P(x)/Q(x)$ may be generated from x by a definite number of operations upon x, including only multiplication of x by itself or a constant, addition of terms thus obtained, and division of one function, obtained by such multiplications and additions, by another. In so far as the variable x is concerned, this procedure is very much like that by which all rational numbers can be obtained from unity, a procedure exemplified in the equation

$$\frac{5}{3}=\frac{1+1+1+1+1}{1+1+1}.$$

Again, *any* function which can be deduced from x by the elementary operations mentioned above, using at each stage of the process functions which have already been obtained from x in the same way, can be reduced to the standard type of rational function. The most general kind of function which can be obtained in this way is sufficiently illustrated by the example

$$\left(\frac{x}{x^2+1}+\frac{2x+7}{x^2+\dfrac{11x-3\sqrt{2}}{9x+1}}\right)\bigg/\left(17+\frac{2}{x^3}\right),$$

which can obviously be reduced to the standard type of rational function.

15. The drawing of graphs of rational functions, even more than that of polynomials, is immensely facilitated by the use of

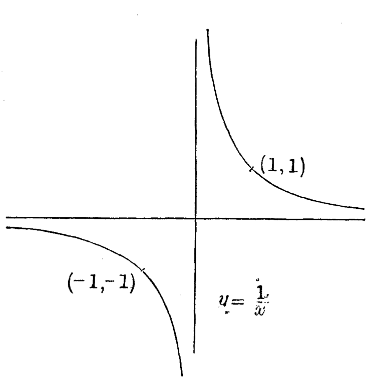

FIG. 13.

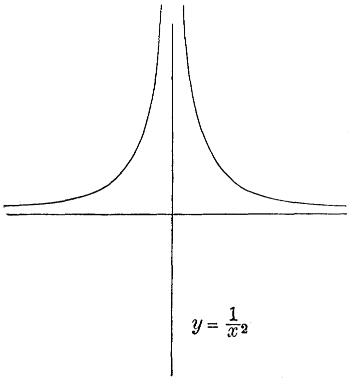

FIG. 14.

methods depending upon the differential calculus. We shall therefore content ourselves at present with a very few examples.

Examples XIII. 1. Draw the graphs of $y=1/x$, $y=1/x^2$, $y=1/x^3$, ….

[The figure shows the graphs of the first two curves. It should be observed that, since $1/0$, $1/0^2$, … are meaningless expressions, these functions are not defined for $x=0$.]

2. Trace $y=x+(1/x)$, $x-(1/x)$, $x^2+(1/x^2)$, $x^2-(1/x^2)$ and $ax+(b/x)$, taking various values, positive and negative, for a and b.

3. Trace

$$y=\frac{x+1}{x-1},\quad \left(\frac{x+1}{x-1}\right)^2,\quad \frac{x^2+1}{x^2-1},\quad \frac{1}{(x-1)^2},\quad (x+1)^2-\frac{1}{(x-1)^2}.$$

4. Trace $y=1/(x-a)(x-b)$, $1/(x-a)(x-b)(x-c)$, where $a<0<b<c$.

5. Sketch the general form assumed by the curves $y=1/x^m$ as m becomes larger and larger, considering separately the cases in which m is odd or even.

16. C. **Explicit Algebraical Functions.** The next important class of functions is that of *explicit algebraical functions.* These are functions which can be generated from x by a definite number of operations such as those used in generating rational functions, together with a definite number of operations of root extraction. Thus

$$\frac{\sqrt{(1+x)}-\sqrt[3]{(1-x)}}{\sqrt{(1+x)}+\sqrt[3]{(1-x)}},\qquad \sqrt{x}+\sqrt{(x+\sqrt{x})},$$
$$\left(\frac{x^2+x+\sqrt{3}}{x\sqrt[3]{2}-\pi}\right)^{\frac{2}{3}},$$

are explicit algebraical functions, and so is $x^{m/n}$ (i.e. $\sqrt[n]{x^m}$) where m and n are any integers.

Functions such as these differ fundamentally from rational functions in two respects. In the first place, a rational function is always defined for all values of x with a certain number of isolated exceptions. But such a function as $\sqrt{x}$ is undefined for *a whole range* of values of x (i.e. all negative values). Secondly, the function, when x has a value for which it is defined, has generally several values. Thus, if $x>0$, $\sqrt{x}$ has *two* values, of opposite signs.

Examples XIV. 1. $\sqrt{\{(x-a)(b-x)\}}$, where $a<b$, is defined only for $a \leqq x \leqq b$. If $a<x<b$ it has two values: if $x=a$ or b only one, viz. 0.

2. Consider similarly $\sqrt{(x-a)(x-b)(c-x)}$ $(a<b<c)$,

$$\sqrt{x(x^2-a^2)}, \quad \sqrt[3]{(x-a)^2(b-x)} \quad (a<b),$$

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}, \quad \sqrt{\{x+\sqrt{x}\}}.$$

3. Trace $y=\sqrt{x}, \quad \sqrt[3]{x}, \quad \sqrt[3]{x^2}, \quad (1+\sqrt{x})/(1-\sqrt{x}).$

4. Trace $y=\sqrt{(a^2-x^2)}, \quad y=b\sqrt{\{1-(x^2/a^2)\}}.$

17. D. **Implicit Algebraical Functions.** It is easy to verify that if

$$y=\frac{\sqrt{(1+x)}-\sqrt[3]{(1-x)}}{\sqrt{(1+x)}+\sqrt[3]{(1-x)}}$$

then
$$\left(\frac{1+y}{1-y}\right)^6=\frac{(1+x)^3}{(1-x)^2}$$

or if
$$y=\sqrt{x}+\sqrt{(x+\sqrt{x})}$$

then
$$y^4-(4y^2+4y+1)x=0.$$

Each of these equations is of the form

$$y^m+R_1y^{m-1}+\ldots+R_m=0 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $R_1, R_2, \ldots, R_m$ are rational functions of x: and the reader will easily verify that, if y is any one of the functions considered in the last set of examples, y satisfies an equation of this form. It is naturally suggested that the same is true of any explicit algebraic function. And this is in fact true, and indeed not difficult to prove, though we shall not delay to write out a formal proof here.

An example should make clear to the reader the lines on which such a proof would proceed. Let

$$y=\frac{x+\sqrt{x}+\sqrt{\{x+\sqrt{x}\}}+\sqrt[3]{(1+x)}}{x-\sqrt{x}+\sqrt{\{x+\sqrt{x}\}}-\sqrt[3]{(1+x)}}.$$

Then we have the equations

$$y=\frac{x+u+v+w}{x-u+v-w},$$

$$u^2=x, \quad v^2=x+u, \quad w^3=1+x,$$

and we have only to eliminate u, v, w between these equations in order to obtain an equation of the form desired.

We are therefore led to give the following definition: *a function* $y=f(x)$ *will be said to be an algebraical function of* x *if it is the root of an equation such as* (1), *i.e. the root of an equation of the* m^{th} *degree in* y, *whose coefficients are rational functions of* x.

This class of functions includes all the explicit algebraical functions considered in § 16. But it also includes other functions which cannot be expressed as explicit algebraical functions. For it is known that such an equation as (1) cannot *as a rule* be solved explicitly for y in terms of x, when m is greater than 4, though such a solution is always possible if $m=1$, 2, 3, or 4 and in special cases for higher values of m.

The definition of an algebraical function should be compared with that of an algebraical number given in the last chapter (Misc. Exs. 30).

Examples XV. 1. If $m=1$, y is a *rational* function.

2. If $m=2$ the equation is $y^2+R_1y+R_2=0$, so that

$$y=\tfrac{1}{2}\{-R_1\pm\sqrt{(R_1^2-4R_2)}\}.$$

This function is defined for all values of x for which $R_1^2\geqq 4R_2$. It has two values if $R_1^2>4R_2$ and one if $R_1^2=4R_2$.

If $m=3$ or 4 we can use the methods explained in treatises on Algebra for the solution of cubic and biquadratic equations. But as a rule the process is complicated and the results inconvenient in form, and we can generally study the properties of the function better by means of the original equation.

3. Consider the functions defined by the equations

$$y^2-2y-x^2=0,\quad y^2-2y+x^2=0,\quad y^4-2y^2+x^2=0,$$

in each case obtaining y as an explicit function of x, and stating for what values of x it is defined.

4. Find algebraical equations, with coefficients rational in x, satisfied by each of the functions

$$\sqrt{x}+\sqrt{(1/x)},\quad \sqrt[3]{x}+\sqrt[3]{(1/x)},\quad \sqrt[4]{x}+\sqrt[4]{(1/x)},\quad \sqrt[3]{(1+x)}+\sqrt[3]{(1-x)},$$
$$\sqrt{\{x+\sqrt{x}\}},\quad \sqrt{[x+\sqrt{\{x+\sqrt{x}\}}]}.$$

5. Consider the equation $y^4=x^2$.

[Here $y^2=\pm x$. If x is positive $y=\pm\sqrt{x}$: if negative $y=\pm\sqrt{(-x)}$. Thus the function has two values for all values of x save $x=0$, when it has the one value 0.]

6. An algebraical function of an algebraical function of x is itself an algebraical function of x.

[For we have

$$y^m + R_1(z)\, y^{m-1} + \dots + R_m(z) = 0,$$

where

$$z^n + S_1(x)\, z^{n-1} + \dots + S_n(x) = 0.$$

Eliminating z we find an equation of the form

$$y^p + T_1(x)\, y^{p-1} + \dots + T_p(x) = 0.$$

Here all the capital letters denote rational functions.]

7. An example should perhaps be given of an algebraical function which *cannot* be expressed in an explicit algebraical form. Such an example is the function y defined by the equation

$$y^5 - y - x = 0.$$

But a proof that we cannot find an explicit algebraical expression for y in terms of x is difficult, and cannot be attempted here.

18. Transcendental Functions. All functions of x which are not rational or even algebraical are called *transcendental* functions. This class of functions, being defined in so purely negative a manner, naturally includes an infinite variety of whole kinds of functions of varying degrees of simplicity and importance. Among these we can at present distinguish two kinds which are particularly interesting.

E. **The direct and inverse trigonometrical or circular functions.** These are the sine and cosine functions of elementary trigonometry, and their inverses, and the functions derived from them. We may assume that the reader is familiar with their most important properties.

Examples XVI. 1. Draw the graphs of $\sin x$, $\cos x$, and $a\cos x + b\sin x$.

[Since $a\cos x + b\sin x = \beta\cos(x-\alpha)$, where $\beta = \sqrt{(a^2+b^2)}$, and α is an angle whose cosine and sine are $a/\sqrt{(a^2+b^2)}$ and $b/\sqrt{(a^2+b^2)}$, the graphs of these three functions are similar in character.]

2. Draw the graphs of $\cos^2 x$, $\sin^2 x$, $a\cos^2 x + b\sin^2 x$.

3. Suppose the graphs of $f(x)$ and $F(x)$ drawn. Then the graph of

$$f(x)\cos^2 x + F(x)\sin^2 x$$

is a wavy curve which oscillates between the curves $y = f(x)$, $y = F(x)$. Draw the graph when $f(x)$, $F(x)$ are any pair of the functions

$$1/x^2,\quad 1/x,\quad 1,\quad x,\quad x^2,\quad ax+b,\quad x+(1/x).$$

4. Discuss in the same manner the form of the graph of

$$f(x)\cos x + F(x)\sin x.$$

5. Draw the graphs of $x+\sin x$, $x^2+\sin x$, $(1/x)+\sin x$, $x\sin x$, $x^2\sin x$, $(\sin x)/x$.

6. Draw the graph of $\sin(1/x)$.

[If $y=\sin(1/x)$, $y=0$ when $x=1/m\pi$, where m is any integer. Similarly $y=1$ when $x=1/(2m+\frac{1}{2})\pi$ and $y=-1$ when $x=1/(2m-\frac{1}{2})\pi$. The curve is entirely comprised between the lines $y=\pm 1$. It oscillates up and down, the rapidity of the oscillations becoming greater and greater as x approaches 0. For $x=0$ the function is undefined. When x is large y is small. The negative half of the curve is an inversion of the positive half (Fig. 15).]

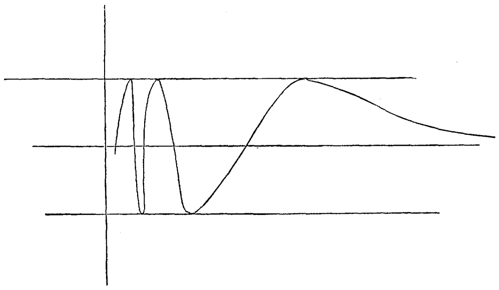

FIG. 15.

7. Draw the graph of $x\sin(1/x)$.

[This curve is comprised between the lines $y=\pm x$ just as the last curve was comprised between the lines $y=\pm 1$ (Fig. 16).]

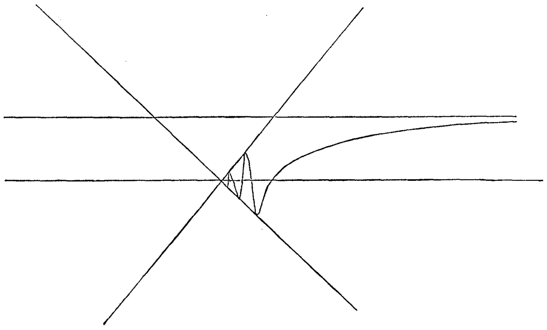

FIG. 16.

8. Draw the graphs of $x^2\sin(1/x)$, $(1/x)\sin(1/x)$, $\sin^2(1/x)$, $\{x\sin(1/x)\}^2$, $a\cos^2(1/x)+b\sin^2(1/x)$, $\sin x+\sin(1/x)$, $\sin x\sin(1/x)$.

9. Draw the graphs of $\cos x^2$, $\sin x^2$, $a\cos x^2+b\sin x^2$.

10. Draw the graphs of $\operatorname{arc}\cos x$ and $\operatorname{arc}\sin x$.

[If $y = \operatorname{arc}\cos x$, $x = \cos y$. This enables us to draw the graph of x, considered as a function of y, and the same curve shows y as a function of x. It is clear that y is only defined for $-1 \leqq x \leqq 1$, and is infinitely many valued for these values of x. As the reader no doubt remembers, there is, when $-1 < x < 1$, a value of y between 0 and π, say α, and the other values of y are given by the formula $2n\pi \pm \alpha$, where n is any integer, positive or negative.]

11. Draw the graphs of

$$\tan x, \quad \cot x, \quad \sec x, \quad \operatorname{cosec} x, \quad \tan^2 x, \quad \cot^2 x, \quad \sec^2 x, \quad \operatorname{cosec}^2 x.$$

12. Draw the graphs of $\operatorname{arc}\tan x$, $\operatorname{arc}\cot x$, $\operatorname{arc}\sec x$, $\operatorname{arc}\operatorname{cosec} x$. Give formulae expressing all the values of each of these functions in terms of any particular value.

13. Draw the graphs of $\tan(1/x)$, $\cot(1/x)$, $\sec(1/x)$, $\operatorname{cosec}(1/x)$.

14. Show that $\sin x$ and $\cos x$ are not rational functions of x.

[It is easy to see that no function which, like the sine or cosine, has a *period*, can possibly be a rational function. For suppose that

$$f(x) = P(x)/Q(x),$$

where P and Q are polynomials, and $f(x) = f(x + 2\pi)$, each of these equations holding for all values of x. Let $f(0) = k$. Then the equation

$$P(x) - kQ(x) = 0$$

is satisfied by an infinite number of values of x, viz. $x = 0$, 2π, 4π, etc., and so it is an *identity*. Thus $f(x) = k$ for all values of x, i.e. $f(x)$ is a mere constant.]

15. Show, more generally, that no function with a period can be an algebraical function of x.

[Let the equation which defines the algebraical function be

$$y^m + R_1 y^{m-1} + \dots + R_m = 0 \quad\dots\dots\dots\dots (1),$$

where R_1, ... are rational functions of x. This may be put in the form

$$P_0 y^m + P_1 y^{m-1} + \dots + P_m = 0,$$

where P_0, P_1, ... are polynomials in x. Arguing as above we see that

$$P_0 k^m + P_1 k^{m-1} + \dots + P_m = 0$$

is an identity. Hence $y = k$ satisfies the equation (1) for all values of x, and one set of values of our algebraical function reduces to a constant.

Now divide (1) by $y - k$ and repeat the argument m times. Our final conclusion is that our algebraical function has, for any value of x, the same m values k, k', ...; i.e. it is composed of m mere constants.]

16. The inverse sine and inverse cosine are not rational or algebraical functions.

[This follows from the fact that for any value of x between -1 and $+1$, $\operatorname{arc}\sin x$ and $\operatorname{arc}\cos x$ have infinitely many values.]

19. F. Other classes of transcendental functions. Next in importance to the trigonometrical functions come the exponential and logarithmic functions, which will be discussed in Chh. IX. and X. But these functions are beyond our range at present. And most of the other classes of transcendental functions whose properties have been studied, such as the elliptic functions, Bessel's and Legendre's functions, Gamma-functions, and so forth, lie altogether beyond the range of this book. There are however some elementary types of functions which, though of much less importance theoretically than the rational, algebraical, or trigonometrical functions, are particularly instructive as illustrations of the possible varieties of the functional relation.

Examples XVII. 1. Let $y=[x]$, where $[x]$ denotes the algebraically greatest integer contained in x. The graph is shown in Fig. 17 (*a*).

2. $y=x-[x]$. (Fig. 17 (*b*).)

3. $y=\surd\{x-[x]\}$. (Fig. 17 (*c*).)

4. $y=[x]+\surd\{x-[x]\}$. (Fig. 17 (*d*).)

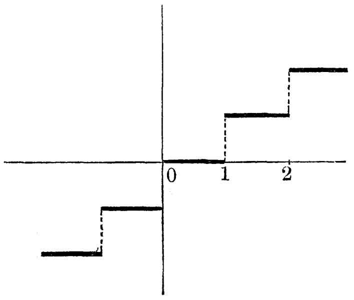

Fig. 17 *a*.

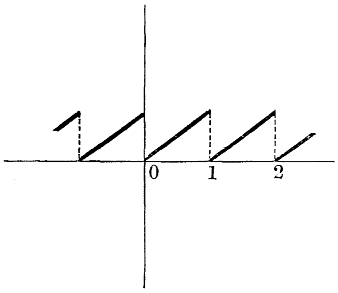

Fig. 17 *b*.

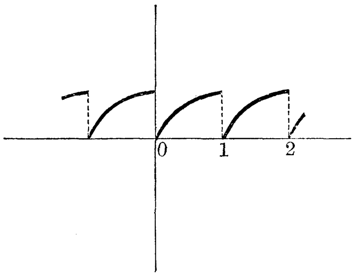

Fig. 17 *c*.

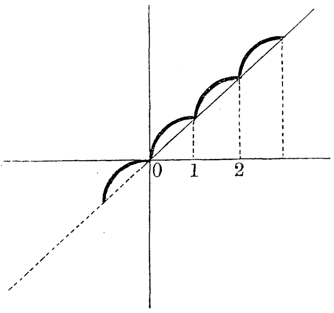

Fig. 17 *d*.

5. $y=(x-[x])^2$, $[x]+(x-[x])^2$.

6. $y=[\surd x]$, $[x^2]$, $\surd x-[\surd x]$, $x^2-[x^2]$, $[1-x^2]$.

7. Let y be defined as *the largest prime factor of* x (cf. Exs. VII. 6). Then y is defined only for integral values of x. When

$$\begin{array}{l}x=\pm 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ \ldots\\ y=\ \ \ 1,\ 2,\ 3,\ 2,\ 5,\ 3,\ 7,\ 2,\ 3,\ \ 5,\ 11,\ \ 3,\ 13,\ \ldots\end{array}$$

The graph consists of a number of isolated points.

8. Let y be *the denominator of* x (Exs. VII. 7). In this case y is defined only for rational values of x. We can mark off as many points on the graph as we please, but the result is not in any ordinary sense of the word a curve, and there are no points corresponding to any irrational values of x.

Draw the straight line joining the points $(N-1, N)$, (N, N). Show that the number of points of the locus which lie on this line is equal to the number of numbers less than and prime to N.

9. Let $y=0$ when x is an integer, $y=x$ when x is not an integer. The graph is derived from the straight line $y=x$ by taking out the points

$$\ldots(-1,\ -1),\quad (0,\ 0),\quad (1,\ 1),\quad (2,\ 2),\ \ldots,$$

and adding the points $(-1, 0)$, $(0, 0)$, $(1, 0)$, ... on the axis of x.

The reader may possibly regard this as an *unreasonable* function. *Why*, he may ask, if y is equal to x for all values of x save integral values, should it not be equal to x for integral values too? The answer is simply, *why should it?* The function y does in point of fact answer to the definition of a function: there is a relation between x and y such that when x is known y is known. We are perfectly at liberty to take this relation to be what we please, however arbitrary and apparently futile. This function y is, of course, a quite different function from that one which is *always* equal to x, whatever value, integral or otherwise, x may have. Let us take an apparently still more arbitrary example.

10. Let

$$\begin{aligned} y&=0 \text{ when } x=-2\tfrac{1}{2},\\ y^2&=1 \text{ when } x=-1,\\ y&=\sin x \text{ if } -\tfrac{1}{2}\leqq x\leqq\tfrac{1}{2},\end{aligned}$$

and

$$y^2=x^2 \quad \text{if} \quad 1<x\leqq 2,$$

except that $y=-1$ when $x=1\frac{1}{2}$. And for $x=3$ let y have all values between -1 and $+1$. Finally, suppose that y is not defined at all except for the various values just enumerated. The graph is shown in Fig. 18. It consists of the curved arc L, the line P, the lines M and N, from which however the middle points and the ends nearest the axis of x must be taken out, and the four isolated points A, B, C, D. We notice further that y has *infinitely many* values for $x=3$, *two* for $x=-1$ and $1<x<1\frac{1}{2}$ and $1\frac{1}{2}<x\leqq 2$, *one* for $x=-2\frac{1}{2}$, $-\frac{1}{2}\leqq x\leqq\frac{1}{2}$ and $x=1\frac{1}{2}$, and *none* for any other value of x.

This example is given merely to illustrate *possibilities*; it is not suggested that such functions as these are likely to be of any practical importance.

The reader should however not be too ready to assume that even from the practical point of view it is only what is obvious and straightforward which is important. If he turns back to Exs. VII. 4, 5, for instance, he will see examples of functions suggested by physical considerations and defined by different formulae for different ranges of values of x.

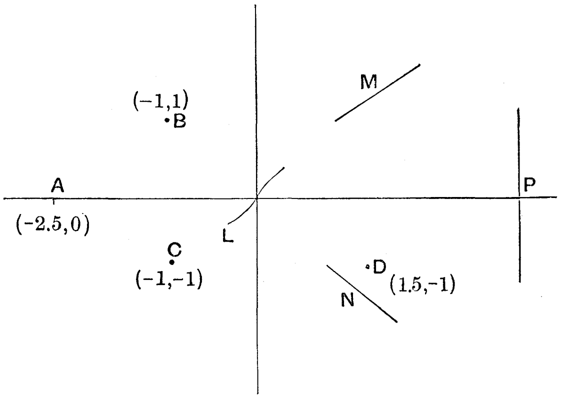

Fig. 18.

11. Let $y=1$ when x is rational, but $y=0$ when x is irrational. The graph consists of two series of points arranged upon the lines $y=1$ and $y=0$. To the eye it is not distinguishable from two continuous straight lines, but in reality an infinite number of points are missing from each line.

12. Let $y=x$ when x is irrational and $y=\sqrt{\{(1+p^2)/(1+q^2)\}}$ when x is a rational fraction p/q.

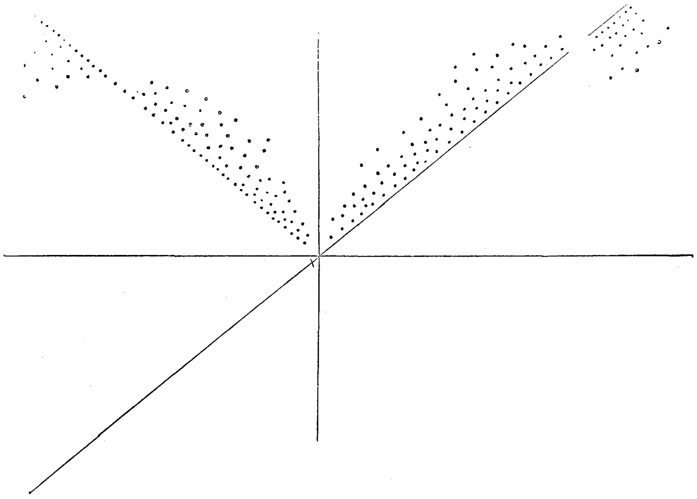

Fig. 19.

The irrational values of x contribute to the graph a curve in reality discontinuous, but apparently not to be distinguished from the straight line $y=x$.

Now consider the rational values of x. First let x be positive. Then $\surd\{(1+p^2)/(1+q^2)\}$ cannot be equal to p/q unless $p=q$, i.e. $x=1$. Thus all the points which correspond to rational values of x lie off the line, except the one point (1, 1). Again, if $p<q$, $\surd\{(1+p^2)/(1+q^2)\}>p/q$; if $p>q$, $\surd\{(1+p^2)/(1+q^2)\}<p/q$. Thus the points lie above the line $y=x$ if $0<x<1$, below if $x>1$. If p and q are large $\surd\{(1+p^2)/(1+q^2)\}$ is nearly equal to p/q. Near any value of x we can find any number of rational fractions with large numerators and denominators. Hence the graph contains a large number of points which crowd round the line $y=x$. Its general appearance (for positive values of x) is that of a line surrounded by a swarm of isolated points which gets denser and denser as the points approach the line.

The part of the graph which corresponds to negative values of x consists of the rest of the discontinuous line together with the reflections of all these isolated points in the axis of y. Thus to the left of the axis of y the swarm of points is not round $y=x$ but round $y=-x$, which is not itself part of the graph. See Fig. 19.

20. Graphical solution of Equations containing a single unknown quantity.

Many equations can be expressed in the form

$$f(x)=\phi(x) \text{..........................} (1),$$

where $f(x)$ and $\phi(x)$ are functions whose graphs are easy to draw. And it is obvious that if the curves

$$y=f(x), \quad y=\phi(x)$$

intersect in a point P whose abscissa is ξ, then ξ is a root of the equation (1).

Examples XVIII. 1. **The quadratic equation** $ax^2+2bx+c=0$. This may be solved graphically in a variety of ways. For instance we may draw the graphs of

$$y=ax+2b, \quad y=-c/x,$$

whose intersections give the roots, if any. Or we may take

$$y=x^2, \quad y=-(2bx+c)/a.$$

But the simplest method is probably to draw the circle

$$a(x^2+y^2)+2bx+c=0,$$

whose centre is $(-b/a, 0)$ and radius $\{\surd(b^2-ac)\}/a$. The abscissae of its intersections with the axis of x are the roots of the equation.

2. Solve by any of these methods

$$x^2+2x-3=0, \quad x^2-7x+4=0, \quad 3x^2+2x-2=0.$$

3. **The equation** $x^m+ax+b=0$. This may be solved by constructing the curves $y=x^m$, $y=-ax-b$.

4. Verify the following table for the number of real roots (if any) of

$$x^m+ax+b=0:$$

$$(a)\quad m \text{ even} \begin{cases} b \text{ positive, } \textit{two or none}, \\ b \text{ negative, } \textit{two}; \end{cases}$$

$$(b)\quad m \text{ odd} \begin{cases} a \text{ positive, } \textit{one}, \\ a \text{ negative, } \textit{three or one}. \end{cases}$$

Construct numerical examples to illustrate all possible cases.

5. Show that the equation $\tan x=ax+b$ has always an infinite number of real roots.

6. Determine the number of real roots of

$$\sin x=x, \quad \sin x=x/3, \quad \sin x=x/8, \quad \sin x=x/120.$$

7. Show that if a is small and positive (e.g. $a=\cdot01$) the equation

$$x-a=\tfrac{1}{2}\pi\sin^2 x$$

has three real roots. Consider also the case in which a is small and negative. Explain how the number of roots varies as a varies.

21. Functions of two variables and their graphical representation. In § 9 we considered two variables connected by a relation. We may similarly consider *three* variables (x, y, and z) connected by a relation such that when the values of x and y are both given, the value or values of z are known. In this case we call z a *function of the two variables* x and y; x and y the *independent* variables, z the *dependent* variable; and we express this dependence of z upon x and y by writing

$$z=f(x, y).$$

The remarks of § 9 may all be applied, *mutatis mutandis*, to this more complicated case.

The method of representing such functions of two variables graphically is exactly the same in principle as in the case of functions of a single variable. We must take *three* axes OX, OY, OZ in space of three dimensions, each axis being perpendicular to the other two. The point (a, b, c) is the point whose distances from the planes YOZ, ZOX, XOY, measured parallel to OX, OY, OZ, are a, b, and c. Regard must of course be paid to sign, lengths measured in the directions OX, OY, OZ being regarded as positive. The definitions of *coordinates, axes, origin* are the same as before.

Now let $$z = f(x, y).$$
As x and y vary the point (x, y, z) will move in space. The aggregate of all these points is called the *locus* of the point (x, y, z) or the *graph* of the function $z = f(x, y)$. When the relation between x, y, and z which defines z can be expressed in an analytical formula this formula is called the *equation* of the locus.

22. Equation of a plane. It may be shown without difficulty that the coordinates of the point which divides PQ in a given ratio $\mu : \lambda$ are

$$\frac{\lambda a + \mu\alpha}{\lambda + \mu}, \quad \frac{\lambda b + \mu\beta}{\lambda + \mu}, \quad \frac{\lambda c + \mu\gamma}{\lambda + \mu},$$

where (a, b, c) are the coordinates of P and (α, β, γ) those of Q (cf. Ex. VIII. 3).

From this we can at once deduce the following important theorem: *the general equation of the first degree represents a plane.* For let

$$ax + by + cz + d = 0$$

be the equation; and let (x_1, y_1, z_1), (x_2, y_2, z_2) be two points P, Q on the graph of the function z (or the locus represented by the equation). Then

$$ax_1 + by_1 + cz_1 + d = 0,$$
$$ax_2 + by_2 + cz_2 + d = 0,$$

and so, multiplying by λ and μ, adding, and dividing by $\lambda + \mu$,

$$a\frac{\lambda x_1 + \mu x_2}{\lambda + \mu} + b\frac{\lambda y_1 + \mu y_2}{\lambda + \mu} + c\frac{\lambda z_1 + \mu z_2}{\lambda + \mu} + d = 0.$$

Thus the locus is such that if P and Q lie upon it, the point R which divides PQ in any ratio lies upon it. That is to say every point of the line PQ lies in the locus. The locus therefore satisfies Euclid's definition of a plane. Conversely *the equation of any plane is of the first degree.* For let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be any three points on the plane. We can choose a, b, c, d so that

$$ax_1 + by_1 + cz_1 + d = 0,$$
$$ax_2 + by_2 + cz_2 + d = 0,$$
$$ax_3 + by_3 + cz_3 + d = 0.$$

We can therefore determine a locus represented by an equation of the type

$$ax + by + cz + d = 0,$$

which passes through the three points. But we have already seen that this locus is a plane; and it can obviously only be the original plane. The equation of that plane is therefore of the first degree.

Examples XIX. 1. Prove that if (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) are three points on a plane, the point whose coordinates are $\frac{\lambda a_1+\mu a_2+\nu a_3}{\lambda+\mu+\nu}$, etc. lies on the plane; and that it may, by choosing the ratios $\lambda : \mu : \nu$ appropriately, be made to coincide with any point in the plane.

2. Hence, by an argument similar to that of § 11, deduce that the general equation of the first degree represents a plane, and conversely.

3. The equation of the sphere whose centre is (a, b, c) and radius r is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2.$$

Conversely, this equation always represents a sphere.

4. Establish results for planes and spheres corresponding to those of Exs. IX. 3–9, 11–14.

23. Curves in a plane. We have hitherto used the notation

$$y=f(x) \quad \dots\dots\dots\dots\dots\dots\dots\dots(1)$$

to express functional dependence of y upon x. It is evident that this notation is most appropriate in the case in which y is expressed explicitly by means of some formula involving x alone, as when for example

$$y=x^2, \quad \sin x, \quad a\cos^2 x+b\sin^2 x.$$

We have however very often to deal with functional relations which cannot be or are most conveniently not expressed in this form. If, for example, $y^5-y-x=0$ or $x^5+y^5-ay=0$ it is known to be impossible to express y explicitly as a simple function of x. If

$$x^2+y^2+2gx+2fy+c=0$$

y can indeed be so expressed, viz. by the formula

$$y=-f\pm\sqrt{f^2-x^2-2gx-c};$$

but the functional dependence of y upon x is better and more simply expressed by the original equation.

It will be observed that in these two cases the functional relation is fully expressed *by equating a function of the two variables x and y to zero,* i.e. by means of an equation

$$f(x, y)=0 \quad \dots\dots\dots\dots\dots\dots\dots\dots(2).$$

We shall adopt this equation as the standard method of expressing the functional relation. It includes the equation (1) as a special case, since $y-f(x)$ is a special form of a function of x and y. We can then speak of the locus of the point (x, y) subject to $f(x, y)=0$, the graph of the function y defined by $f(x, y)=0$, the curve or locus $f(x, y)=0$, and the equation of this curve or locus.

There is another method of representing curves which is often useful. Suppose that x and y are both functions of a third variable t, which is to be regarded as essentially auxiliary and devoid of any particular geometrical significance. We may write

$$x=f(t), \quad y=F(t) \quad \text{.....................(3).}$$

If t has any arbitrary value assigned to it, the value (or values) of x and of y are known. Each pair of such values defines a point (x, y). If we construct all the points which thus correspond to all the different values of t we obtain *the graph of the locus defined by the equations* (3). Suppose for example

$$x=a\cos t, \quad y=a\sin t.$$

Let t vary from 0 to 2π. Then it is easy to see that the point (x, y) describes the circle whose centre is the origin and radius is a. If t varies beyond these limits (x, y) describes the circle over and over again. We can in this case at once obtain a direct relation between x and y by squaring and adding: we find that $x^2+y^2=a^2$, t being now eliminated.

Examples XX. 1. The points of intersection of the two curves whose equations are $f(x, y)=0$, $\phi(x, y)=0$ are given by solving this pair of simultaneous equations.

2. Trace the curves $(x+y)^2=1$, $xy=1$, $x^2-y^2=1$.

3. The curve $f(x, y)+\lambda\phi(x, y)=0$ represents a curve passing through the points of intersection of $f=0$, $\phi=0$.

4. What loci are represented by

(α) $x=at+b$, $y=ct+d$, (β) $x/a=2t/(1+t^2)$, $y/a=(1-t^2)/(1+t^2)$,

when t varies through all real values?

24. Loci in space. In space of three dimensions there are two fundamentally different kinds of loci, of which the simplest examples are the plane and the straight line.

A particle which moves along a straight line has only *one degree of freedom.* Its direction of motion is fixed; if its velocity is given (velocity being regarded as a quantity capable of sign) its mode of motion is completely determined. Or again the position of a point on a line can be completely fixed by one measurement of position, e.g. by its distance from a fixed point on the line. If we take the line as our fundamental line L of Ch. I., the position of any of its points is determined by a *single coordinate* x. A particle which moves in a plane, on the other hand, has *two* degrees of freedom. In order to determine its mode of motion completely we require a knowledge of its component velocities in *two* different directions. Or again the position of a point on a plane requires the determination of *two* coordinates in order to fix it.

Now let us look at these loci from the point of view of their equations. The plane is represented by a single equation $\alpha x + \beta y + \gamma z + \delta = 0$. *Two* of the three coordinates y and z may be chosen arbitrarily, and the third is then fixed. The straight line on the other hand is the intersection of two planes. Let these two planes be

$$ax + by + cz + d = 0, \quad \alpha x + \beta y + \gamma z + \delta = 0 \text{.........}(1).$$

Then if *one* of the three coordinates is chosen arbitrarily, both of the others and the position of the point are fixed.

We can of course draw *any number* of planes through the line. Hence it might appear that the coordinates of the point on the line are subject to *more than two relations.* And so in fact they are, but the relations are not all independent. Any other plane through the line could be expressed in the form

$$ax + by + cz + d + \lambda (\alpha x + \beta y + \gamma z + \delta) = 0$$

and any equation of this type is a mere consequence of the equations (1).

The locus represented by a single equation

$$z = f(x, y)$$

is called a *surface.* It may or may not (in the obvious simple cases it will) satisfy our common-sense notion of what a surface should be.

The considerations of § 21 may evidently be generalised so as to give definitions of a function $f(x, y, z)$ of *three* variables (or of functions of any number of variables). And as in § 23 we agreed to adopt $f(x, y) = 0$ as the standard form of the equation of a plane curve, so now we shall agree to adopt

$$f(x, y, z) = 0,$$

as the standard form of equation of a surface.

The locus represented by *two* equations of the form $z = f(x, y)$ or $f(x, y, z) = 0$ is called a *curve*. Thus a *straight line* may be represented by two equations of the type $\alpha x + \beta y + \gamma z + \delta = 0$. A *circle* in space may be regarded as the intersection of a sphere and a plane; it may therefore be represented by two equations of the forms

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2, \quad \alpha x + \beta y + \gamma z + \delta = 0.$$

Examples XXI. 1. What is represented by *three* equations of the type $f(x, y, z) = 0$?

[Three equations in three variables are capable of solution (practically or theoretically). The solution consists of a finite or infinite number of isolated sets of values (x, y, z). The three equations therefore represent a number of isolated points.

Or we may regard the question thus. *Two* of the equations determine a *curve*, which meets the *surface* represented by the third equation in a number of points.]

2. Three linear equations represent a single point.

3. The equations of a curve differ from the equation of a surface in that their mode of expression is not unique, since either may be transformed by means of the other. Thus the curve $y = 1$, $x^2 + y^2 + z^2 = 2$, (a circle) may also be represented by $y = 1$, $x^2 + z^2 = 1$.

4. What are the equations of a plane curve $f(x, y) = 0$ in the plane XOY, when regarded as a curve in space? [$f(x, y) = 0$, $z = 0$.]

5. **Cylinders.** What is the meaning of a single equation $f(x, y) = 0$, considered as a locus in space of three dimensions?

[All points on the surface satisfy $f(x, y) = 0$ *whatever be the value* of z. The curve $f(x, y) = 0$, $z = 0$ is the curve in which the locus cuts the plane XOY. Draw the plane $z = a$, cutting ZOX, YOZ in OX', OY', and take OX', OY' as axes of coordinates in this plane (Fig. 20). Obviously $x' = x$, $y' = y$ and so $f(x', y') = 0$. The curves in which the two planes $z = 0$, $z = a$ cut the locus are therefore repetitions of the same plane curve: if one curve were moved a distance a parallel to the axis of z it would coincide with the other. The locus is the surface formed by drawing lines parallel to OZ through all points of the plane curve $f(x, y) = 0$, $z = 0$. Such a surface is called a *cylinder*.]

6. Interpret the equations: (*a*) $y = mx + c$, (*b*) $y = mx + c$, $z = a$, (*c*) $x^2 + y^2 = 1$, (*d*) $x^2 + y^2 = 1$, $z = a$, as loci in three-dimensional space.

7. **Graphical representation of a surface on a plane. Contour Maps.** It might seem to be impossible to adequately represent a surface by a drawing on a plane; and so indeed it is: but a very fair notion of the nature of the surface may often be obtained as follows. Let the equation of the surface be $z=f(x, y)$.

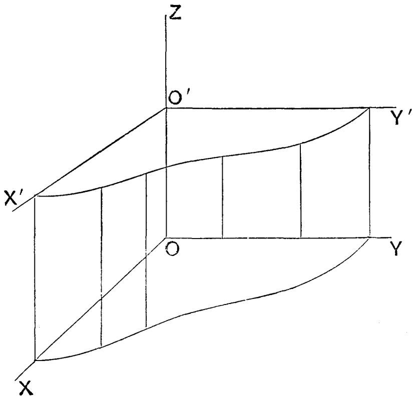

FIG. 20.

If we give z a particular value a, we have an equation $a=f(x, y)$, which we may regard as determining a plane curve on the paper. We trace this curve and mark it (a). Actually the curve (a) is the projection on the plane XOY of the section of the surface by the plane $z=a$ (Fig. 20). We do this for all values of a (practically, of course, for a selection of values of a). We obtain some such figure as is shown in Fig. 21. It will at once suggest a contoured Ordnance Survey map: and in fact this is the principle on which such maps are constructed. The contour line 1000 is the projection on the plane of the sea level of the section of the surface of the land by the plane parallel to the plane of the sea level and 1000 ft. above it*.

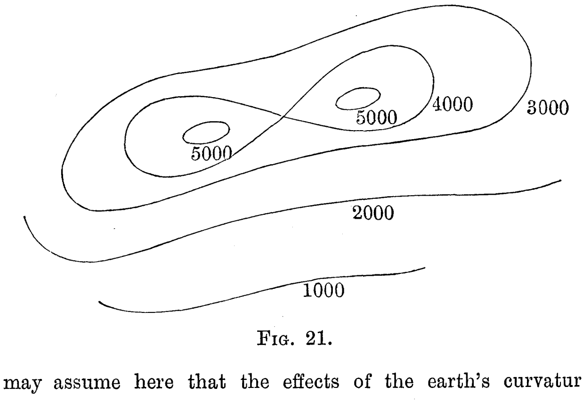

FIG. 21.

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* We may assume here that the effects of the earth's curvature may be neglected.

8. Draw a series of contour lines to illustrate the form of the surface $2z=3xy$.

9. **Right circular cones.** Take the origin of coordinates at the vertex of the cone and the axis of z along the axis of the cone (Figs. 22, 23). Let α be the semi-vertical angle of the cone, P any point on it, C and P' its projections on the axis OZ and the plane XOY. Then if x, y, z are the coordinates of P, we have $x^2+y^2=OP'^2=CP^2=OC^2\tan^2\alpha=z^2\tan^2\alpha$. The equation of the cone (which must be regarded as extending both ways from its vertex) is therefore $x^2+y^2-z^2\tan^2\alpha=0$.

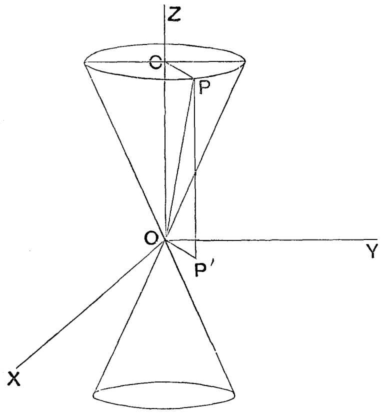

FIG. 22.

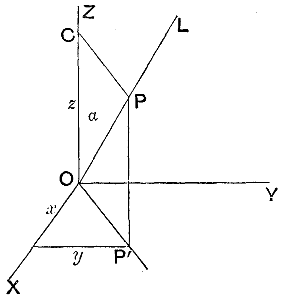

FIG. 23.

10. **Surfaces of revolution in general.** We notice that the cone of Ex. 9 cuts ZOX in the lines $x=\pm z\tan\alpha$, which may be combined in the equation $x^2=z^2\tan^2\alpha$. That is to say, the equation of the surface generated by the revolution of the curve $y=0$, $x^2=z^2\tan^2\alpha$ round the axis of z is derived from the second of these equations by changing x^2 into x^2+y^2.

Show generally that the equation of the surface generated by the revolution of the curve $y=0$, $x=f(z)$, round the axis of z, is $\sqrt{(x^2+y^2)}=f(z)$, or

$$x^2+y^2=\{f(z)\}^2.$$

Verify in the case of (1) the line $y=0$, $x=1$; (2) the circle $y=0$, $x^2+z^2=1$.

11. **Cones in general.** A surface formed by straight lines passing through a fixed point is called a *cone*: the point is called the *vertex*. A particular case is given by the right circular cone of Ex. 9. Show that the equation of a cone whose vertex is 0 is of the form

$$f\left(\frac{z}{x},\ \frac{z}{y}\right)=0,$$

and that any equation of this form represents a cone.

[If (x, y, z) lies on the cone, so must $(\lambda x, \lambda y, \lambda z)$, for any value of λ.]

12. **Ruled surfaces.** Cylinders and cones are special cases of *surfaces composed of straight lines.* Such surfaces are called *ruled surfaces.*

The two equations

$$\left.\begin{aligned} x &= az + b \\ y &= cz + d \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

represent the intersection of two planes, i.e. a straight line. Now suppose that a, b, c, d instead of being fixed, are *functions of an auxiliary variable t.* For any particular value of t the equations (1) give a line. As t varies this line moves, and generates a surface, whose equation may be found by eliminating t between the two equations (1). For instance, in Fig. 23 the line OL is inclined at a fixed angle α to OZ. PP' is perpendicular to the plane XOY and $XOP' = t$. The equations of the line are

$$\left.\begin{aligned} x &= z \tan \alpha \cos t \\ y &= z \tan \alpha \sin t \end{aligned}\right\}.$$

As t varies the line turns round OZ and generates the cone $x^2 + y^2 = z^2 \tan^2 \alpha$.

Another simple example of a ruled surface may be constructed as follows. Take two sections of a right circular cylinder perpendicular to the axis and at a distance l apart (Fig. 24 *a*). We can imagine the surface of the cylinder to be made up of a number of thin parallel rigid rods of length l, such as PQ, the ends of the rods being fastened to two circular rods of radius a.

Now let us take a third circular rod of the same radius and place it round the surface of the cylinder at a distance h from one of the first two rods (Fig. 24 *a*). Unfasten the end Q of the rod PQ and turn PQ about P until Q can be fastened to the third circular rod in the position Q'. The angle $qOQ' = \alpha$ in the figure is evidently given by

$$l^2 - h^2 = qQ'^2 = (2a \sin \tfrac{1}{2}\alpha)^2.$$

Let all the other rods of which the cylinder was composed be treated in the same way. We obtain a ruled surface whose form is indicated in Fig. 24 *b*. It is entirely built up of straight lines; but the surface is curved everywhere, and is in general shape not unlike certain forms of table-napkin rings (Fig. 24 *c*).

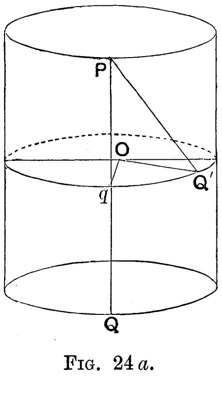

Fig. 24 *a*.

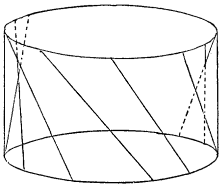

Fig. 24 *b*.

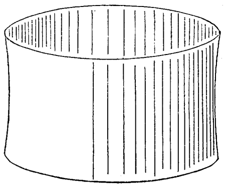

Fig. 24 *c*.

MISCELLANEOUS EXAMPLES ON CHAPTER II.

1. If $y=f(x)=(ax+b)/(cx-a)$, show that $x=f(y)$.

2. If $f(x)=f(-x)$ for all values of x, $f(x)$ is called an *even* function. If $f(x)=-f(-x)$ it is called an *odd* function. Show that any function of x, defined for all values of x, is the sum of an even and an odd function of x. [Use the identity $f(x)=\frac{1}{2}\{f(x)+f(-x)\}+\frac{1}{2}\{f(x)-f(-x)\}$.]

3. Find all the values of x for which $y=\dfrac{x-\sqrt{2}}{\sqrt{3}-x}$ has a rational value.

4. Draw the graphs of the functions

$$3\sin x+4\cos x, \quad \sin\left(\frac{\pi}{\sqrt{2}}\sin x\right). \qquad (\textit{Math. Trip.}\ 1896.)$$

5. Draw the graphs of the functions

$$\sin x\,(a\cos^2 x+b\sin^2 x), \quad \frac{\sin x}{x}(a\cos^2 x+b\sin^2 x), \quad \left(\frac{\sin x}{x}\right)^2.$$

6. Draw graphs of the functions

$$\text{(i)} \quad \text{arc}\cos(2x^2-1)-2\,\text{arc}\cos x,$$

$$\text{(ii)} \quad \text{arc}\tan\frac{a+x}{1-ax}-\text{arc}\tan a-\text{arc}\tan x,$$

where the symbols arc cos a, arc tan a denote, for any value of a, the least positive (or zero) angle, whose cosine or tangent is a.

7. Verify the following method of constructing the graph of $f\{\phi(x)\}$ by means of the line $y=x$ and the graphs of $f(x)$ and $\phi(x)$: take $OA=x$ along OX, draw AB parallel to OY to meet $y=\phi(x)$ in B, BC parallel to OX to meet $y=x$ in C, CD parallel to OY to meet $y=f(x)$ in D, and DP parallel to OX to meet AB in P: then P is a point on the graph required.

8. Show that the roots of $x^3+px+q=0$ are the abscissae of the points of intersection (other than the origin) of the parabola $y=x^2$ and the circle

$$x^2+y^2+(p-1)y+qx=0.$$

9. The roots of $x^4+nx^3+px^2+qx+r=0$ are the abscissae of the points of intersection of the parabola $x^2=y-\frac{1}{2}nx$, and the circle

$$x^2+y^2+(\tfrac{1}{8}n^2-\tfrac{1}{2}pn+\tfrac{1}{2}n+q)x+(p-1-\tfrac{1}{4}n^2)y+r=0.$$

10. Discuss the graphical solution of the equation

$$x^m+ax^2+bx+c=0$$

by means of the curves $y=x^m$, $y=-ax^2-bx-c$. Draw up a table of the various possible numbers of real roots.

11. Show that the equation

$$2x=(2n+1)\pi(1-\cos x)$$

where n is a positive integer, has $2n+3$ real roots and no more, roughly indicating their localities. (*Math. Trip.* 1896.)

12. Discuss the number and value of the real roots of the equations

(1) $\cot x+x-\frac{3}{2}\pi=0$, (2) $x^2+\sin^2 x=1$, (3) $\tan x=2x/(1+x^2)$,
(4) $\sin x-x+\frac{1}{6}x^3=0$, (5) $(1-\cos x)\tan \alpha-x+\sin x=0$.

13. Determine a polynomial of the 5th degree which has, for $x=-1$, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1 the values 3, 7, 2, 0, 4.

14. The polynomial of the second degree which assumes, when $x=a, b, c$, the values α, β, γ, is

$$\alpha\frac{(x-b)(x-c)}{(a-b)(a-c)}+\beta\frac{(x-c)(x-a)}{(b-c)(b-a)}+\gamma\frac{(x-a)(x-b)}{(c-a)(c-b)}.$$

Give a similar formula for the polynomial of the $(n-1)$-th degree which assumes, when $x=a_1, a_2, \dots a_n$, the values $\alpha_1, \alpha_2, \dots \alpha_n$.

15. If x is a rational function of y, and y is a rational function of x, show that $Axy+Bx+Cy+D=0$.

16. If y is a rational function of x, with rational coefficients, then y has a rational value for all rational values of x.

17. If y is an algebraical function of x, x is an algebraical function of y.

18. Verify that for values of x between 0 and 1 the equation

$$\cos\tfrac{1}{2}\pi x=1-\frac{x^2}{x+(x-1)\sqrt{\left(\dfrac{2-x}{3}\right)}}$$

is approximately true. [Take $x=0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1$, and use tables. For which of these values is the formula exact?]

19. The equation

$$a_0x^n+na_1x^{n-1}y+\frac{n(n-1)}{1.2}a_2x^{n-2}y^2+\dots+a_ny^n=0$$

represents n straight lines through the origin.

20. Show that the line $Ax+By+C=0$ and the two lines represented by

$$(Ax+By)^2-3(Ay-Bx)^2=0$$

form the sides of an equilateral triangle. (*Math. Trip.* 1906.)

21. The equation of the circle described on the line joining (x, y) and (x', y') as diameter is $(x-x')(x-x'')+(y-y')(y-y'')=0$.

22. The general equation of a circle through (x', y') and (x'', y'') may be expressed in either of the forms

(1) $$(x-x')(y-y'')-(x-x'')(y-y')$$
$$=\{(x-x')(x-x'')+(y-y')(y-y'')\}\tan\alpha,$$

$$(2)\quad (x-x')\left(x-x''+\frac{\lambda}{x''-x'}\right)+(y-y')\left(y-y''-\frac{\lambda}{y''-y'}\right)=0.$$

Here α is the angle contained in one of the segments of the circle. Express λ in terms of α.

23. The general equation of a circle cutting $x^2+y^2+2\lambda x+c=0$ orthogonally, for all values of λ, is $x^2+y^2+2\mu y-c=0$. Sketch the two sets of circles.

24. The general equation of all circles cutting at right angles the two circles $x^2+y^2-2a_1x-2b_1y+c_1=0$, $x^2+y^2-2a_2x-2b_2y+c_2=0$ is

$$\begin{vmatrix} x^2+y^2 & x & y \\ c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \end{vmatrix} + \lambda \begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0.$$

(*Math. Trip.* 1906.)

25. Show that the intersection of the two circular cylinders $x^2+y^2=1$, $x^2+z^2=1$, consists of two plane curves. Give a sketch of the cylinders and their line of intersection.

26. **Sections of a right circular cone by a plane.** Show that the cone $x^2+y^2=z^2\tan^2\alpha$ and the plane $z=x\tan\theta+c$ intersect in a curve whose projection on the plane XOY is $x^2+y^2=(x\tan\theta+c)^2\tan^2\alpha$.

Taking axes $O'\xi$, $O'\eta$ in the plane of section, O' being on OZ and $O'\eta$ parallel to OY, show that the equation of the curve of section is

$$\xi^2\cos^2\theta+\eta^2=(\xi\sin\theta+c)^2\tan^2\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Show that this curve consists of a single closed branch, a single infinite branch, or two infinite branches, according as $\theta \overset{<}{\underset{>}{=}} \frac{1}{2}\pi-\alpha$, and that in any case it is symmetrical about $O'\xi$.

27. Show that the equation (1) of the last example may be expressed in the form

$$(\xi-\gamma)^2+\eta^2=e^2(\xi-\kappa)^2,$$

where $$e=\sin\theta\sec\alpha,\quad \gamma=-\frac{c\sin\alpha\sin\theta}{\sin\alpha\pm\cos\theta},$$

and $$\kappa=-\frac{c\sin\alpha}{\sin\theta}\,\frac{1\pm\sin\alpha\cos\theta}{\sin\alpha\pm\cos\theta};$$

unless $\theta=\frac{1}{2}\pi-\alpha$, in which case

$$e=1,\quad \gamma=-\tfrac{1}{2}c\cos\alpha,\quad \kappa=-\tfrac{1}{2}c\sec\alpha(1+\sin^2\alpha).$$

28. Deduce that the section is a curve such that the distance of any point on the curve from a fixed point $(\gamma, 0)$ is e times its distance from a fixed line $\xi-\kappa=0$, i.e. that the curve is a conic, having the focus and directrix property which is usually adopted as the definition of a conic in books on Conic Sections. The conic is an ellipse, parabola, or hyperbola, according as $e \overset{<}{\underset{>}{=}} 1$; and, except in the special cases when $e=1$ or $e=0$, has two foci $(\gamma, 0)$ and two corresponding directrices.

29. Let A, A' be the vertices of the conic, i.e. the points where the conic cuts the axis of symmetry $O'\xi$, S, S' the foci, and K, K' the points where the directrices cut $O'\xi$. Show that A, A' are given by $\xi = \mp c \sin \alpha / \cos(\alpha \mp \theta)$, that A lies between S and K and A' between S' and K', and that the length of the 'major axis' AA' is $2c \sin \alpha \cos \alpha \cos \theta / (\cos^2 \theta - \sin^2 \alpha)$.

30. Show further that if we take axes parallel to $O'\xi$, $O'\eta$ through C, the middle point of AA', the equation of the curve becomes of the form

$$(x^2/a^2) + (y^2/b^2) = 1,$$

where $a = \frac{1}{2}AA'$ and $b = a\sqrt{(1-e^2)}$, or

$$(x^2/a^2) - (y^2/b^2) = 1,$$

where $a = \frac{1}{2}AA'$ and $b = a\sqrt{(e^2-1)}$, according as $e \lessgtr 1$. Sketch the forms of the curves.

31. In the case when $e = 1$, show that the one point A where the curve cuts $O'\xi$ is given by $\xi = -\frac{1}{2}c \sec \alpha$, and that by taking axes through this point we can reduce the equation of the curve to the form $y^2 = 4ax$, where $a = \frac{1}{2}c \sin \alpha \tan \alpha$.

[For an account of the simplest properties of the conic sections, deducible from the equations $(x^2/a^2) \pm (y^2/b^2) = 1$ or $y^2 = 4ax$, we must refer to treatises dealing specially with this subject.]

32. Show that the most general equation of the second degree, viz.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a conic. [It is this property which accounts for the importance of the conic sections.]

33. Show that the equation represents an ellipse, parabola, or hyperbola, according as $h^2 \underset{>}{\overset{<}{=}} ab$.

34. The equation

$$ax^2 + 2hxy + by^2 + 2(gx + fy)(lx + my) + c(lx + my)^2 = 0$$

represents the two lines joining the origin to the points in which the line $lx + my = 1$ cuts the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

35. Show directly that if the cylinder $x^2 + y^2 = 1$ is cut by a plane neither parallel nor perpendicular to its axis, the intersection is a curve possessing the focus and directrix property of a conic.

36. The curve

$$\lambda f(x, y)\, \phi(x, y) + \mu F(x, y)\, \Phi(x, y) = 0$$

passes through all points of intersection of $f = 0$ and $F = 0$, of $f = 0$ and $\Phi = 0$, of $\phi = 0$ and $F = 0$, and of $\phi = 0$ and $\Phi = 0$.

37. If $\lambda_r = a_r x + b_r y + c_r$, the equation $\lambda L_1 L_3 + \mu L_2 L_4 = 0$ is the general equation of a conic circumscribing the quadrangle formed by the four lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$, $L_4 = 0$ taken in order.

38. Determine the equation of the surface generated when the circle

$$y=0, \quad (x-a)^2+z^2=1$$

rotates round the axis of z. Sketch the form of the surface for different values of a.

39. What is the form of the graph of the functions

$$z=[x]+[y], \quad z=x+y-[x]-[y]\,?$$

40. What is the form of the graph of the functions $z=\sin x+\sin y$, $z=\sin x \sin y$, $z=\sin xy$, $z=\sin(x^2+y^2)$?

41. **Geometrical Constructions for irrational numbers.** In Chapter I. we indicated one or two simple geometrical constructions for a length equal to $\sqrt{2}$, starting from a given unit length. We also showed how to construct the roots of any quadratic equation $ax^2+2bx+c=0$, it being supposed that we can construct lines whose lengths are equal to any of the ratios of the quantities a, b, c. All these constructions were what may be called Euclidean constructions; they depended on the ruler and compass only.

It is fairly obvious that any irrational expression, however complicated, can be constructed by means of these methods, *provided it only contains square roots.* Thus

$$\sqrt[4]{\left\{\sqrt{\left(\frac{17+3\sqrt{11}}{17-3\sqrt{11}}\right)}-\sqrt{\left(\frac{17-3\sqrt{11}}{17+3\sqrt{11}}\right)}\right\}}$$

is a case in point. This contains a fourth root, but this is of course the square root of a square root. We should begin by constructing $\sqrt{11}$, e.g. as the mean between 1 and 11: then $17+3\sqrt{11}$, and so on. Or these two mixed surds might be constructed directly as the roots of $x^2-34x+190=0$.

Conversely, *only* irrationals of this kind *can* be constructed by Euclidean methods. Starting from a unit length we can construct any *rational* length. And hence we can construct the line $ax+by+c=0$, or the circle

$$(x-a)^2+(y-\beta)^2=r^2, \quad (\text{or } x^2+y^2+2gx+2fy+d=0)$$

provided the constants which occur in these equations are rational.

Now in any Euclidean construction, each new point introduced into the figure is determined as the intersection of two lines or circles, or a line and a circle. But if the coefficients are rational, such a pair of equations as

$$ax+by+c=0, \quad x^2+y^2+2gx+2fy+d=0$$

give, on solution, values of x and y of the form $m+n\sqrt{p}$, where m, n, p are rational: for if we substitute for x in terms of y in the second equation we obtain a quadratic in y with rational coefficients. Hence the coordinates of all points obtained by means of lines and circles with rational coefficients are expressible by rational numbers and quadratic surds. And so the same is true of the distance $\sqrt{\{(x_1-x_2)^2+(y_1-y_2)^2\}}$ between any two points so obtained.

With the irrational distances thus constructed we may proceed to construct a number of lines and circles whose coefficients may now themselves involve quadratic surds. It is evident, however, that by the use of such lines and circles we can still only construct lengths expressible by square roots only, though our surd expressions may now be of a more complicated form. And it is clear that this remains true however far we may go. Hence *Euclidean methods will construct any surd expression involving square roots, and no others.* In particular they will not construct $\sqrt[3]{2}$, i.e. they will not solve the problem of *the duplication of the cube*, which was one of the famous problems of antiquity.

42. **Approximate quadrature of the circle.** Let O be the centre of a circle of radius R. On the tangent at A take $AP = \frac{11}{5}R$ and $AQ = \frac{13}{5}R$, in the same direction. On AO take $AN = OP$ and draw NM parallel to OQ and cutting AP in M. Show that

$$AM = \tfrac{13}{25}\sqrt{146} \,.\, R,$$

and that to take AM as being equal to the circumference of the circle would lead to a value of π correct to five places of decimals.

If R is the earth's radius, the error in supposing AM to be its circumference is less than 11 yards.

43. Show that the only lengths which can be constructed *with the ruler only*, starting from a given unit length, are *rational* lengths.

44. **Constructions for $\sqrt[3]{2}$.** O is the vertex and S the focus of the parabola $y^2 = 4x$, and P is one of its points of intersection with the parabola $x^2 = 2y$. Show that OP meets the latus rectum of the first parabola in a point Q such that $SQ = \sqrt[3]{2}$.

45. Take a circle of unit diameter, a diameter OA and the tangent at A. Draw a chord OBC cutting the circle at B and the tangent at C. On this line take $OM = BC$. Taking O as origin and OA as axis of x, show that the locus of M is the curve

$$(x^2 + y^2)\,x - y^2 = 0$$

(the *Cissoid of Diocles*). Sketch the curve. Take along the axis of y a length $OD = 2$. Let AD cut the curve in P and OP cut the tangent at A in Q. Show that $AQ = \sqrt[3]{2}$.

CHAPTER III.

COMPLEX NUMBERS.

25. Displacements along a line and in a plane. The 'real number' x, with which we have been concerned in the two preceding chapters, may be regarded from a considerable number of different points of view. It may be regarded as a *pure number*, destitute of geometrical significance, or a geometrical significance may be attached to it in at least three different ways. It may be regarded as *the measure of a length*, viz. the length A_0P along the line L of Chap. I. It may be regarded as *the mark of a point*, viz. the point P whose distance from A_0 is x. Or it may be regarded as *the measure of a displacement* or *change of position* on the line L. It is on this last point of view that we shall now concentrate our attention.

Let a small particle be placed at P on the line L and then displaced to Q. We shall call the displacement or change of position which is needed to transfer the particle from P to Q *the displacement* $\overline{PQ}$. To completely specify a displacement three things are needed, its *magnitude*, its *sense* (forwards or backwards along the line), and what may be called its *point of application*, i.e. the original position P of the particle. But, when we are thinking merely of the change of position produced by the displacement, it is natural to disregard the point of application and to consider all displacements as equivalent whose lengths and senses are the same. Then the displacement is completely specified by the length $PQ = x$, the sense of the displacement being fixed by the sign of x. We may therefore, without ambiguity, speak of *the displacement* $[x]$, and we may write

$$\overline{PQ} = [x].$$

We use the square bracket to distinguish the displacement $[x]$ from the length or number x*. If the coordinate of P is a, that of Q will be $a+x$; the displacement $[x]$ therefore transfers a particle from the point x to the point $a+x$.

We come now to consider *displacements in a plane.* We may define the displacement $\overline{PQ}$ as before. But now more data are required in order to specify it completely. We require to know: (i) the *magnitude* of the displacement, i.e. the length of the straight line PQ; (ii) the *direction* of the displacement, which is determined by the angle which PQ makes with some fixed line in the plane; (iii) the *sense* of the displacement; and (iv) its *point of application.* Of these requirements we may disregard the fourth, if we consider two displacements as equivalent if they are the same in magnitude, direction, and sense. In other words, if PQ and RS are equal and parallel, and the sense of motion from P to Q is the same as that of motion from R to S, we regard the displacements $\overline{PQ}$ and $\overline{RS}$ as equivalent, and write

$$\overline{PQ} = \overline{RS}.$$

Now let us take any pair of coordinate axes in the plane

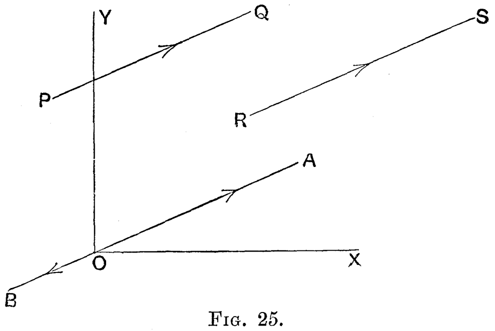

Fig. 25.

(such as OX, OY in Fig. 25). Draw a line OA equal and parallel to PQ, the sense of motion from O to A being the same as that from P to Q. Then $\overline{PQ}$ and $\overline{OA}$ are equivalent displacements. Let x and y be the coordinates of A. Then it is evident that $\overline{OA}$ is completely specified if x and y are given. We call $\overline{OA}$ *the displacement* $[x, y]$ and write

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* Strictly speaking we ought, by some similar difference of notation, to distinguish the actual length x from the number x which measures it. The reader will perhaps be inclined to consider such distinctions futile and pedantic. But increasing experience of mathematics will reveal to him the great importance of distinguishing clearly between things which, however intimately connected, are *not the same.* If cricket were a mathematical science it would be very important to distinguish between the *motion* of the batsman between the wickets, the *run* which he scores, and the *mark* which is put down in the score-book.

$$\overline{OA} = \overline{PQ} = \overline{RS} = [x, y].$$

26. Equivalence of displacements. Multiplication of displacements by numbers. If ξ and η are the coordinates of P, ξ' and η' those of Q, it is evident that

$$x = \xi' - \xi, \quad y = \eta' - \eta.$$

The displacement from (ξ, η) to (ξ', η') is therefore

$$[\xi' - \xi, \ \eta' - \eta].$$

It is evident that two displacements $[x, y]$, $[x', y']$ are equivalent if, and only if, $x = x'$, $y = y'$. Thus $[x, y] = [x', y']$ if

$$x = x', \quad y = y' \quad \text{........................(1)}.$$

The reverse displacement $\overline{QP}$ would be $[\xi - \xi', \ \eta - \eta']$, and it is natural to agree that

$$[\xi - \xi', \ \eta - \eta'] = -[\xi' - \xi, \ \eta' - \eta],$$

$$\overline{QP} = -\overline{PQ},$$

these equations being really definitions of the meaning of the symbols $-[\xi' - \xi, \ \eta' - \eta]$, $-\overline{PQ}$.

Having thus agreed that

$$-[x, y] = [-x, -y],$$

it is natural to agree further that

$$\alpha[x, y] = [\alpha x, \alpha y] \quad \text{.....................(2)}$$

where α is any real number, positive or negative. Thus (Fig. 25) if $OB = -\frac{1}{2}OA$,

$$\overline{OB} = -\tfrac{1}{2}\overline{OA} = -\tfrac{1}{2}[x, y] = [-\tfrac{1}{2}x, -\tfrac{1}{2}y].$$

The equations (1) and (2) define the first two important ideas connected with displacements, viz. *equivalence* of displacements, and *multiplication of displacements by numbers.*

27. Addition of displacements. We have not yet given any definition which enables us to attach any meaning to the expressions

$$\overline{PQ}+\overline{P'Q'},\quad [x,\ y]+[x',\ y'].$$

Common sense at once suggests that we should define the sum of two displacements as the displacement which is the result of the successive application of the two given displacements. In other words, it suggests that if QQ_1 be drawn equal and parallel to $P'Q'$, so that the result of successive displacements $\overline{PQ}$, $\overline{P'Q'}$ on a particle at P is to transfer it first to Q and then to Q_1, we should define the sum of $\overline{PQ}$ and $\overline{P'Q'}$ as being $\overline{PQ_1}$. Or, if we draw OB equal and parallel to $P'Q'$, and complete the parallelogram $OACB$,

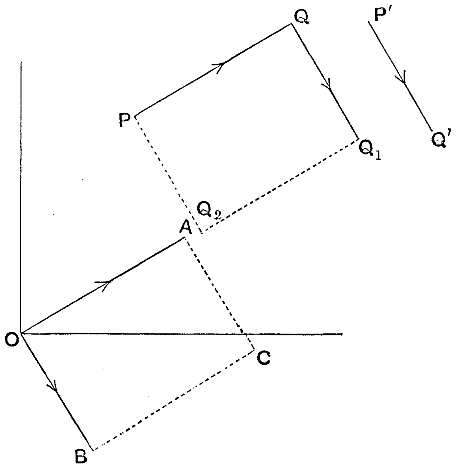

FIG. 26.

$$\overline{PQ}+\overline{P'Q'}=\overline{OA}+\overline{OB}=\overline{OC}=\overline{PQ_1}.$$

Let us consider the consequences of adopting this definition. If the coordinates of B are x', y', those of the middle point of AB are $\frac{1}{2}(x+x')$, $\frac{1}{2}(y+y')$, and those of C are $x+x'$, $y+y'$. Hence

$$[x,\ y]+[x',\ y']=[x+x',\ y+y'] \quad \text{............}(3),$$

which may be regarded as the symbolic definition of *addition of displacements.* We observe that

$$\begin{aligned}[x',\ y']+[x,\ y]&=[x'+x,\ y'+y]\\ &=[x+x',\ y+y']=[x,\ y]+[x',\ y'].\end{aligned}$$

In other words, *addition of displacements obeys the commutative law* expressed in ordinary algebra by the equation $a + b = b + a$.

Looked at geometrically, this expresses the obvious fact that if we move from P *first* through a distance PQ_2 equal and parallel to $P'Q'$, and then through a distance equal and parallel to PQ, we shall arrive at the same point Q_1 as before. Again, since

$$[x, y] + [x, y] = [2x, 2y] = 2\,[x, y],$$

our definition of addition agrees with that previously adopted for multiplication by a number.

In particular

$$[x, y] = [x, 0] + [0, y] \quad\text{.....................}(4).$$

Here $[x, 0]$ denotes a displacement through a distance x in a direction parallel to OX. It is in fact what we previously denoted by $[x]$, when we were considering only displacements along a line. We call $[\underset{x}{\cancel{y}}, 0]$ and $[0, y]$ the *components* of $[x, y]$, and $[x, y]$ their *resultant.*

When we have once defined addition of two displacements there is no further difficulty in the way of defining addition of any number. Thus (by definition)

$$\begin{aligned}[x, y] + [x', y'] + [x'', y''] &= ([x, y] + [x', y']) + [x'', y''] \\ &= [x + x', y + y'] + [x'', y''] = [x + x' + x'', y + y' + y''].\end{aligned}$$

We define *subtraction* of displacements by the equation

$$[x, y] - [x', y'] = [x, y] + (-[x', y']) \quad\text{...........}(5),$$

which is the same thing as $[x, y] + [-x', -y']$ or as $[x - x', y - y']$.

In particular

$$[x, y] - [x, y] = [0, 0].$$

The displacement $[0, 0]$ *leaves the particle where it was*; it is the *zero displacement,* and we agree to write $[0, 0] = 0$.

Examples XXII. 1. Addition of displacements, and multiplication of displacements by numbers, obey all the ordinary laws of algebra, expressed by the equations,

(i) $\alpha[\beta x, \beta y] = \beta[\alpha x, \alpha y] = [\alpha\beta x, \alpha\beta y]$,

(ii) $([x, y] + [x', y']) + [x'', y''] = [x, y] + ([x', y'] + [x'', y''])$,

(iii) $[x, y] + [x', y'] = [x', y'] + [x, y]$,

(iv) $(\alpha + \beta)[x, y] = \alpha[x, y] + \beta[x, y]$,

(v) $\alpha\{[x, y] + [x', y']\} = \alpha[x, y] + \alpha[x', y']$.

[We have already proved (iii). The remaining equations follow with equal ease from the definitions. The reader should in each case consider the geometrical significance of the equation, as we did above in the case of (iii).]

2. If M is the middle point of PQ, $\overline{OM}=\frac{1}{2}(\overline{OP}+\overline{OQ})$. More generally if M divides PQ in the ratio $\mu:\lambda$

$$\overline{OM}=\frac{\lambda}{\lambda+\mu}\overline{OP}+\frac{\mu}{\lambda+\mu}\overline{OQ}.$$

3. If G is the centre of mass of equal particles at $P_1, P_2, \dots, P_n$

$$\overline{OG}=(\overline{OP}_1+\overline{OP}_2+\dots+\overline{OP}_n)/n.$$

4. If P, Q, R are collinear points in the plane, it is possible to find real numbers α, β, γ, not all zero, and such that

$$\alpha\,.\,\overline{OP}+\beta\,.\,\overline{OQ}+\gamma\,.\,\overline{OR}=0\,;$$

and conversely. [This is really only another way of stating Ex. 2.]

5. If $\overline{AB}$ and $\overline{AD}$ are two displacements not in the same straight line, and

$$\alpha\,.\,\overline{AB}+\beta\,.\,\overline{AD}=\gamma\,.\,\overline{AB}+\delta\,.\,\overline{AD},$$

then $\alpha=\gamma$ and $\beta=\delta$.

[Take $AB_1=\alpha\,.\,AB$, $AD_1=\beta\,.\,AD$. Complete the parallelogram $AB_1P_1D_1$. Then $\overline{AP}_1=\alpha\,.\,\overline{AB}+\beta\,.\,\overline{AD}$. It is evident that $\overline{AP}_1$ can only be expressed in this form in one way, whence the theorem follows.]

6. $ABCD$ is a parallelogram. Through Q, a point inside the parallelogram, RQS and TQU are drawn parallel to the sides. Show that RU, TS intersect on AC.

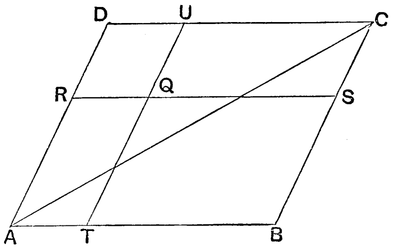

Fig. 27.

[Let the ratios $AT:AB$, $AR:AD$ be denoted by α, β. Then

$$\overline{AT}=\alpha\,.\,\overline{AB},\quad \overline{AR}=\beta\,.\,\overline{AD},$$

$$\overline{AU}=\alpha\,.\,\overline{AB}+\overline{AD},\quad \overline{AS}=\overline{AB}+\beta\,.\,\overline{AD}.$$

Let RU meet AC in P. Then, since R, U, P are collinear

$$\overline{AP}=\frac{\lambda}{\lambda+\mu}\overline{AR}+\frac{\mu}{\lambda+\mu}\overline{AU},$$

where μ/λ is the ratio in which P divides RU. That is to say

$$\overline{AP}=\frac{\alpha\mu}{\lambda+\mu}\overline{AB}+\frac{\beta\lambda+\mu}{\lambda+\mu}\overline{AD}.$$

But since P lies on AC, $\overline{AP}$ is a numerical multiple of $\overline{AC}$; say

$$\overline{AP}=k\,.\,\overline{AC}=k\,.\,\overline{AB}+k\,.\,\overline{AD}.$$

Hence (Ex. 5) $\alpha\mu=\beta\lambda+\mu=(\lambda+\mu)k$, from which we deduce

$$k=\alpha\beta/(\alpha+\beta-1).$$

The symmetry of this result shows that a similar argument would also give

$$\overline{AP'} = \frac{\alpha\beta}{\alpha+\beta-1}\,\overline{AC},$$

if P' is the point where TS meets AC. Hence P and P' are the same point.]

7. $ABCD$ is a parallelogram, and M the middle point of AB. Show that DM trisects and is trisected by AC*.

28. Multiplication of displacements. So far we have made no attempt to attach any meaning whatever to the notion of the *product* of two displacements. The only kind of multiplication which we have considered is that in which a displacement is multiplied by *a mere number*. The expression

$$[x, y] \times [x', y']$$

so far means nothing, and we are at liberty to define it to mean anything we like. It is, however, fairly clear that if any definition of such a product is to be of any use, the product of two displacements *must itself be a displacement.*

We might, for example, define it as being equal to

$$[x + x', y + y'];$$

in other words, we might agree that the product of two displacements was to be always equal to their sum. But there would be two serious objections to such a definition. In the first place our definition would be futile. We should only be introducing a new method of expressing something which we can perfectly well express without it. In the second place our definition would be inconvenient and misleading for the following reasons. If α is a real number, we have already defined $\alpha[x, y]$ as $[\alpha x, \alpha y]$. Now, as we saw in § 25, the real number α may itself from one point of view be regarded as a displacement, viz. the displacement $[\alpha]$ along the axis OX, or, in our later notation, the displacement $[\alpha, 0]$. It is therefore, if not absolutely necessary, at any rate most desirable, that our definition should be such that

$$[\alpha, 0][x, y] = [\alpha x, \alpha y],$$

and the suggested definition does not give this result.

A more reasonable definition might appear to be

$$[x, y][x', y'] = [xx', yy'].$$

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* The two preceding examples are taken from Willard Gibbs' *Vector Analysis*.

But this would give

$$[\alpha, 0]\,[x, y] = [\alpha x, 0],$$

and so this also would be open to the second objection.

In fact, it is by no means obvious what is the best meaning to attach to the product $[x, y]\,[x', y']$. All that is clear is (1) that, if our definition is to be of any use, this product must itself be a displacement whose coordinates depend on x and y, or in other words that we must have

$$[x, y]\,[x', y'] = [X, Y],$$

where X and Y are functions of x, y, x', and y'; (2) that the definition must be such as to agree with the equation

$$[x, 0]\,[x', y'] = [xx', xy'],$$

and (3) that the definition must obey the ordinary commutative, distributive, and associative laws of multiplication, so that

$$[x, y]\,[x', y'] = [x', y']\,[x, y],$$

$$([x, y] + [x', y'])\,[x'', y''] = [x, y]\,[x'', y''] + [x', y']\,[x'', y''],$$

$$[x, y]\,([x', y'] + [x'', y'']) = [x, y]\,[x', y'] + [x, y]\,[x'', y''],$$

and $$[x, y]\,([x', y']\,[x'', y'']) = ([x, y]\,[x', y'])\,[x'', y''].$$

29. The right definition to take is suggested as follows. We know that, if OAB, OCD are two similar triangles, the angles corresponding in the order in which they are written, then

$$OB/OA = OD/OC,$$

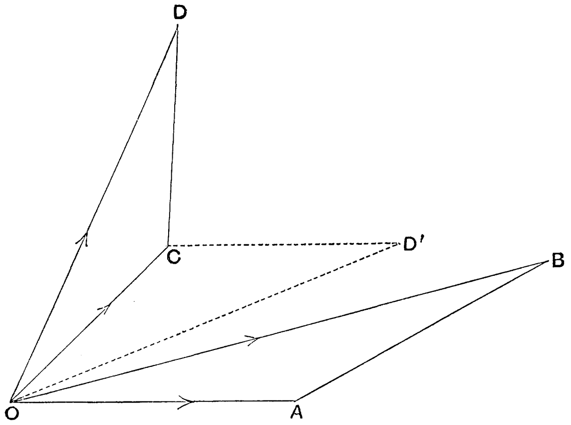

FIG. 28.

or $OB \,.\, OC = OA \,.\, OD$. This suggests that we should try to define multiplication and division of displacements in such a way that

$$\overline{OB}/\overline{OA} = \overline{OD}/\overline{OC}, \quad \overline{OB} \,.\, \overline{OC} = \overline{OA} \,.\, \overline{OD}.$$

Now let

$$\overline{OB} = [x, y], \quad \overline{OC} = [x', y'], \quad \overline{OD} = [X, Y],$$

and suppose that A is the point $(1, 0)$, so that $\overline{OA} = [1, 0]$. Then

$$\overline{OA} \,.\, \overline{OD} = [1, 0]\,[X, Y] = [X, Y],$$

and so

$$[x, y]\,[x', y'] = [X, Y].$$

The product $\overline{OB} \,.\, \overline{OC}$ is therefore to be defined as $\overline{OD}$, D being obtained by constructing on OC a triangle similar to OAB. In order to free this definition from ambiguity, it should be observed that on OC we can describe *two* such triangles, OCD and OCD'. We choose that for which the angle COD is equal to AOB in sign as well as in magnitude. We say that the two triangles are then *similar in the same sense.*

If the polar coordinates of B and C are (ρ, θ) and (σ, ϕ), so that

$$x = \rho \cos\theta, \quad y = \rho \sin\theta, \quad x' = \sigma\cos\phi, \quad y' = \sigma \sin\phi,$$

the polar coordinates of D are evidently $\rho\sigma$ and $\theta + \phi$. Hence

$$X = \rho\sigma \cos(\theta + \phi) = xx' - yy',$$
$$Y = \rho\sigma \sin(\theta + \phi) = xy' + yx'.$$

The required definition is therefore

$$[x, y]\,[x', y'] = [xx' - yy', xy' + yx'] \quad \text{............(6).}$$

We observe (1) that if $y = 0$, $X = xx'$, $Y = xy'$, as we desired; (2) that the right-hand side is not altered if we interchange x and x', and y and y', so that

$$[x, y]\,[x', y'] = [x', y']\,[x, y];$$

and (3) that

$$\begin{aligned}\{[x, y] + [x', y']\}\,[x'', y''] &= [x + x', y + y']\,[x'', y''] \\ &= [(x + x')\,x'' - (y + y')\,y'', (x + x')\,y'' + (y + y')\,x''] \\ &= [xx'' - yy'', xy'' + yx''] + [x'x'' - y'y'', x'y'' + y'x''] \\ &= [x, y]\,[x'', y''] + [x', y']\,[x'', y''].\end{aligned}$$

Similarly we can verify that all the equations at the end of § 28 are satisfied. Thus the definition (6) fulfils all the requirements which we made of it in § 28.

Example. Show directly from the geometrical definition given above that multiplication of displacements obeys the commutative and distributive laws. [Take the commutative law for example. The product $\overline{OB}\,.\,\overline{OC}$ is $\overline{OD}$ (Fig. 28), COD being similar to AOB. To construct the product $\overline{OC}\,.\,\overline{OB}$ we should have to *construct on* OB *a triangle* BOD_1 *similar to* AOC; and so what we want to prove is that D and D_1 coincide, or that BOD is similar to AOC. This is an easy piece of elementary geometry.]

30. Complex numbers. Just as to a displacement $[x]$ along OX correspond a point (x) and a real number x, so to a displacement $[x, y]$ in the plane correspond a point (x, y) and *a pair of real numbers* x, y.

We shall find it convenient to denote this pair of real numbers x, y by the symbol

$$x + yi.$$

The *reason* for the choice of this notation will appear later. For the present the reader must regard $x + yi$ as *simply another way of writing* $[x, y]$. The expression $x + yi$ is called a *complex number*.

We proceed next to define *equivalence, addition,* and *multiplication* of complex numbers. To every complex number corresponds a displacement. Two complex numbers are equivalent if the corresponding displacements are equivalent. The sum or product of two complex numbers is the complex number which corresponds to the sum or product of the two corresponding displacements. Thus

$$x + yi = x' + y'i \quad \textit{if} \quad x = x',\ y = y' \qquad \text{............(1)},$$

$$(x + yi) + (x' + y'i) = (x + x') + (y + y')\,i \qquad \text{.........(2)},$$

$$(x + yi)(x' + y'i) = xx' - yy' + (xy' + yx')\,i \qquad \text{.........(3)}.$$

In particular, if α is any real number, $\alpha(x + yi) = \alpha x + \alpha yi$.

The complex numbers of the particular form $x + 0i$ may be regarded as equivalent to the corresponding real numbers x; thus

$$x + 0i = x,$$

and in particular $0 + 0i = 0$.

Positive integral powers and polynomials of complex numbers are then defined as in ordinary algebra. Thus, by putting $x = x'$, $y = y'$ in (3), we obtain

$$(x + yi)^2 = (x + yi)(x + yi) = x^2 - y^2 + 2xyi,$$

$$(x + yi)^2 + 2(x + yi) + 3 = x^2 - y^2 + 2x + 3 + (2xy + 2y)i.$$

The reader will easily verify for himself that addition and multiplication of complex numbers obey the ordinary laws of algebra, expressed by the equations

$$x + yi + (x' + y'i) = (x' + y'i) + (x + yi),$$

$$\{(x + yi) + (x' + y'i)\} + (x'' + y''i) = (x + yi) + \{(x' + y'i) + (x'' + y''i)\},$$

$$(x + yi)(x' + y'i) = (x' + y'i)(x + yi),$$

$$(x + yi)\{(x' + y'i) + (x'' + y''i)\} = (x + yi)(x' + y'i) + (x + yi)(x'' + y''i),$$

$$\{(x + yi) + (x' + y'i)\}(x'' + y''i) = (x + yi)(x'' + y''i) + (x' + y'i)(x'' + y''i),$$

$$(x + yi)\{(x' + y'i)(x'' + y''i)\} = \{(x + yi)(x' + y'i)\}(x'' + y''i),$$

the proofs of these equations being practically the same as those of the corresponding equations for the corresponding displacements.

Subtraction and division of complex numbers are defined as in ordinary algebra. Thus we may define $(x + yi) - (x' + y'i)$ as

$$(x + yi) + \{-(x' + y'i)\} = x + yi + (-x' - y'i) = (x - x') + (y - y')i;$$

or again, as the number $\xi + \eta i$ such that

$$(x' + y'i) + (\xi + \eta i) = x + yi,$$

which leads to the same result.

And $(x + yi)/(x' + y'i)$ is defined as being the complex number $\xi + i\eta$ such that

$$(x' + y'i)(\xi + \eta i) = x + yi,$$

or

$$x'\xi - y'\eta + (x'\eta + y'\xi)i = x + yi,$$

or

$$x'\xi - y'\eta = x, \quad x'\eta + y'\xi = y \quad \text{...............(4)}.$$

Solving these equations for ξ and η, we obtain

$$\xi = \frac{xx' + yy'}{x'^2 + y'^2}, \quad \eta = \frac{yx' - xy'}{x'^2 + y'^2}.$$

This solution fails if x' and y' are both zero, i.e. if $x' + y'i = 0$. Thus subtraction is always possible; division is always possible unless the divisor is zero.

Examples. (1) From a geometrical point of view the problem of the division of the displacement $\overline{OB}$ by $\overline{OC}$ is that of finding D so that the triangles COB, AOD are similar, and this is evidently possible (and the solution unique) *unless C coincides with O*, or $\overline{OC}=0$.

(2) The numbers $x+yi$, $x-yi$ are said to be *conjugate.* Verify that

$$(x+yi)(x-yi)=x^2+y^2,$$

so that the product of two conjugate numbers is real, and that

$$\frac{x+yi}{x'+y'i}=\frac{(x+yi)(x'-y'i)}{(x'+y'i)(x'-y'i)}$$
$$=\frac{xx'+yy'+i(x'y-xy')}{x'^2+y'^2}.$$

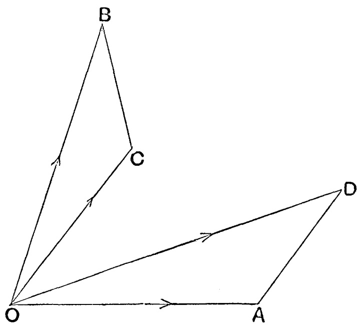

FIG. 29.

31. One most important property of real numbers is that known as *the factor theorem,* which asserts that *the product of two numbers cannot be zero unless one of the two is itself zero.* To prove that this is also true of complex numbers we put $x=0$, $y=0$ in the equations (4) of the preceding section. Then

$$x'\xi-y'\eta=0,\quad x'\eta+y'\xi=0.$$

These equations give $\xi=0$, $\eta=0$, i.e.

$$\xi+\eta i=0,$$

unless $x'=0$ and $y'=0$, or $x'+y'i=0$. Thus $x+yi$ cannot vanish unless either $x'+y'i$ or $\xi+\eta i$ vanishes.

32. The equation $i^2=-1$. We agreed to use, instead of $x+0i$, the simpler notation x. Similarly, instead of $0+yi$, we shall use yi. The particular complex number $1i$ we shall denote simply by i. It is the number which corresponds to *a unit displacement along OY.* Also

$$i^2=ii=(0+1i)(0+1i)=(0.0-1.1)+(0.1+1.0)i=-1.$$

Similarly $(-i)^2=-1$. Thus the complex numbers $\pm i$ satisfy the equation $x^2=-1$.

Now the reader will easily satisfy himself that the upshot of the rules for addition and multiplication of complex numbers is this, that *we operate with complex numbers in exactly the same* *way as with real numbers, treating the symbol i as itself a number, but replacing the product* $ii = i^2$ *by* -1 *whenever it occurs.* Thus, for example,

$$\begin{aligned}(x + yi)(x' + y'i) &= xx' + xy'i + yx'i + yy'i^2 \\ &= (xx' - yy') + (xy' + yx')\,i.\end{aligned}$$

33. The geometrical interpretation of multiplication by i. Since

$$(x + iy)\,i = -y + ix,$$

it follows that if $x + iy$ corresponds to $\overline{OP}$, and OQ is drawn equal to OP, and so that POQ is a positive right angle, then $(x + iy)\,i$ corresponds to $\overline{OQ}$. In other words, *multiplication of a complex number by i turns the corresponding displacement through a right angle.*

We might, had we so chosen, have started from this point of view. We might have regarded x as a length measured along OX, and xi as the same length measured along OY, and regarded i as a symbol of operation equivalent to turning the length x through a right angle round O. We should then naturally have been led to regard $xi^2 = xii$ as denoting the result of *twice* turning x through a right angle. The result of this is to bring it into a position again lying along OX but pointing in the opposite direction, so that we should have been led to the equation

$$xi^2 = -x.$$

Then, denoting $1i$ (a unit length along OY) simply by i, we should have found $i^2 = -1$, and the rule for multiplication of complex numbers would have followed immediately.

34. The equations $x^2 + 1 = 0$, $ax^2 + 2bx + c = 0$. There is no real number x such that $x^2 + 1 = 0$; this is expressed by saying that the equation has *no real roots.* But, as we have just seen, the two complex numbers $\pm i$ satisfy this equation. We express this by saying that the equation has *the two complex roots* $\pm i$. Since i satisfies $x^2 = -1$, it is sometimes written in the form $\surd(-1)$.

Complex numbers are sometimes called *imaginary,* to distinguish them from *real* numbers. The expression is by no means a happily chosen one, but it is firmly established and has to be accepted. It cannot, however, be too strongly impressed upon the reader that in reality an 'imaginary number' is neither 'imaginary' nor 'a number' at all. The 'real' numbers would be better described as 'common' or 'ordinary' numbers; they are the numbers of arithmetic. A 'complex' or 'imaginary number' is really not a number at all, but, as should be clear from the following discussion, *a pair of numbers* (x, y), united symbolically, for purposes purely of convenience, in the form $x+yi$. And such a pair of numbers is no less 'real' than any ordinary number such as $\frac{1}{2}$, or than the paper on which this is printed, or than the Solar System.

In reality

$$i = 0 + 1i$$

stands for the pair of numbers $(0, 1)$, and may be represented geometrically by a point or by the displacement $[0, 1]$. And when we say that i is a root of the equation $x^2+1=0$, what we mean is simply that we have defined a method of combining such pairs of numbers (or displacements) which we call 'multiplication,' and which, when we so combine $(0, 1)$ with itself, gives the result $(-1, 0)$.

Now let us consider the more general equation

$$ax^2 + 2bx + c = 0,$$

where a, b, c are real numbers.

If $b^2 > ac$, the ordinary method of solution gives two real roots

$$\{-b \pm \sqrt{(b^2 - ac)}\}/a.$$

If $b^2 < ac$, the equation has no real roots. It may be written in the form

$$\{x + (b/a)\}^2 = -(ac - b^2)/a^2,$$

an equation which is evidently satisfied if we substitute for $x + (b/a)$ the complex number $\pm i\sqrt{(ac-b^2)}/a$. We express this by saying that the equation has *the two complex roots*

$$\{-b \pm i\sqrt{(ac - b^2)}\}/a.$$

If we agree as a matter of convention to say that when $b^2 = ac$ (in which case the equation is satisfied by *one* value of x only, viz. $-b/a$), the equation has *two equal roots*, we can say that *a quadratic equation with real coefficients has two roots in all* *cases, two distinct real roots, two equal real roots, or two distinct complex roots.*

The question is naturally suggested whether a quadratic equation may not, when complex roots are once admitted, have more than two roots. It is easy to see that this is not possible. In fact, its impossibility may be proved by precisely the same chain of reasoning as is used in elementary algebra to prove that an equation of the nth degree cannot have more than n real roots. Let $z = x + yi$, and let $f(z)$ denote any polynomial in z*, with real or complex coefficients. Then we prove in succession:

(1) that the remainder, when $f(z)$ is divided by $z - a$ (a being any real or complex number), is $f(a)$;

(2) if a is a root of the equation $f(z) = 0$, then $f(z)$ is divisible by $z - a$;

(3) if $f(z)$ is of the nth degree, and $f(z) = 0$ has the n roots $a_1, a_2, \ldots, a_n$, then

$$f(z) \equiv A(z - a_1)(z - a_2) \ldots (z - a_n),$$

where A is a constant (real or complex), in fact the coefficient of z^n in $f(z)$. From the last result it follows at once that $f(z)$ cannot have more than n roots.

We conclude that a quadratic equation with real coefficients has exactly two roots. We shall see later on that a similar theorem is true for an equation of any degree and with either real or complex coefficients: *an equation of the nth degree has exactly n roots.* The only point in the proof which presents any difficulty is the first, viz. the proof that any equation must have *at least one* root. This we must postpone for the present. We may, however, at once call attention to one very interesting result of this theorem. In the theory of number we start from the positive integers, and from the ideas of addition and multiplication, and the converse operations of subtraction and division. We find that these operations are not always possible unless we admit new kinds of numbers. We can only attach a meaning to $3 - 7$ if we admit *negative* numbers, or to $\frac{3}{7}$ if we admit *rational fractions.* When we extend our list of arithmetical operations so as to include root extraction and the solution of equations, we find that some of them, such as the extraction of the square root of a number which (like 2) is not a perfect square, are not possible unless we widen our conception of a number, and admit the irrational numbers of Chap. I.

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* A polynomial in $z = x + yi$ is of course defined in exactly the same way as a polynomial in x, i.e. as an expression of the form $a_0 z^n + a_1 z^{n-1} + \ldots + a_n$.

Others, such as the extraction of the square root of -1, are not possible unless we go still further, and admit the *complex* numbers of this chapter. And it would not be unnatural to suppose that, when we come to consider equations of higher degree, some might prove to be insoluble even by the aid of complex numbers, and that thus we might be led to the considerations of higher and higher types of, so to say, *hyper-complex* numbers. The fact that any algebraical equation whatever can be solved by means of ordinary complex numbers shows that this is not the case. The application of any of the ordinary algebraical operations to complex numbers will yield only complex numbers. In technical language 'the field of the complex numbers is closed for algebraical operations.'

Before we pass on to other matters, let us add that all theorems of elementary algebra which are proved merely by the application of the rules of addition and multiplication are true *whether the numbers which occur in them are real or complex,* since the rules referred to apply to complex as well as real numbers. For example, we know that if α and β are the roots of

$$ax^2 + 2bx + c = 0,$$

then

$$\alpha + \beta = -(2b/a), \quad \alpha\beta = (c/a).$$

Similarly, if α, β, γ are the roots of

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

then

$$\alpha + \beta + \gamma = -(3b/a), \quad \beta\gamma + \gamma\alpha + \alpha\beta = (3c/a), \quad \alpha\beta\gamma = -(d/a).$$

All such theorems as these are true whether $a, b, \ldots \alpha, \beta, \ldots$ are real or complex.

35. The Argand diagram. Let P be the point (x, y) in Fig. 30, r, θ its polar coordinates, so that

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{(x^2 + y^2)}, \quad \cos\theta : \sin\theta : 1 :: x : y : r.$$

We shall denote the complex number $x + yi$ by z, and we shall call z the *complex variable*. We shall call P *the point* z, or the point corresponding to z; and z the *argument of* P. We shall call x the *real part*, y the *imaginary part*, r the *modulus*, and θ the *amplitude* of z, and we shall write

$$x = R(z), \quad y = I(z),$$
$$r = |z|, \quad \theta = \operatorname{am} z.$$

It should be observed that r is essentially *positive* (except when $z = 0$).

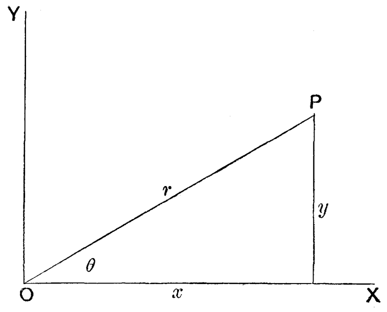

FIG. 30.

When $y = 0$ we shall say that *z is real*, when $x = 0$ that *z is purely imaginary*. Two numbers $x + yi$, $x - yi$ which differ only in the signs of their imaginary parts, we shall call *conjugate*. It will be observed that the sum $(2x)$ of two conjugate numbers and their product $(x^2 + y^2)$ are both real, that their moduli $(\sqrt{x^2 + y^2})$ are equal, and that their product is equal to the square of the modulus of either. The roots of a quadratic with real coefficients, for example, are conjugate, when not real.

It must be observed that θ or $\operatorname{am} z$ is a many-valued function of z, having an infinity of values differing by multiples of 2π. Any one of its values is an angle by turning through which about O a line originally lying along OX will come to lie along OP. We shall denote that one of these values which lies between $-\pi$ and $+\pi$ as the *principal value* of the amplitude of z. This definition is unambiguous except when one of the values is π, in which case $-\pi$ is also a value. In this case we must make some special provision as to which value is to be regarded as the principal value. In general, when we speak of the amplitude of z we shall, unless the contrary is stated, mean the principal value of the amplitude.

Complex numbers were first studied from a geometrical point of view by Wessel, Gauss and Argand, and the figure is usually known as the Argand diagram.

36. De Moivre's Theorem. The following statements follow immediately from the definitions of addition and multiplication.

(1) The real (or imaginary) part of the sum of two complex numbers is the sum of their real (or imaginary) parts.

(2) The modulus of the product of two complex numbers is the product of their moduli.

(3) One value of the amplitude of the product of two complex numbers is the sum of their amplitudes.

It should be observed that it is not true that the principal value of $\operatorname{am}(zz')$ is the sum of the principal values of $\operatorname{am} z$ and $\operatorname{am} z'$. For example, if $z=z'=-1+i$, the principal values of the amplitudes of z and z' are each $\frac{3}{4}\pi$. But $zz'=-2i$, and the principal value of $\operatorname{am}(zz')$ is $-\frac{1}{2}\pi$ and not $\frac{3}{2}\pi$.

The two last theorems may be expressed in the equation

$$r(\cos\theta+i\sin\theta)\times\rho(\cos\phi+i\sin\phi)$$
$$=r\rho\{\cos(\theta+\phi)+i\sin(\theta+\phi)\},$$

which may be proved at once by multiplying out and using the ordinary trigonometrical formulae for $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$. More generally

$$r_1(\cos\theta_1+i\sin\theta_1)\times r_2(\cos\theta_2+i\sin\theta_2)\times\dots\times r_n(\cos\theta_n+i\sin\theta_n)$$
$$=r_1r_2\dots r_n\{\cos(\theta_1+\theta_2+\dots+\theta_n)+i\sin(\theta_1+\theta_2+\dots+\theta_n)\}.$$

A particularly interesting case is that in which

$$r_1=r_2=\dots=r_n=1,\quad \theta_1=\theta_2=\dots=\theta_n=\theta.$$

We then obtain the equation

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta,$$

where n is any positive integer: a result known as *De Moivre's Theorem*[*].

Again, if
$$z=r(\cos\theta+i\sin\theta),$$
$$1/z=(\cos\theta-i\sin\theta)/r.$$

Thus the modulus of the reciprocal of z is the reciprocal of the modulus of z, and the amplitude of the reciprocal is the amplitude of z with its sign changed. Hence we deduce from (2) and (3):

(4) The modulus of the quotient of two complex numbers is the quotient of their moduli.

(5) One value of the amplitude of the quotient of two complex numbers is the difference of their amplitudes.

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[*] It will sometimes be convenient, for the sake of brevity, to denote $\cos\theta+i\sin\theta$ by $\operatorname{Cis}\theta$: in this notation, suggested by Profs. Harkness and Morley, De Moivre's theorem is expressed by the equation $(\operatorname{Cis}\theta)^n=\operatorname{Cis} n\theta$.

Again $$(\cos\theta + i\sin\theta)^{-n} = (\cos\theta - i\sin\theta)^n$$
$$= \{\cos(-\theta) + i\sin(-\theta)\}^n$$
$$= \cos(-n\theta) + i\sin(-n\theta).$$

Hence *De Moivre's Theorem holds for all integral values of n, positive or negative.* A large number of important applications of this theorem will be given later on in this chapter (§§ 38 *et seq.*).

To the theorems (1)—(5) we may add the following theorem, which is also of very great importance.

(6) The modulus of the sum of any number of complex quantities is *not greater than* the sum of their moduli.

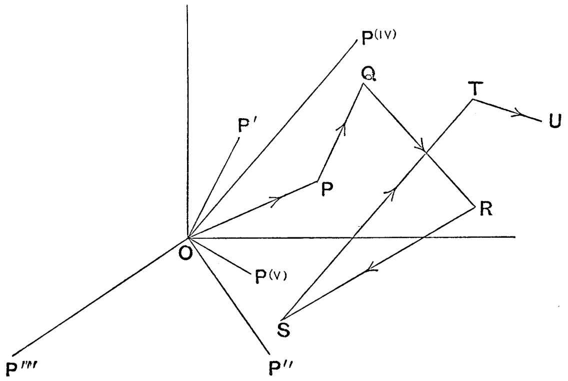

FIG. 31.

Let $\overline{OP}$, $\overline{OP'}$, ... be the displacements corresponding to the various complex quantities. Draw PQ equal and parallel to OP', QR equal and parallel to OP'', and so on. Finally we reach a point U, such that

$$\overline{OU} = \overline{OP} + \overline{OP'} + \overline{OP''} + \dots.$$

The length OU is the modulus of the sum of the complex quantities, whereas the sum of their moduli is the total length of the broken line $OPQR\dots U$. The truth of the theorem is now obvious (see also Ex. XXIII. 1).

37. We add some theorems concerning rational functions of complex numbers. A *rational function* of the complex variable z is defined exactly as is a rational function of a real variable x, viz. as the quotient of two polynomials in z.

THEOREM 1. *If* $R(x+yi)$ *is a rational function of* $x+yi$, *it can be reduced to the form* $X+Yi$, *where* X *and* Y *are rational functions of* x *and* y *with real coefficients.*

In the first place it is evident that any polynomial $P(x+yi)$ can be reduced, in virtue of the definitions of addition and multiplication, to the form $A+Bi$, where A and B are polynomials in x and y with real coefficients. Similarly $Q(x+yi)$ can be reduced to the form $C+Di$. Hence

$$R(x+yi)=P(x+yi)/Q(x+yi)$$

can be expressed in the form

$$(A+Bi)/(C+Di)=(A+Bi)(C-Di)/(C+Di)(C-Di)$$
$$=\frac{AC+BD}{C^2+D^2}+\frac{BC-AD}{C^2+D^2}i,$$

which proves the theorem.

THEOREM 2. *If* $R(x+yi)=X+Yi$, R *denoting a rational function as before, but with* **real** *coefficients, then* $R(x-yi)=X-Yi$.

In the first place this is easily verified for a power $(x+yi)^n$ by actual expansion.

It follows by addition that the theorem is true for any polynomial with real coefficients. Hence, in the notation used above,

$$R(x-yi)=\frac{A-Bi}{C-Di}=\frac{AC+BD}{C^2+D^2}-\frac{BC-AD}{C^2+D^2}i,$$

the reduction being the same as before except that the sign of i is changed throughout. It is evident that results similar to those of Theorems 1 and 2 hold for functions of any number of complex variables.

THEOREM 3. *The roots of an equation*

$$a_0z^n+a_1z^{n-1}+\ldots+a_n=0,$$

whose coefficients are real, may, in so far as they are not themselves real, be arranged in conjugate pairs.

For it follows from Theorem 2 that if $x+yi$ is a root, so is $x-yi$. A particular case of this theorem is the result (§ 34) that the roots of a quadratic equation with real coefficients are either real or conjugate.

This theorem is sometimes stated as follows—*in an equation* *with real coefficients complex roots occur in conjugate pairs.* It should be compared with the result of Exs. IV. 10, which may be stated as follows—*in an equation with rational coefficients irrational roots occur in conjugate pairs**.

Examples XXIII. 1. Prove theorem (6) of § 36 directly from the definitions and without the aid of geometrical considerations.

[First, to prove that $|z+z'| \leqq |z|+|z'|$ is to prove that

$$\cancel{(x+y)^2+(x'+y')^2} \leqq \{\sqrt{(x^2+y^2)}+\sqrt{(x'^2+y'^2)}\}^2.$$

The theorem is then easily extended to the general case.]

2. The one and only case in which

$$|z|+|z'|+\dots=|z+z'+\dots|,$$

is that in which the numbers $z, z', \dots$ have all the same amplitude. Prove this both geometrically and analytically.

3. The modulus of the sum of any number of complex numbers is not less than the sum of their real (or imaginary) parts.

4. If the sum and product of two complex numbers are both real the two numbers must either be real or conjugate.

5. If $\quad a+b\sqrt{2}+(c+d\sqrt{2})\,i=A+B\sqrt{2}+(C+D\sqrt{2})\,i,$
where a, b, c, d, A, B, C, D are real rational numbers, then

$$a=A, \quad b=B, \quad c=C, \quad d=D.$$

6. Express the following numbers in the form $A+Bi$, where A and B are real numbers: $(1+i)^2$, $(1-i)^2$, $(3-2i)/(2+3i)$, $(\lambda+\mu i)/(\lambda-\mu i)$,

$$\left(\frac{1+i}{1-i}\right)^2, \quad \left(\frac{1-i}{1+i}\right)^2, \quad \left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3, \quad \left(\frac{\lambda+\mu i}{\lambda-\mu i}\right)^2-\left(\frac{\lambda-\mu i}{\lambda+\mu i}\right)^2,$$

λ and μ denoting any real numbers.

7. Express the following functions of $z=x+yi$ in the form $X+Yi$, where X and Y are real functions of x and y: z^2, z^3, z^n, $1/z$, $z+(1/z)$, $(1+z)/(1-z)$, $(\alpha+\beta z)/(\gamma+\delta z)$, $\alpha, \beta, \gamma, \delta$ denoting real numbers.

8. Find the moduli of the numbers and functions in the two preceding examples.

9. The two lines joining the points $z=a$, $z=b$ and $z=c$, $z=d$ will be perpendicular if

$$\operatorname{am}\left(\frac{a-b}{c-d}\right)=\pm\tfrac{1}{2}\pi;$$

i.e. if $(a-b)/(c-d)$ is purely imaginary. What is the condition that the lines should be parallel?

10. The three angular points of a triangle are given by $z=\alpha$, $z=\beta$, $z=\gamma$, where α, β, γ are complex quantities. Establish the following propositions:

(i) *The centre of gravity is given by* $z=\frac{1}{3}(\alpha+\beta+\gamma)$.

(ii) *The circum-centre is given by* $|z-\alpha|=|z-\beta|=|z-\gamma|$.

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* The numbers $a+\sqrt{b}$, $a-\sqrt{b}$, where a, b are rational, are sometimes said to be 'conjugate.'

(iii) *The three perpendiculars from the angular points on the opposite sides meet in a point given by*

$$R\left(\frac{z-\alpha}{\beta-\gamma}\right)=R\left(\frac{z-\beta}{\gamma-\alpha}\right)=R\left(\frac{z-\gamma}{\alpha-\beta}\right)=0.$$

[If A, B, C are the vertices, and P any point z, the condition that AP should be perpendicular to BC (Ex. 9) is that $(z-\alpha)/(\beta-\gamma)$ should be purely imaginary, or that $R(z-\alpha)R(\beta-\gamma)+I(z-\alpha)I(\beta-\gamma)=0$.

This equation and the two similar equations, obtained by permuting α, β, γ cyclically, are satisfied by the same value of z, as appears from the fact that the sum of the three left-hand sides is zero (so that the third equation is a consequence of the first two). This proves the theorem.]

(iv) *There is a point P inside the triangle such that*

$$CBP=ACP=BAP=\omega.$$

Also $$\cot\omega=\cot A+\cot B+\cot C.$$

[From the equations*

$$\omega=CBP=\operatorname{am}(z-\beta)-\operatorname{am}(\gamma-\beta),$$
$$\cot\operatorname{am}(z-\beta)=\{R(z-\beta)/I(z-\beta)\},\text{ etc.,}$$

we deduce

$$\cot\omega\,\{I(z-\beta)R(\gamma-\beta)-R(z-\beta)I(\gamma-\beta)\}$$
$$=R(z-\beta)R(\gamma-\beta)+I(z-\beta)I(\gamma-\beta)\ldots\ldots\ldots(1).$$

This equation, and the two similar equations obtained by permuting α, β, γ cyclically, suffice to determine $\cot\omega$ and the real and imaginary parts of z. If we add the three equations z disappears and we are left with

$$\cot\omega\,\Sigma\{I(\beta)R(\gamma)-R(\beta)I(\gamma)\}$$
$$=\Sigma\{R(\beta)R(\gamma-\beta)+I(\beta)I(\gamma-\beta)\}\ldots\ldots\ldots\ldots(2),$$

the sign of summation referring to the three terms produced by cyclical interchange of α, β, γ.

Now $$\cot A=\cot\{\operatorname{am}(\gamma-\alpha)-\operatorname{am}(\beta-\alpha)\}$$
$$=\frac{R(\gamma-\alpha)R(\beta-\alpha)+I(\gamma-\alpha)I(\beta-\alpha)}{I(\gamma-\alpha)R(\beta-\alpha)-R(\gamma-\alpha)I(\beta-\alpha)},$$

and a little reduction shows that the denominator of this fraction is equal to the coefficient of $\cot\omega$ in equation (2), with its sign changed. Hence we can deduce that $\cot\omega=\cot A+\cot B+\cot C$.]

11. The two triangles whose vertices are the points a, b, c and x, y, z respectively will be similar if

$$\begin{vmatrix}1&1&1\\a&b&c\\x&y&z\end{vmatrix}=0.$$

[The condition required is that $\overline{AB}/\overline{AC}=\overline{XY}/\overline{XZ}$ (large letters denoting the points whose arguments are the corresponding small letters), or $(b-a)/(c-a)=(y-x)/(z-x)$, which is the same as the given condition.]

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* We suppose that as we go round the triangle in the direction ABC we leave it on our left.

12. Deduce from the last example that if the points x, y, z are collinear we can find *real* numbers α, β, γ such that $\alpha+\beta+\gamma=0$ and $\alpha x+\beta y+\gamma z=0$, and conversely (cf. Exs. XXII. 4). [Use the fact that in this case the triangle formed by x, y, z is similar to a certain line-triangle on the axis OX, and apply the result of the last example.]

13. **The general equation of the first degree with complex coefficients.** The equation $az+b=0$ has the one solution $z=-(b/a)$, unless $a=0$. If we put

$$a=\alpha+i\alpha', \quad b=\beta+i\beta', \quad z=x+iy,$$

and equate real and imaginary parts, we obtain two equations to determine the two real quantities x and y. The equation will have a *real* root if $y=0$, which gives $\alpha x+\beta=0$, $\alpha' x+\beta'=0$, and the condition that these equations should be consistent is $\alpha\beta'-\alpha'\beta=0$.

14. **The general quadratic equation with complex coefficients.** This equation is

$$(a+iA)\,z^2+2\,(b+iB)\,z+(c+iC)=0.$$

Unless a and A are both zero we can divide through by $a+iA$. Hence we may consider

$$z^2+2\,(b+iB)\,z+(c+iC)=0 \quad \text{........................(1)},$$

as the standard form of our equation. Putting $z=x+iy$ and equating real and imaginary parts we obtain a pair of simultaneous equations for x and y, viz.

$$x^2-y^2+2\,(bx-By)+c=0, \quad 2xy+2\,(by+Bx)+C=0.$$

If we put

$$x+b=\xi, \quad y+B=\eta, \quad b^2-B^2-c=h, \quad 2bB-C=k,$$

these equations become $\xi^2-\eta^2=h, \quad 2\xi\eta=k.$

Squaring and adding we obtain

$$\xi^2+\eta^2=\surd(h^2+k^2), \quad \xi=\pm\sqrt{\tfrac{1}{2}\{\surd(h^2+k^2)+h\}}, \quad \eta=\pm\sqrt{\tfrac{1}{2}\{\surd(h^2+k^2)-h\}}.$$

We must choose the signs so that $\xi\eta$ has the sign of k: i.e. if k is positive we must take like signs, if k is negative unlike signs.

Conditions for equal roots. The two roots can only be equal if both the square roots above vanish, i.e. if $h=0$, $k=0$, or if $c=b^2-B^2$, $C=2bB$. These conditions are equivalent to the single condition $c+iC=(b+iB)^2$ which obviously expresses the fact that the left-hand side of (1) is a perfect square.

Condition for a real root. If $x^2+2\,(b+iB)\,x+(c+iC)=0$, where x is real, then $x^2+2bx+c=0$, $2Bx+C=0$. Eliminating x we find that the required condition is $C^2-4bBC+4cB^2=0.$

Condition for a purely imaginary root. This is easily found to be

$$C^2-4bBC-4b^2c=0.$$

Conditions for a pair of conjugate complex roots. Since the sum and the product of two conjugate complex quantities are both real, $b+iB$ and $c+iC$ must both be real, i.e. $B=0$, $C=0$. Thus the equation (1) can have a pair of conjugate complex roots only if its coefficients are real. The reader should verify this conclusion by means of the explicit expressions of the roots. Moreover, even in this case, if $b^2 \geqq c$ the roots will be real. Hence for a pair of conjugate roots we must have $B=0$, $C=0$, $b^2<c$.

15. **The Cubic equation.** Consider the cubic equation

$$z^3+3Hz+G=0,$$

where G and H are complex quantities, it being given that the equation has (a) a real root, (b) a purely imaginary root, (c) a pair of conjugate roots.

If $H=\lambda+i\mu$, $G=\rho+i\sigma$, we arrive at the following conclusions.

(a) *A real root.* If $\mu \neq 0$ the real root is $-\sigma/3\mu$, and $\sigma^3+27\lambda\mu^2\sigma-27\mu^3\rho=0$. On the other hand, if $\mu=0$, we must also have $\sigma=0$, and the coefficients of the equation are real. In this case there may be three real roots.

(b) *A purely imaginary root.* If $\mu \neq 0$ the purely imaginary root is $(\rho/3\mu)\,i$, and $\rho^3-27\lambda\mu^2\rho-27\mu^3\sigma=0$. If $\mu=0$, then also $\rho=0$, and the root is iy, where y is given by the equation $y^3-3\lambda y-\sigma=0$, which has real coefficients. In this case there may be three purely imaginary roots.

(c) *A pair of conjugate roots.* Let these be $x \pm yi$. Then since the sum of the three roots is zero the third root must be $-2x$. From the relations between the coefficients and the roots of an equation we deduce

$$y^2-3x^2=3H, \qquad 2x(x^2+y^2)=G.$$

Hence G and H must both be real.

In each case we can either find a root (in which case the equation can be reduced to a quadratic by dividing by a known factor) or we can reduce the solution of the equation to the solution of a cubic equation with real coefficients.

16. The cubic equation $x^3+a_1x^2+a_2x+a_3=0$, where $a_1=A_1+iA_1', \ldots$ has a pair of conjugate imaginary roots. Prove that provided $A_3' \neq 0$ the remaining root is $-A_1'a_3/A_3'$, and two identical relations hold between $A_1, A_1', A_2, \ldots$. Examine the case in which $A_3'=0$.

17. Prove that if $z^3+3Hz+G=0$ has two imaginary roots, the equation

$$8\alpha^3+6\alpha H-G=0$$

has one real root which is the real part α of the imaginary roots of the original equation; and show that α has the same sign as G.

18. An equation of any order with complex coefficients will *in general* have no real roots, nor pairs of conjugate complex roots. How many conditions must be satisfied by the coefficients in order that the equation should have (a) a real root, (b) a pair of conjugate roots?

19. **Coaxal circles.** In Fig. 32, let a, b, z be the arguments of A, B, P. Then

$$\operatorname{am}\frac{z-b}{z-a}=APB,$$

the principal value of the amplitude being taken, and APB being a positive angle less than π. If the two circles shown in the figure are equal, and

z', z_1, z_1' are the arguments of P', P_1, P_1', and $APB=\theta$, it is easy to see that

$$\operatorname{am}\frac{z'-b}{z'-a}=\pi-\theta, \quad \operatorname{am}\frac{z_1-b}{z_1-a}=-\theta,$$

and
$$\operatorname{am}\frac{z_1'-b}{z_1'-a}=-\pi+\theta.$$

The locus defined by the equation

$$\operatorname{am}\frac{z-b}{z-a}=\theta,$$

where θ is constant, is the arc APB. By writing $\pi-\theta$, $-\theta$, $-\pi+\theta$ for θ we obtain the other three arcs shown.

The system of equations obtained by supposing that θ is a parameter, varying from $-\pi$ to $+\pi$, represents *the system of circles which can be drawn through the points A, B*. It should however be observed that each circle has to be divided into two parts to which correspond different values of θ.

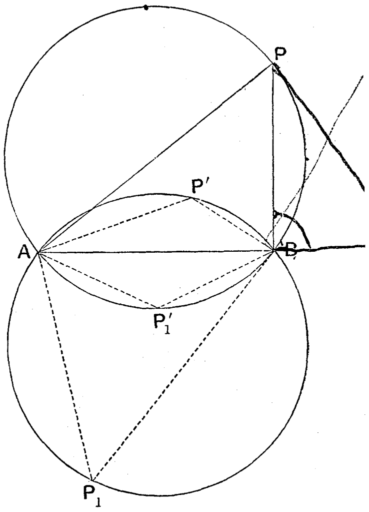

Fig. 32.

20. Now let us consider the equation

$$|(z-b)/(z-a)|=\lambda \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where λ is a constant.

Take the point K on BA produced so that $KPA=KBP$. Then the triangles KPA, KBP are similar, and so

$$AP/BP=KP/KB=KA/KP=\lambda.$$

Hence $KA/KB=\lambda^2$, and therefore K is a fixed point for all positions of P which satisfy the equation (1). Also $KP^2=KA\,.\,KB=\text{const.}$ Hence *the locus of P is a circle whose centre is K*. For different values of λ the equation (1) therefore represents a system of circles.

Every circle of this system cuts at right angles every circle of the system of Ex. 19. For the equation $KP^2=KA\,.\,KB$ shows that KP is a tangent to the circle APB.

The system of Ex. 19 is *a system of coaxal circles of the common point kind*. The system of Ex. 20 is called *a system of coaxal circles of the limiting point kind*; if λ is very small the circle is a very small circle containing B in its interior, if λ is very large a very small circle containing A in its interior: it is from this fact that the name is derived.

21. **Bilinear Transformations.** Consider the equation

$$z=Z+a\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $z=x+iy$ and $Z=X+iY$ are two complex variables which we may suppose to be represented in two planes xoy, XOY. To every value of z

corresponds one of Z, and conversely. If $a=\alpha+i\beta$, then

$$x=X+\alpha, \quad y=Y+\beta,$$

and to the point (x, y) corresponds the point (X, Y). If (x, y) describes a curve of any kind in its plane, (X, Y) describes a curve in its plane. Thus to any figure in one plane corresponds a figure in the other. A passage of this kind from a figure in the plane xoy to a figure in the plane XOY by means of a relation such as (1) between z and Z is called a *transformation.* In this particular case the relation between corresponding figures is very easily defined. The (X, Y) figure is the same in size, shape, and orientation as the (x, y) figure, but is shifted a distance α to the left, and a distance β downwards. Such a transformation is called a *translation.*

Now consider the equation

$$z=\rho Z \text{ ....(2)},$$

where ρ is real. This gives $x=\rho X$, $y=\rho Y$. The two figures are similar and similarly situated about their respective origins, but the scale of the (X, Y) figure is $(1/\rho)$ times that of the (x, y) figure. Such a transformation is called a *magnification.*

Finally consider the equation

$$z=(\cos\phi+i\sin\phi)\,Z \text{ ....(3)}.$$

It is clear that $|z|=|Z|$, $\operatorname{am} z=\operatorname{am} Z+\phi$, and that the two figures differ only in that the (X, Y) figure is the (x, y) figure turned about the origin through an angle ϕ in the negative direction. Such a transformation is called a *rotation.*

The general linear transformation

$$z=aZ+b \text{ ....(4)}$$

is a combination of the three transformations (1), (2), (3). For if $|a|=\rho$ and $\operatorname{am} a=\phi$ we can replace (4) by the three equations

$$z=z'+b, \quad z'=\rho Z', \quad Z'=(\cos\phi+i\sin\phi)\,Z.$$

Thus *the general linear transformation is equivalent to the combination of a translation, a magnification, and a rotation.*

Next let us consider the transformation

$$z=1/Z \text{ ....(5)}.$$

If $|Z|=R$ and $\operatorname{am} Z=\Theta$, then $|z|=1/R$ and $\operatorname{am} z=-\Theta$, and to pass from the (x, y) figure to the (X, Y) figure we *invert* the former with respect to o, with unit radius of inversion, and then construct the *image* of the new figure in the axis ox (i.e. the symmetrical figure on the other side of ox). We thus obtain a figure in the (x, y) plane, similar in every respect to the (X, Y) figure.

Finally consider the transformation

$$z=(aZ+b)/(cZ+d) \text{ ....(6)}.$$

This is equivalent to the combination of the transformations

$$z=(a/c)+(bc-ad)(z'/c), \quad z'=1/Z', \quad Z'=cZ+d,$$

i.e. to a certain combination of transformations of the types already considered.

The transformation $z=(aZ+b)/(cZ+d)$

is called the *general bilinear transformation.* Solving for Z we obtain

$$Z=(dz-b)/(cz-a).$$

This is the most general type of transformation for which one and only one value of z corresponds to each value of Z, and conversely.

22. *The general bilinear transformation transforms circles into circles.* This may be proved in a variety of ways. We may assume the well known theorem in pure geometry, that *inversion* transforms circles into circles (which may of course in particular cases be *straight lines*). Or we may take the equation

$$x+iy=\frac{(\alpha+i\alpha')(X+iY)+(\beta+i\beta')}{(\gamma+i\gamma')(X+iY)+(\delta+i\delta')},$$

assume that x and y satisfy the equation of a circle, calculate x and y in terms of X and Y, and so find the relation between X and Y by straightforward algebra. Or finally we may use the results of Exs. 19 and 20. This is the best and simplest method. If, e.g., the (x, y) circle is

$$|(z-\sigma)/(z-\rho)|=\lambda,$$

and we substitute for z in terms of Z, we obtain

$$|(Z-\sigma')/(Z-\rho')|=\lambda',$$

where $$\sigma'=-\frac{b-\sigma d}{a-\sigma c},\quad \rho'=-\frac{b-\rho d}{a-\rho c},\quad \lambda'=\lambda\left|\frac{a-\rho c}{a-\sigma c}\right|.$$

23. Consider the transformations $z=1/Z$, $z=(1+Z)/(1-Z)$, and draw the (X, Y) curves which correspond to (1) circles whose centre is the origin, (2) straight lines through the origin, in the (x, y) plane.

24. The condition that the transformation $z=(aZ+b)/(cZ+d)$ should make the circle $x^2+y^2=1$ correspond to a straight line in the (X, Y) plane, is $|a|=|c|$.

25. **Cross ratios.** The cross ratio (z_1z_2, z_3z_4) is defined to be

$$\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}.$$

If the four points z_1, z_2, z_3, z_4 are on the same line, this agrees with the definition adopted in elementary geometry. There are 24 cross ratios which can be formed from z_1, z_2, z_3, z_4 by permuting the suffixes. These consist of six groups of four equal cross ratios. If one ratio is λ the six distinct cross ratios are λ, $1-\lambda$, $1/\lambda$, $1/(1-\lambda)$, $1-(1/\lambda)$, $\lambda/(\lambda-1)$. The four points are said to be *harmonic* or *harmonically related* if any one of these is equal to -1. In this case the six ratios are -1, -1, $\frac{1}{2}$, $\frac{1}{2}$, 2, 2.

If any cross ratio is real all are real and the four points lie on a circle. For in this case

$$\operatorname{am}\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}=0 \text{ (or } \pi),$$

so that am $\{(z_1-z_3)/(z_1-z_4)\}$ and am $\{(z_2-z_3)/(z_2-z_4)\}$ are either equal or differ by π (cf. Ex. 19).

If $(z_1z_2,\ z_3z_4)=-1$, we have the two equations

$$\operatorname{am}\frac{z_1-z_3}{z_1-z_4}=\pi+\operatorname{am}\frac{z_2-z_3}{z_2-z_4},\qquad \left|\frac{z_1-z_3}{z_1-z_4}\right|=\left|\frac{z_2-z_3}{z_2-z_4}\right|.$$

The four points A_1, A_2, A_3, A_4 (Fig. 33) lie on a circle, A_1 and A_2 being separated by A_3 and A_4. Also $A_1A_3/A_1A_4=A_2A_3/A_2A_4$. Let O be the middle point of A_3A_4. The equation

$$\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}=-1$$

may be put in the form

$$(z_1+z_2)(z_3+z_4)=2(z_1z_2+z_3z_4),$$

or, what is the same thing,

$$\{z_1-\tfrac{1}{2}(z_3+z_4)\}\{z_2-\tfrac{1}{2}(z_3+z_4)\}=\{\tfrac{1}{2}(z_3-z_4)\}^2.$$

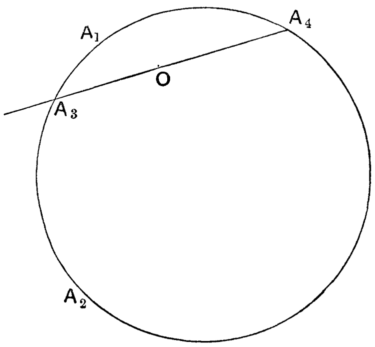

Fig. 33.

But this is equivalent to $\overline{OA_1}\,.\,\overline{OA_2}=\overline{OA_3}^2=\overline{OA_4}^2$. Hence OA_1 and OA_2 make equal angles with A_3A_4, and $OA_1\,.\,OA_2=OA_3^2=OA_4^2$. It will be observed that the relation between the pairs A_1, A_2 and A_3, A_4 is symmetrical. Hence if O' is the middle point of A_1A_2, $O'A_3$ and $O'A_4$ are equally inclined to A_1A_2, and $O'A_3\,.\,O'A_4=O'A_1^2=O'A_2^2$.

26. If the points A_1, A_2 are given by $az^2+2bz+c=0$, and the points A_3, A_4 by $a'z^2+2b'z+c'=0$, and O is the middle point of A_3A_4, and $ac'+a'c-2bb'=0$, then OA_1, OA_2 are equally inclined to A_3A_4 and $OA_1\,.\,OA_2=OA_3^2=OA_4^2$. (*Math. Trip.* 1901.)

[The pairs A_1, A_2 and A_3, A_4 are harmonically related.]

27. **The condition that four points should lie on a circle.** A sufficient condition is that one (and therefore all) of the cross ratios should be real (Ex. 25); this condition is also necessary. Another form of the condition is that it should be possible to choose real quantities α, β, γ such that

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ z_1z_4+z_2z_3 & z_2z_4+z_3z_1 & z_3z_4+z_1z_2 \end{vmatrix}=0.$$

To prove this we observe that the transformation $Z=1/(z-z_4)$ is equivalent to an inversion with respect to the point z_4, coupled with a certain reflexion (Ex. 21). If z_1, z_2, z_3 lie on a circle through z_4 the corresponding points $Z_1=1/(z_1-z_4)$, $Z_2=1/(z_2-z_4)$, $Z_3=1/(z_3-z_4)$ lie on a straight line. Hence (Ex. 12) we can find real quantities α', β', γ' such that $\alpha'+\beta'+\gamma'=0$ and $\alpha'/(z_1-z_4)+\beta'/(z_2-z_4)+\gamma'/(z_3-z_4)=0$, and it is easy to prove that this is equivalent to the given condition.

28. Prove the following analogue of De Moivre's Theorem for real quantities:—if $\phi_1, \phi_2, \phi_3, \dots$ is a series of positive acute angles such that

$$\tan \phi_{m+1} = \tan \phi_m \sec \phi_1 + \sec \phi_m \tan \phi_1,$$

then

$$\tan \phi_{m+n} = \tan \phi_m \sec \phi_n + \sec \phi_m \tan \phi_n,$$

$$\sec \phi_{m+n} = \sec \phi_m \sec \phi_n + \tan \phi_m \tan \phi_n,$$

and

$$\tan \phi_m + \sec \phi_m = (\tan \phi_1 + \sec \phi_1)^m.$$

[Use the method of mathematical induction.]

29. **The transformation** $z = Z^m$. In this case $r = R^m$, and θ and $m\Theta$ differ by a multiple of 2π. If Z describes a circle round the origin, z describes a circle round the origin m *times*.

The whole (x, y) plane corresponds to any one of m sectors in the (X, Y) plane, each of angle $2\pi/m$. To each point in the (x, y) plane correspond m points in the (X, Y) plane.

30. **Complex functions of a real variable.** If $f(t)$, $\phi(t)$ are two real functions of a real variable t defined for a certain range of values of t, we call

$$z = f(t) + i\phi(t) \quad \dots\dots\dots\dots\dots\dots(1)$$

a complex function of t. We can represent it graphically by drawing the curve

$$x = f(t), \quad y = \phi(t);$$

the equation of the curve may be obtained by eliminating t between these equations. If z is a polynomial in t, or rational function of t, with complex coefficients, we can express it in the form (1) and so determine the curve represented by the function.

(i) Let $$z = a + (b - a)t,$$
where a and b are complex numbers. If $a = \alpha + i\alpha'$, $b = \beta + i\beta'$, then

$$x = \alpha + (\beta - \alpha)t, \quad y = \alpha' + (\beta' - \alpha')t.$$

The curve is the straight line joining the points $z = a$ and $z = b$. The segment between the points corresponds to the range of values of t from 0 to 1. Find the values of t which correspond to the two produced segments of the line.

(ii) If $$z = c + \rho\{(1 + it)/(1 - it)\},$$
the curve is the circle of centre c and radius ρ. As t varies through all real values z describes the circle once.

(iii) In general the equation $z = (a + bt)/(c + dt)$ represents a circle. This can be proved by calculating x and y and eliminating: but this process is rather cumbrous. A simpler method is obtained by using the result of Ex. 22. Let $z = (a + bZ)/(c + dZ)$, $Z = t$. As t varies Z describes a straight line, viz. the axis of X. Hence z describes a circle.

(iv) The equation $$z = a + 2bt + ct^2$$
represents a parabola generally, a straight line if b/c is real.

(v) The equation $z=(a+2bt+ct^2)/(\alpha+2\beta t+\gamma t^2)$, where α, β, γ are real, represents a conic section.

[Eliminate t from

$$x=(A+2Bt+Ct^2)/(\alpha+2\beta t+\gamma t^2), \quad z=(A'+2B't+C't^2)/(\alpha+2\beta t+\gamma t^2)$$

(where $a=A+iA'$, $b=B+iB'$, $c=C+iC'$).]

38. Formulae for sin $n\theta$ **and cos** $n\theta$. De Moivre's Theorem enables us to express $\sin n\theta$ and $\cos n\theta$, where n is a positive integer, in terms of $\sin\theta$ and $\cos\theta$. For from the formula

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

(where n is a positive integer) we deduce

$$\cos n\theta = R\,[(\cos\theta + i\sin\theta)^n]$$
$$= (\cos\theta)^n\left\{1-\binom{n}{2}\tan^2\theta+\binom{n}{4}\tan^4\theta-\dots\right\},$$
$$\sin n\theta = I\,[(\cos\theta + i\sin\theta)^n]$$
$$= (\cos\theta)^n\left\{\binom{n}{1}\tan\theta-\binom{n}{3}\tan^3\theta+\dots\right\},$$

where $\binom{n}{r}$ is the general binomial coefficient

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}$$

(sometimes written ${}^nC_r$).

Then

$$\frac{\cos n\theta}{(\cos\theta)^n}=1-\binom{n}{2}t^2+\binom{n}{4}t^4-\dots \quad\dots\dots\dots\dots(1),$$
$$\frac{\sin n\theta}{(\cos\theta)^n}=\binom{n}{1}t-\binom{n}{3}t^3+\dots \quad\dots\dots\dots\dots\dots\dots(2),$$

where $t=\tan\theta$.

By division

$$\tan n\theta=\left\{\binom{n}{1}t-\binom{n}{3}t^3+\dots\right\}\Big/\left\{1-\binom{n}{2}t^2+\dots\right\} \quad\dots(3).$$

From (1) and (2) we can deduce further formulae expressing

$$\cos n\theta, \quad \frac{\sin n\theta}{\sin\theta}$$

in terms of $\cos\theta$ only. For

$$\cos^n\theta\tan^{2r}\theta=\sin^{2r}\theta\cos^{n-2r}\theta=\cos^{n-2r}\theta\,(1-\cos^2\theta)^r;$$

and on substituting in (1), after multiplying up by $\cos^n\theta$, we see that

$$\cos n\theta = a_n \cos^n\theta + a_{n-2}\cos^{n-2}\theta + \dots$$

where $a_n, a_{n-2}, \dots$ are constants, and the last term is a constant or a multiple of $\cos\theta$, according as n is even or odd.

Similarly, from (2), we deduce

$$\frac{\sin n\theta}{\sin\theta} = b_{n-1}\cos^{n-1}\theta + b_{n-3}\cos^{n-3}\theta + \dots.$$

To determine the actual values of the coefficients generally and directly, and by means of really elementary methods, is a matter of some little difficulty. The formulae are

$$2\cos n\theta = (2\cos\theta)^n - \frac{n}{1\,!}(2\cos\theta)^{n-2} + \frac{n(n-3)}{2\,!}(2\cos\theta)^{n-4}$$

$$-\dots+(-)^r\frac{n(n-r-1)\dots(n-2r+1)}{r\,!}(2\cos\theta)^{n-2r}+\dots\quad\dots(4),$$

$$\frac{\sin n\theta}{\sin\theta} = (2\cos\theta)^{n-1} - \frac{n-2}{1\,!}(2\cos\theta)^{n-3} + \frac{(n-3)(n-4)}{2\,!}(2\cos\theta)^{n-5}$$

$$-\dots+(-)^r\frac{(n-r-1)\dots(n-2r)}{r\,!}(2\cos\theta)^{n-2r-1}+\dots\quad\dots\dots(5).$$

That these formulae are correct is easily verified by induction. For

$$2\cos(n+1)\theta = \cos\theta\,(2\cos n\theta) - 2(1-\cos^2\theta)\frac{\sin n\theta}{\sin\theta},$$

$$\frac{\sin(n+1)\theta}{\sin\theta} = \cos\theta\left(\frac{\sin n\theta}{\sin\theta}\right) + \cos n\theta,$$

and if we assume that the formulae hold for $n = 1, 2, \dots k$ (and they are easily verified for $n = 1, 2, 3$), we can at once show that they hold for $n = k+1$. We leave this as an exercise for the reader.

39. When $\tan n\theta$ is given we can regard the equation (3), which we may write for brevity in the form $\tan n\theta = f(t)$, as an equation of the nth degree in t, one of whose roots is $t = \tan\theta$.

Similarly, one of the roots of

$$\tan\left\{n\left(\theta+\frac{k\pi}{n}\right)\right\} = f(t)$$

is $t = \tan\{\theta + (k\pi/n)\}$ (k being any integer). But since

$$\tan(n\theta + k\pi) = \tan n\theta,$$

the two equations are the same. Hence $\tan\{\theta+(k\pi/n)\}$ is a root of (3) for all values of k.

As we give k all integral values, this expression assumes n and only n distinct values, viz.

$$\tan\theta,\ \tan\left(\theta+\frac{\pi}{n}\right),\ \tan\left(\theta+\frac{2\pi}{n}\right),\ \dots,\ \dots\ \tan\left\{\theta+\frac{(n-1)\pi}{n}\right\}.$$

These are therefore the roots of (3), considered as an equation in t. It follows that any symmetric function of these quantities can be expressed in terms of the coefficients of (3), i.e. in terms of $\tan n\theta$. The equations (1), (2), (4), (5) can of course be considered from the same point of view. Some illustrations will be found in the examples which follow.

Examples XXIV. 1. The equation (3) may be written in the form

$$t^n-\binom{n}{2}t^{n-2}+\binom{n}{4}t^{n-4}-\dots-\begin{matrix}\tan\\ \cot\end{matrix}\,n\theta\left\{\binom{n}{1}t^{n-1}-\binom{n}{3}t^{n-3}+\dots\right\}=0,$$

where $\tan n\theta$ or $\cot n\theta$ is to be chosen according as n is odd or even.

2. Show that

$$\sec^2\theta+\sec^2\left(\theta+\frac{\pi}{n}\right)+\dots+\sec^2\left\{\theta+\frac{(n-1)\pi}{n}\right\}$$

is equal to $n^2\sec^2 n\theta$ or $n^2\operatorname{cosec}^2 n\theta$ according as n is odd or even.

(*Math. Trip.* 1900.)

[The expression given is $n+\Sigma t_r^2$, where t_r is a root of the equation in Ex. 1.]

3. Prove that $\sec^2\frac{\pi}{9}+\sec^4\frac{2\pi}{9}+\sec^4\frac{3\pi}{9}+\sec^4\frac{4\pi}{9}=1120$.

4. If n is odd

$$t^n-\binom{n}{2}t^{n-2}+\binom{n}{4}t^{n-4}-\dots\equiv t\left(t^2-\tan^2\frac{\pi}{n}\right)\left(t^2-\tan^2\frac{2\pi}{n}\right)\dots\left(t^2-\tan^2\frac{r\pi}{n}\right),$$

where $r=\frac{1}{2}(n-1)$. State and prove the corresponding formula when n is even.

5. The roots of the equation $2\cos n\theta=x^n-\frac{n}{1!}x^{n-2}+\frac{n(n-3)}{2!}x^{n-4}-\dots$ are

$$2\cos\theta,\quad 2\cos\left(\theta+\frac{2\pi}{n}\right),\ \dots\quad 2\cos\left\{\theta+\frac{2(n-1)\pi}{n}\right\}.$$

6. The roots of $64x^3-112x^2+56x-7=0$ are $\sin^2\frac{\pi}{7}$, $\sin^2\frac{2\pi}{7}$, $\sin^2\frac{4\pi}{7}$.

Deduce that $\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}-\sin\frac{\pi}{7}=\frac{1}{2}\sqrt{7}$.

7. Show that $4\cos^2(\pi/7)$ is a root of $x^3-5x^2+6x-1=0$, and find the other roots. (*Math. Trip.* 1898.)

8. **Factor-formulae for** $\cos n\theta$ **and** $(\sin n\theta)/(\sin\theta)$*. Show that $2\cos n\theta$ is equal to

$$2^n\left(\cos^2\theta-\cos^2\frac{\pi}{2n}\right)\left(\cos^2\theta-\cos^2\frac{3\pi}{2n}\right)\ldots\ldots\left\{\cos^2\theta-\cos^2\frac{(n-2)\pi}{2n}\right\}\cos\theta,$$

or to $$2^n\left(\cos^2\theta-\cos^2\frac{\pi}{2n}\right)\left(\cos^2\theta-\cos^2\frac{3\pi}{2n}\right)\ldots\ldots\left\{\cos^2\theta-\cos^2\frac{(n-1)\pi}{2n}\right\},$$

according as n is odd or even.

9. Show that $$\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\ldots\sin\frac{r\pi}{2n}=2^{-\frac{1}{2}(n-1)},$$

where r is equal to $n-2$ or to $n-1$ according as n is odd or even.

[Put $\theta=0$ in Ex. 8: consider the sign carefully, when taking the square root of each side.]

10. Show that $(\sin n\theta)/(\sin\theta)$ is equal to

$$2^{n-1}\left(\cos^2\theta-\cos^2\frac{\pi}{n}\right)\left(\cos^2\theta-\cos^2\frac{2\pi}{n}\right)\ldots\ldots\left\{\cos^2\theta-\cos^2\frac{(n-2)\pi}{2n}\right\}\cos\theta,$$

or to $$2^{n-1}\left(\cos^2\theta-\cos^2\frac{\pi}{n}\right)\left(\cos^2\theta-\cos^2\frac{2\pi}{n}\right)\ldots\ldots\left\{\cos^2\theta-\cos^2\frac{(n-1)\pi}{2n}\right\},$$

according as n is even or odd.

11. Show from Ex. 10, and equation (2) of § 38, that if n is odd

$$2^{n-1}\left(\cos^2\theta-\cos^2\frac{\pi}{n}\right)\left(\cos^2\theta-\cos^2\frac{2\pi}{n}\right)\ldots\left\{\cos^2\theta-\cos^2\frac{(n-1)\pi}{2n}\right\}$$
$$=n\cos^{n-1}\theta-\frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^2\theta+\ldots,$$

and deduce, by putting $\theta=0$, that

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{2n}=2^{-\frac{1}{2}(n-1)}\sqrt{n}.$$

Obtain the corresponding formula for the case in which n is even.

40. Roots of complex numbers. We have not, up to the present, attributed any meaning to symbols such as $\sqrt[n]{a}$, $a^{m/n}$, when a is a complex number, and m and n integers. It is, however, natural to adopt the definitions which are given in elementary algebra for real values of a. Thus we define $\sqrt[n]{a}$ or $a^{1/n}$, where n is a positive integer, as any number a which satisfies the equation $z^n=a$; and $a^{m/n}$, where m is an integer, as $(a^{1/n})^m$. These definitions do not prejudge the question as to whether there are or are not more than one (or any) roots of the equation.

41. Solution of the equation $z^n=a$. Let

$$a=\rho(\cos\phi+i\sin\phi),$$

where ρ is positive and ϕ is an angle such that $-\pi<\phi\leqq\pi$.

---

* The results of Exs. 8—11 are of considerable importance in Higher Trigonometry.

Then if $z = r(\cos\theta + i\sin\theta)$, the equation takes the form

$$r^n(\cos n\theta + i\sin n\theta) = \rho(\cos\phi + i\sin\phi),$$

so that $$r^n = \rho, \quad \cos n\theta = \cos\phi, \quad \sin n\theta = \sin\phi \;\ldots\ldots\ldots(1).$$

The only possible value of r is $\sqrt[n]{\rho}$, the ordinary arithmetical nth-root of ρ; and in order that the last two equations should be satisfied it is necessary and sufficient that $n\theta = \phi + 2k\pi$, where k is an integer, or

$$\theta = (\phi + 2k\pi)/n.$$

If $k = pn + q$, where p and q are integers, and $0 \leqq q < n$, the value of θ is $2p\pi + (\phi + 2q\pi)/n$, and in this the value of p is a matter of indifference. Hence *the equation*

$$z^n = a = \rho(\cos\phi + i\sin\phi)$$

has n roots and n only, given by $z = r(\cos\theta + i\sin\theta)$, *where*

$$r = \sqrt[n]{\rho}, \quad \theta = (\phi + 2q\pi)/n \quad (q = 0, 1, 2, \ldots n-1).$$

That these n roots are in reality all distinct is easily seen by plotting them on the Argand Diagram. The figure (Fig. 34) shows the four fourth roots of

$$(1{\cdot}6)(\cos 55^\circ + i\sin 55^\circ).$$

The particular root

$$\sqrt[n]{\rho}\,\{\cos(\phi/n) + \sin(\phi/n)\}$$

is called the *principal value* of $\sqrt[n]{a}$.

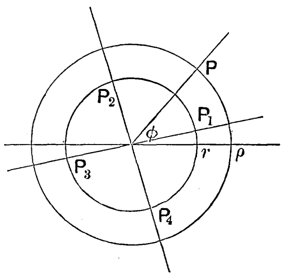

FIG. 34.

The case in which $a = 1$, $\rho = 1$, $\phi = 0$ is of particular interest. The n roots of the equation $z^n = 1$ are

$$\cos(2q\pi/n) + i\sin(2q\pi/n), \quad (q = 0, 1, \ldots n-1).$$

These quantities are called the nth roots of unity; the principal value is unity itself. If we write ω_n for $\cos(2\pi/n) + i\sin(2\pi/n)$ we see that the n roots of unity are

$$1, \ \omega_n, \ \omega_n^2, \ldots \omega_n^{n-1}.$$

Examples XXV. 1. The two square roots of 1 are $1, -1$; the three cube roots are 1, $\frac{1}{2}(-1+i\sqrt{3})$, $\frac{1}{2}(-1-i\sqrt{3})$; the four fourth roots are 1, i, -1, $-i$; and the five fifth roots are

$$1, \ \tfrac{1}{4}\{\sqrt{5}-1+i\sqrt{10+2\sqrt{5}}\}, \ \tfrac{1}{4}\{-\sqrt{5}-1+i\sqrt{10-2\sqrt{5}}\},$$
$$\tfrac{1}{4}\{-\sqrt{5}-1-i\sqrt{10-2\sqrt{5}}\}, \ \tfrac{1}{4}\{\sqrt{5}-1-i\sqrt{10+2\sqrt{5}}\}.$$

2. Prove that $$1+\omega_n+\omega_n{}^2+\dots+\omega_n{}^{n-1}=0.$$

3. Prove that $(x+y\omega_3+z\omega_3{}^2)(x+y\omega_3{}^2+z\omega_3)=x^2+y^2+z^2-yz-zx-xy$.

4. The nth roots of a are $\sqrt[n]{a}$, $\omega_n\sqrt[n]{a}$, $\omega_n{}^2\sqrt[n]{a}, \dots \omega_n{}^{n-1}\sqrt[n]{a}$, where $\sqrt[n]{a}$ denotes the *principal* nth root.

5. It follows from Exs. XXIII. 14 that the roots of
$$z^2=\alpha+i\beta$$
are $$\pm\sqrt{[\tfrac{1}{2}\{\sqrt{(\alpha^2+\beta^2)}+\alpha\}]}\pm i\sqrt{[\tfrac{1}{2}\{\sqrt{(\alpha^2+\beta^2)}-\alpha\}]},$$
like or unlike signs being chosen according as β is positive or negative. Show that this result agrees with the result of § 41.

6. Show that $(x^{2m}-a^{2m})/(x^2-a^2)$ is equal to
$$\left(x^2-2ax\cos\frac{\pi}{m}+a^2\right)\left(x^2-2ax\cos\frac{2\pi}{m}+a^2\right)\dots\left(x^2-2ax\cos\frac{(m-1)\pi}{m}+a^2\right).$$

[The factors of $x^{2m}-a^{2m}$ are
$$(x-a),\quad (x-a\omega_{2m}),\quad (x-a\omega^2{}_{2m}),\quad \dots(x-a\omega_{2m}^{2m-1}).$$
The factor $x-a\omega_{2m}{}^m$ is $x+a$. The factors $(x-a\omega_{2m}{}^s)$, $(x-a\omega_{2m}^{2m-s})$ taken together give a factor $x^2-2ax\cos\dfrac{s\pi}{m}+a^2$.]

7. Resolve $x^{2m+1}-a^{2m+1}$, $x^{2m}+a^{2m}$, and $x^{2m+1}+a^{2m+1}$ into factors in a similar way.

8. Show that $x^{2n}-2x^na^n\cos\theta+a^{2n}$ is equal to
$$\left(x^2-2xa\cos\frac{\theta}{n}+a^2\right)\left(x^2-2xa\cos\frac{\theta+2\pi}{n}+a^2\right)\dots$$
$$\dots\left(x^2-2xa\cos\frac{\theta+2(n-1)\pi}{n}+a^2\right).$$

[Use the formula
$$x^{2n}-2x^na^n\cos\theta+a^{2n}=\{x^n-a^n(\cos\theta+i\sin\theta)\}\{x^n-a^n(\cos\theta-i\sin\theta)\},$$
and split up each of the last two expressions into n factors.]

9. The problem of finding the accurate value of ω_n in a numerical form involving only square roots, as in the formula $\omega_3=\frac{1}{2}(-1+i\sqrt{3})$, is the algebraical equivalent of the geometrical problem of inscribing in a circle a regular polygon of n sides by Euclidean methods, i.e. by ruler and compasses. We saw in fact in Chapters I. and II. that irrationals involving square roots could always be so constructed, and are the only irrationals which can be so constructed.

Euclid gives constructions for $n=3$, 4, 5, 6, 8, 10, 12, and 15. It is evident that the construction is possible for any value of n which can be found from these by multiplication by any power of 2. There are other special values of n for which such constructions are possible, the most interesting being $n=17$.

Approximate constructions for regular polygons of any number of sides will be found in books of practical geometry.

42. The general form of De Moivre's Theorem. It follows from the results of the last section that, if q is a positive integer, one of the values of $(\cos\theta + i\sin\theta)^{1/q}$ is

$$\cos(\theta/q) + i\sin(\theta/q).$$

Raising each of these expressions to the power p (where p is any integer positive or negative), we obtain the theorem that one of the values of $(\cos\theta + i\sin\theta)^{p/q}$ is $\cos(p\theta/q) + i\sin(p\theta/q)$, or *if* α *is any rational quantity, one of the values of* $(\cos\theta + i\sin\theta)^{\alpha}$ *is*

$$\cos\alpha\theta + i\sin\alpha\theta.$$

This is a generalised form of De Moivre's theorem stated in § 36.

MISCELLANEOUS EXAMPLES ON CHAPTER III.

1. The condition that a triangle (xyz) should be equilateral is that

$$x^2+y^2+z^2-yz-zx-xy=0.$$

[Let XYZ be the triangle. The displacement $\overline{ZX}$ is $\overline{YZ}$ turned through an angle $\frac{2}{3}\pi$ in the positive or negative direction: or, as $\text{Cis}\,\frac{2}{3}\pi=\omega_3$, $\text{Cis}\,(-\frac{2}{3}\pi)=1/\omega_3=\omega_3{}^2$, we have $x-z=(z-y)\,\omega_3$ or $x-z=(z-y)\,\omega_3{}^2$. Hence $x+y\omega_3+z\omega_3{}^2=0$ or $x+y\omega_3{}^2+z\omega_3=0$. The result follows from Ex. XXV. 3.]

2. If XYZ, $X'Y'Z'$ are two triangles, and

$$\overline{YZ}\,.\,\overline{Y'Z'}=\overline{ZX}\,.\,\overline{Z'X'}=\overline{XY}\,.\,\overline{X'Y'},$$

then both triangles are equilateral.

[From the equations

$$(y-z)\,(y'-z')=(z-x)\,(z'-x')=(x-y)\,(x'-y')=\kappa^2$$

say, we deduce $\Sigma\,1/(y'-z')=0$, or $\Sigma x'^2-\Sigma y'z'=0$. Now apply the last example.]

3. On the sides of a triangle ABC similar triangles BCX, CAY, ABZ are described. Show that the centres of gravity of ABC, XYZ are coincident.

[We have $(x-c)/(b-c)=(y-a)/(c-a)=(z-b)/(a-b)=\lambda$, say. Express $\frac{1}{3}(x+y+z)$ in terms of a, b, c.]

4. If X, Y, Z are points on the sides of the triangle ABC, such that

$$BX/XC=CY/YA=AZ/ZB=r,$$

and if ABC, XYZ are similar, then either $r=1$ or both triangles are equilateral.

5. Deduce Ptolemy's Theorem concerning cyclic quadrilaterals from the fact that the cross ratios of four concyclic points are real.

[Start from the identity

$$(x_2-x_3)\,(x_1-x_4)+(x_3-x_1)\,(x_2-x_4)+(x_1-x_2)\,(x_3-x_4)=0.]$$

6. If $z^2+z'^2=1$, the points z, z' are ends of conjugate diameters of an ellipse whose foci are the points 1, -1. [If CP, CD are conjugate semi-diameters of an ellipse and S, H its foci, then CD is parallel to the exterior bisector of the angle SPH, and $SP\,.\,HP=CD^2$.]

7. Prove that $|a+b|^2+|a-b|^2=2\{|a|^2+|b|^2\}$. [This is the analytical equivalent of the geometrical theorem that, if M is the middle point of PQ, $OP^2+OQ^2=2OM^2+2MP^2$.]

8. Deduce from Ex. 7 that

$$|a+\sqrt{(a^2-b^2)}|+|a-\sqrt{(a^2-b^2)}|=|a+b|+|a-b|.$$

[If $a+\sqrt{(a^2-b^2)}=z_1$, $a-\sqrt{(a^2-b^2)}=z_2$, we have

$$|z_1|^2+|z_2|^2=\tfrac{1}{2}|z_1+z_2|^2+\tfrac{1}{2}|z_1-z_2|^2=2|a|^2+2|a^2-b^2|,$$

and so $(|z_1|+|z_2|)^2=2\{|a|^2+|a^2-b^2|+|b|^2\}=|a+b|^2+|a-b|^2+2|a^2-b^2|$.

Another way of stating the result is: if z_1 and z_2 are the roots of $\alpha z^2+2\beta z+\gamma=0$, then

$$|z_1|+|z_2|=(1/|\alpha|)\{(|-\beta+\sqrt{\alpha\gamma}|)+(|-\beta-\sqrt{\alpha\gamma}|)\}.]$$

9. If $x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ is an equation with real coefficients and has two real and two complex roots, concyclic in the Argand diagram, then

$$a_3^2+a_1^2a_4+a_2^3-a_2a_4-2a_1a_2a_3=0.$$

10. The four roots of $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ will be harmonically related if

$$a_0a_3^2+a_1^2a_4+a_2^3-a_0a_2a_4-2a_1a_2a_3=0.$$

[Express $Z_{23,14}Z_{31,24}Z_{12,34}$, where $Z_{23,14}=(z_1-z_2)(z_3-z_4)+(z_1-z_3)(z_2-z_4)$ and z_1, z_2, z_3, z_4 are the roots of the equation, in terms of the coefficients.]

11. **Imaginary points and straight lines.** Let $ax+by+c=0$ be an equation with complex coefficients (which of course may be real in special cases).

If we give x any particular real or complex value, we can find the corresponding value of y. The aggregate of pairs of real or complex values of x and y which satisfy the equation is called an *imaginary straight line*; the pairs of values are called *imaginary points*, and are said *to lie on the line*. The values of x and y are called the *coordinates* of the point (x, y). When x and y are real, the point is called a *real point*: when a, b, c are all real (or can be made all real by division by a factor), the line is called a *real line*. The points $x=\alpha+i\beta$, $y=\gamma+i\delta$ and $x=\alpha-i\beta$, $y=\gamma-i\delta$ are said to be *conjugate*; and so are the lines

$$(A+iA')x+(B+iB')y+C+iC'=0,\quad (A-iA')x+(B-iB')y+C-iC'=0.$$

Verify the following assertions:—every real line contains infinitely many pairs of conjugate imaginary points; every imaginary line contains one and only one real point; an imaginary line cannot contain a pair of conjugate imaginary points:—and find the conditions (a) that the line joining two given imaginary points should be real, and (b) that the point of intersection of two imaginary lines should be real.

12. Sum the series

$$\cos a+\cos(a+b)+\cos(a+2b)+\dots,\quad \sin a+\sin(a+b)+\sin(a+2b)+\dots$$

to n terms.

[Sum the geometrical series

$$(\cos a+i\sin a)\{1+(\cos b+i\sin b)+(\cos b+i\sin b)^2+\dots\}$$

to n terms, and equate real and imaginary parts.]

13. Also sum

$$1+\binom{n}{1}\cos a+\binom{n}{2}\cos 2a+\dots,\quad 1+\binom{n}{1}\sin a+\binom{n}{2}\sin 2a+\dots$$

to $n+1$ terms.

14. Sum the series $\cos \alpha+x\cos(\alpha+\beta)+\dots+x^{n-1}\cos\{\alpha+(n-1)\beta\}$.

(*Math. Trip.* 1905.)

15. Find the modulus and the amplitude of

$$1+\cos\theta+i\sin\theta,\quad 1+\cos\theta-i\sin\theta,\quad 1-\cos\theta+i\sin\theta,\quad 1-\cos\theta-i\sin\theta.$$

16. Find the square roots of the numbers in the preceding example.

17. Prove that

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n=\cos(\tfrac{1}{2}n\pi-n\theta)+i\sin(\tfrac{1}{2}n\pi-n\theta).$$

18. Prove the identities

$$(x+y+z)(x+\omega_3y+\omega_3{}^2z)(x+\omega_3{}^2y+\omega_3z)=x^3+y^3+z^3-3xyz,$$

$$(x+y+z)(x+\omega_5y+\omega_5{}^4z)(x+\omega_5{}^2y+\omega_5{}^3z)(x+\omega_5{}^3y+\omega_5{}^2z)(x+\omega_5{}^4y+\omega_5z)$$
$$=x^5+y^5+z^5-5x^3yz+5xy^2z^2.$$

19. Solve the equations

$$x^3-3ax+(a^3+1)=0 \text{ and } x^5-5ax^3+5a^2x+(a^5+1)=0.$$

20. If $f(x)=a_0+a_1x+\dots+a_kx^k$, then

$$\{f(x)+f(\omega x)+\dots+f(\omega^{n-1}x)\}/n=a_0+a_nx^n+a_{2n}x^{2n}+\dots+a_{\lambda n}x^{\lambda n},$$

ω being any root of $x^n=1$ (except $x=1$), and λn the greatest multiple of n contained in λ. Find a similar formula for $a_\mu+a_{\mu+n}x^n+a_{\mu+2n}x^{2n}+\dots$.

21. If $$(1+x)^n=p_0+p_1x+p_2x^2+\dots$$

(n being a positive integer), prove that

$$p_0-p_2+p_4-\dots=2^{\frac{1}{2}n}\cos\tfrac{1}{4}n\pi,\quad p_1-p_3+p_5-\dots=2^{\frac{1}{2}n}\sin\tfrac{1}{4}n\pi.$$

22. Sum the series

$$\frac{x}{2!\,n-2!}+\frac{x^2}{5!\,n-5!}+\frac{x^3}{8!\,n-8!}+\dots+\frac{x^{n/3}}{n-1!},$$

n being a multiple of 3. (*Math. Trip.* 1899.)

23. Show how to deduce the formulae given in § 38 from the addition theorems for $\cos x$ and $\sin x$, using no complex quantities. [See Hobson's *Trigonometry*, Ch. VII.]

24. Prove that, when m is odd, $3\Sigma \operatorname{cosec}^2(r\pi/m) = m^2-1$, the summation extending to $r=1, 2, \dots, m-1$.

The corresponding sum, extended only to values of r less than m and prime to it, is denoted by S. Show that, if m is the product of unequal odd primes $a, b, c, \dots k$, then $3S=(a^2-1)(b^2-1)(c^2-1)\dots(k^2-1)$.

(*Math. Trip.* 1902.)

25. Prove that, if m is odd,

$$\sum_{r=0}^{m-1} \tan^4\{(2r+1)\pi/m\} = \tfrac{1}{3}m(m-1)(m^2+m-3).$$

(*Math. Trip.* 1903.)

26. If $\alpha_r = a + b\cos\{\theta+(2r\pi/n)\}$, for $r=1, 2, \dots, n$, show that

$$3n(n-2)\Sigma\alpha_1\Sigma\alpha_1\alpha_2 = 3n^2\Sigma\alpha_1\alpha_2\alpha_3 + (n-1)(n-2)(\Sigma\alpha_1)^3,$$

if $n>3$, and find the corresponding equation when $n=3$. (*Math. Trip.* 1906.)

27. The roots of

$$1 - nx - \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + (-1)^{\frac{1}{2}n(n+1)}x^n = 0$$

are
$$\tan\frac{\pi}{4n}, \quad \tan\frac{5\pi}{4n}, \dots, \quad \tan\frac{(4n-3)\pi}{4n}.$$

[The equation is $(1-xi)^n + i(1+xi)^n = 0$.]

28. If $\alpha = \pi/4n$, show (cf. Ex. 27) that

$$\cot\alpha, \quad -\cot 3\alpha, \quad \cot 5\alpha, \dots, \quad (-)^{n-1}\cot(2n-1)\alpha$$

are the roots of

$$x^n - nx^{n-1} - \frac{n(n-1)}{2!}x^{n-2} + \frac{n(n-1)(n-2)}{3!}x^{n-3} - \dots = 0.$$

Deduce that

$$\cot\alpha\operatorname{cosec}^2\alpha - \cot 3\alpha\operatorname{cosec}^2 3\alpha + \dots \text{ to } n \text{ terms}$$

is equal to $2n^3$. (*Math. Trip.* 1901.)

29. There are in general two points unaltered by the transformation $z=(aZ+b)/(cZ+d)$. If these points are α, β, the transformation can be put in the form $(z-\alpha)/(z-\beta) = k(Z-\alpha)/(Z-\beta)$.

In particular, reduce the transformation $z=(1+Z)/(1-Z)$ to this form. Divide the Z-plane into 8 regions by means of the axes and the unit circle. Find the region in the z-plane which corresponds to each of these regions.

30. If $z=Z^2-1$, then as z describes the circle $|z|=\kappa$, the two corresponding positions of Z each describe the Cassinian oval $\rho_1\rho_2=\kappa$, where ρ_1, ρ_2 are the distances from the points ± 1. Trace the ovals for different values of κ.

31. If t is a complex number such that $|t|=1$, then as t varies, the point $x=(at+b)/(t-c)$ describes a circle, unless $|c|=1$, when it describes a straight line.

32. If t varies as in the last example, the point $x=\frac{1}{2}\{at+(b/t)\}$ in general describes an ellipse whose foci are given by $x^2=ab$, and whose axes are $|a|+|b|$ and $|a|-|b|$. But if $|a|=|b|$, x describes the finite straight line joining the points $\pm\sqrt{(ab)}$.

33. If $z=2Z+Z^2$, the circle $|Z|=1$ corresponds to a cardioid in the plane of z.

34. Discuss the transformation $z=\frac{1}{2}\{Z+(1/Z)\}$, showing in particular that to the circles $X^2+Y^2=a^2$ correspond the confocal ellipses

$$\frac{x^2}{\left\{\frac{1}{2}\left(a+\frac{1}{a}\right)\right\}^2}+\frac{y^2}{\left\{\frac{1}{2}\left(a-\frac{1}{a}\right)\right\}^2}=1.$$

35. If $(z+1)^2=4/Z$, the unit circle in the z-plane corresponds to the parabola $R\cos^2\frac{1}{2}\Theta=1$ in the Z-plane, and the inside of the circle to the outside of the parabola.

36. Show that, by means of the transformation $z=\{(Z-ic)/(Z+ic)\}^2$, the upper half of the z-plane may be made to correspond to the interior of a certain semicircle in the Z-plane.

37. Consider the relation $az^2+2hzZ+bZ^2+2gz+2fZ+c=0$.

Show that there are two values of Z for which the corresponding values of z are equal, and *vice versa*. We call these the *branch points* in the Z and z-planes respectively. Show that, if z describes an ellipse whose foci are the branch points, so does Z.

[We can, without loss of generality, take the given relation in the form

$$z^2+2zZ\cos\omega+Z^2=1$$

—the reader should satisfy himself that this is the case. The branch points in either plane are $\pm\operatorname{cosec}\omega$. An ellipse of the form specified is given by

$$|z+\operatorname{cosec}\omega|+|z-\operatorname{cosec}\omega|=\text{const.}$$

This is equivalent (Ex. 8) to

$$|z+\sqrt{(z^2-\operatorname{cosec}^2\omega)}|+|z-\sqrt{(z^2-\operatorname{cosec}^2\omega)}|=\text{const.}$$

Express this in terms of Z.]

38. If $z=aZ^m+bZ^n$, where m, n are positive integers and a, b real, show that as Z describes the unit circle, z describes a hypo- or epi-cycloid.

39. Prove that

$$\frac{\sin(2n+1)\theta}{(2n+1)\sin\theta}=\prod_{r=1}^{n}\left(1-\frac{\sin^2\theta}{\sin^2\{r\pi/(2n+1)\}}\right). \quad (\textit{Math. Trip.}\ 1907.)$$

40. By putting $\theta=\frac{1}{2}\pi$ in the last example, prove that

$$\cot\frac{\pi}{2n+1}\cot\frac{2\pi}{2n+1}\dots\cot\frac{n\pi}{2n+1}=\frac{1}{\sqrt{(2n+1)}}.$$

41. Prove that

$$\sin n\theta = 2^{n-1} \sin\theta \sin\left(\theta + \frac{\pi}{n}\right) \dots \sin\left\{\theta + \frac{(n-1)\pi}{n}\right\}.$$

[Put $x=a=1$ in Ex. XXV. 8, and change θ into 2θ.]

42. Prove that $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\dots\cos\frac{7\pi}{15} = \frac{1}{128}$.

43. Prove that $\tan\frac{\pi}{2n}\tan\frac{2\pi}{2n}\dots\tan\frac{(n-1)\pi}{2n} = 1$.

44. Prove that

$$\frac{(1+x)^n-(1-x)^n}{2x} = A\left(x^2+\tan^2\frac{\pi}{n}\right)\left(x^2+\tan^2\frac{2\pi}{n}\right)\dots\left(x^2+\tan^2\frac{r\pi}{n}\right),$$

where $A=1$, $r=\frac{1}{2}(n-1)$ if n is odd, and $A=n$, $r=\frac{1}{2}n-1$ if n is even.

45. If $1\Big/\prod\limits_{r=1}^{n}\left(x+\tan^2\frac{r\pi}{2n+1}\right)$ is expressed in the form

$$\sum_{r=1}^{n} A_r\Big/\left(x+\tan^2\frac{r\pi}{2n+1}\right),$$

n being a positive integer, show that

$$A_r = \frac{(-1)^{r-1}2}{2n+1}\sin^2\frac{r\pi}{2n+1}\cos^{2n-3}\frac{r\pi}{2n+1}.$$

(*Math. Trip.* 1905.)

[Apply the ordinary rule for partial fractions: it will be found that

$$A_r = (-1)^{r-1}2\sin^2\frac{r\pi}{2n+1}\cos^{2n-3}\frac{r\pi}{2n+1}\prod_{k=1}^{n}\cot^2\frac{k\pi}{2n+1},$$

and Ex. 40 can be used to obtain the given result.]

46. Show that

$$\sum_{r=0}^{n-1}\sin\frac{(2r+1)\pi}{2n}\operatorname{cosec}\left\{\frac{(2r+1)\pi}{2n}-a\right\} = n\cos(n-1)a\sec na.$$

(*Math. Trip.* 1907.)

[The right-hand side is

$$n\frac{x^{n-1}+x^{-(n-1)}}{x^n+x^{-n}} = n\frac{x^{2n-1}+x}{x^{2n}+1},$$

where $x=\cos a+i\sin a=\operatorname{Cis} a$. The roots of $x^{2n}+1=0$ are

$$\operatorname{Cis}\frac{(2r+1)\pi}{2n}\quad(r=0,\ 1,\ \dots,\ 2n-1).$$

Split up the right-hand side into partial fractions of the form

$$A_r\Big/\left\{x-\operatorname{Cis}\frac{(2r+1)\pi}{2n}\right\}.$$

It will be found that $A_r = -i\sin\frac{(2r+1)\pi}{2n}\operatorname{Cis}\frac{(2r+1)\pi}{2n}$. To get the result in the form given we must associate the terms in pairs $(r,\ n+r)$ where $r=0,\ 1,\ \dots,\ n-1$.]

47. Show that, if m and n are positive integers, and $m \leqq n$, then $x^{m-1}/(1+x^n) = (1/n)\,\Sigma\,\omega^{m-n}/(x-\omega)$, where ω is a root of $x^n+1=0$: and hence show that, if n is even,

$$\frac{1}{1+x^n} = \frac{2}{n}\sum_{0}^{\frac{1}{2}n-1}\left\{1-x\cos\frac{(2r+1)\,\pi}{n}\right\}\Big/\left\{1-2x\cos\frac{(2r+1)\,\pi}{n}+x^2\right\},$$

and find the corresponding formula when n is odd.

48. Express $x^{m-1}/(1-x^n)$, where m and n are positive integers, and $m \leqq n$, in partial fractions, and obtain the formulae for $1/(1-x^n)$ corresponding to those of Ex. 47.

49. Show that

$$\frac{x^n - a^n\cos n\theta}{x^{2n}-2x^n a^n\cos n\theta + a^{2n}} = \frac{1}{nx^{n-1}}\sum_{r=0}^{n-1}\frac{x-a\cos\left(\theta+\dfrac{2r\pi}{n}\right)}{x^2-2xa\cos\left(\theta+\dfrac{2r\pi}{n}\right)+a^2}.$$

50. If $\rho_1, \rho_2, \ldots \rho_n$ are the distances of a point P, in the plane of a regular polygon, from the vertices, prove that

$$\sum_1^n \frac{1}{\rho_\nu^{\,2}} = \frac{n}{r^2-a^2}\,\frac{r^{2n}-a^{2n}}{r^{2n}-2r^n a^n\cos n\theta + a^{2n}},$$

where O is the centre and a the radius of the circumcircle of the polygon, r the length OP, and θ the angle between OP and the radius from O to any vertex of the polygon.

51. If $A_1A_2\ldots A_n$, $B_1B_2\ldots B_m$ are concentric regular polygons, m and n being prime to one another, prove that

$$\sum_{r=1}^{n}\sum_{s=1}^{m}\frac{1}{(A_rB_s)^2} = \frac{mn}{b^2-a^2}\,\frac{b^{2mn}-a^{2mn}}{b^{2mn}-2b^{mn}a^{mn}\cos^{mn}\theta + a^{2mn}},$$

where a and b are the radii of the circumcircles of the polygons, and θ the angle between any two radii drawn one to a vertex of each polygon.

(*Math. Trip.* 1903.)

52. If p and q are integers, and q prime to p, and k is an odd positive integer less than $2p$, and $\theta = q\pi/p$, show that

$$\sum_{n=0}^{p-1}\frac{\cos k\,(a+n\theta)}{\sin\,(a+n\theta)} = p\cot pa, \qquad \sum_{n=0}^{p-1}\frac{\sin k\,(a+n\theta)}{\sin\,(a+n\theta)} = p.$$

[We have
$$\frac{px^{\lambda-1}}{x^p-1} - \frac{1}{x-1} = \sum_1^{p-1}\frac{t^{n\lambda}}{x-t^n},$$
where
$$t = \cos 2\theta + i\sin 2\theta, \quad 1 \leqq \lambda \leqq p.$$
In this equation write $\frac{1}{2}(k+1)$ for λ and $\cos 2a - i\sin 2a$ for x.]

CHAPTER IV.

LIMITS OF FUNCTIONS OF A POSITIVE INTEGRAL VARIABLE.

43. Functions of a positive integral variable. In Chapter II. we discussed the notion of a function of a real variable x, and illustrated the discussion by a large number of examples of such functions. And the reader will remember that there was one important particular with regard to which the functions which we took as illustrations differed very widely. Some were defined for *all* values of x, some for *rational* values only, some for *integral* values only, and so on.

Consider, for example, the following functions: (i) $y=x$, (ii) $y=\sqrt{x}$, (iii) $y=$ the denominator of x, (iv) $y=$ the square root of the product of the numerator and the denominator of x, (v) $y=$ the largest prime factor of x, (vi) $y=$ the product of $\sqrt{x}$ and the largest prime factor of x, (vii) $y=$ the xth prime number, (viii) $y=$ the height measured in inches of convict x in Dartmoor prison.

Then the aggregates of values of x for which these functions are defined or, as we may say, the *fields of definition* of the functions, consist of (i) *all* values of x, (ii) *all positive* values of x, (iii) *all rational* values of x, (iv) *all positive rational* values of x, (v) *all integral* values of x, (vi), (vii) *all positive integral* values of x, (viii) a certain number of positive integral values of x, viz., $1, 2, \dots, N$, where N is the total number of convicts at Dartmoor at the present moment of time[*].

Now let us consider a function, such as (vii) above, which is defined for all positive integral values of x and no others. This function may be regarded from two slightly different points of view. We may consider it, as has so far been our custom, as a function of the real variable x defined for some only of the values of x, viz. positive integral values, and say that for all other values of x the definition fails. Or we may, as in Chap. I. § 8, leave values of x other than positive integral values entirely out of account, and regard our function as a function of the *positive integral variable* n, whose values are the positive integers

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[*] In the last case N depends on the time, and convict x, where x has a definite value, is a different individual at different moments of time. Thus if we take different moments of time into consideration we have a simple example of a function $y=F(x, t)$ of *two* variables, defined for a certain range of values of t, viz. from the time of the establishment of Dartmoor prison to the time of its abandonment, and for a certain number of positive integral values of x, this number varying with t.

$$1, 2, 3, 4, \ldots.$$

In this case we may write

$$y = \phi(n)$$

and regard y now as a function of n defined for *all* values of n.

It is obvious that any function of x defined for all values of x gives rise to a function of n defined for all values of n. Thus from the function $y = x^2$ we deduce the function $y = n^2$ by merely omitting from consideration all values of x other than positive integers, and the corresponding values of y. On the other hand from any function of n we can deduce any number of functions of x by merely assigning values to y, corresponding to values of x other than positive integral values, in any way we please.

44. Interpolation. The problem of determining a function of x which shall assume, for all positive integral values of x, values agreeing with those of a given function of n, is of extreme importance in higher mathematics. It is called the *problem of functional interpolation.*

Were the problem however merely that of finding *some* function of x to fulfil the condition stated it would of course present no difficulty whatever. We could, as explained above, simply fill in the missing values as we pleased: we might indeed simply regard the given values of the function of n as *all* the values of the function of x and say that the definition of the latter function failed for all other values of x. But such purely theoretical solutions are obviously not what is usually wanted. What is usually wanted is some *formula* involving x (of as simple a kind as possible) which assumes the given values for $x = 1, 2, \ldots$.

In some cases, especially when the function of n is itself defined by a formula, there is an obvious solution. If for example $y = \phi(n)$, where $\phi(n)$ is a function of n which would have a meaning even were n not a positive integer (e.g. n, n^2, $(n-1)/(n+1)$), we naturally take our function of x to be $y = \phi(x)$. But even in this very simple case it is easy to write down other almost equally obvious solutions of the problem. For example

$$y = \phi(x) + \sin x\pi,$$

assumes the value $\phi(n)$ for $x = n$, since $\sin n\pi = 0$.

In other cases $\phi(n)$ may be defined by a formula, such as $(-1)^n$, which ceases to define for some values of x at any rate (as here in the case of fractional values of x with even denominators, or irrational values). But it may be possible to transform the formula in such a way that it does define for all values of x. In this case, for example,

$$(-1)^n = \cos n\pi,$$

if n is an integer, and the problem of interpolation is solved by the function $\cos x\pi$.

In other cases $\phi(x)$ may be defined for some values of x other than positive integers, but not for all. Thus from $y=n^n$ we are led to $y=x^x$. This expression has a meaning for some only of the remaining values of x. If for simplicity we confine ourselves to *positive* values of x, x^x has a meaning for all *rational* values of x, since

$$(p/q)^{p/q} = \sqrt[q]{(p/q)^p},$$

according to the definition of fractional indices adopted in elementary algebra. But when x is *irrational* x^x has (so far as we are in a position to say at the present moment) no meaning at all. Thus in this case the problem of interpolation at once leads us to consider the question of extending our definitions in such a way that x^x shall have a meaning even when x is irrational. We shall see later on how the desired extension may be effected.

Again consider the case in which

$$y = 1 \,.\, 2 \ldots n = n!$$

In this case there is no obvious formula in x which reduces to $n!$ for $x=n$, as $x!$ means nothing for values of x other than the positive integers. This is a case in which attempts to solve the problem of interpolation have led to important advances in mathematics. For mathematicians have succeeded in discovering a function (the Gamma-function) which possesses the desired property and many other interesting and important properties besides.

45. Finite and infinite classes. Before we proceed further it is necessary to make a few remarks about certain ideas of an abstract and logical nature which are of constant occurrence in Pure Mathematics.

In the first place, the reader is probably familiar with the notion of **a class**. It is unnecessary to discuss here any logical difficulties which may be involved in the notion of a 'class': roughly speaking we may say that a class is the aggregate or collection of all the entities or objects which possess a certain property, simple or complex. Thus we have the class of British subjects, or red-headed Germans, or positive integers, or real numbers.

Moreover, the reader has probably an idea of what is meant by a **finite** or **infinite** class. Thus the class of *British subjects* is a finite class: the aggregate of all British subjects, past, present, and future, has a certain definite number n, though of course we cannot tell at present the actual value of n. The class of *present British subjects*, on the other hand, has a number n which could be ascertained by counting, were the methods of the census effective enough.

On the other hand the class of positive integers is *not* finite but infinite. This may be expressed more precisely as follows. If n is any positive integer (e.g. 1000, 1,000,000 or any number we like to think of), there are more than n positive integers. Thus if the number we think of is 1,000,000, there are obviously at least 1,000,001 positive integers. Similarly the class of real numbers, or of points on a line, is infinite. It is convenient to express this by saying that there are **an infinite number** of positive integers, or real numbers, or points on a line. But the reader must be careful always to remember that by saying this we mean **simply** that the class in question is not a class with a definite number of members, such as 1000 or 1,000,000.

46. Properties possessed by a function of n for large values of n. We may now return to the 'functions of n' which we were discussing in §§ 43, 44. They have many points of difference from the functions of x which we discussed in Chap. II. But there is one fundamental characteristic which the two classes of functions have in common—*the values of the variable for which they are defined form an infinite class.* It is this fact which forms the basis of all the considerations which follow and which, as we shall see in the next chapter, apply, *mutatis mutandis*, to functions of x as well.

Suppose that $\phi(n)$ is any function of n, and that P is any property which $\phi(n)$ may or may not have, such as that of being a positive integer or of being greater than 1. Consider, for each of the values $n=1, 2, 3, \ldots$, whether $\phi(n)$ has the property P or not. Then there are three possibilities—

(a) $\phi(n)$ may have the property P for *all* values of n, or for all values of n except a definite number N of such values:

(*b*) $\phi(n)$ may have the property for *no* values of n, or *only* for a definite number N of such values:

(*c*) neither (*a*) nor (*b*) may be true.

If (*b*) is true the values of n for which $\phi(n)$ has the property form a finite class. If (*a*) is true the values of n for which $\phi(n)$ has *not* the property form a finite class. In the third case neither class is finite. Let us consider some particular cases.

(1) Let $\phi(n)=n$, and let P be the property of being a positive integer. Then $\phi(n)$ has the property P for *all* values of n.

If on the other hand P denotes the property of being a positive integer greater than or equal to 1000, $\phi(n)$ has the property for all values of n except a definite number of values of n, viz. 1, 2, 3, …, 999. In either of these cases (*a*) is true.

(2) If $\phi(n)=n$, and P is the property of being less than 1000, (*b*) is true.

(3) If $\phi(n)=n$, and P is the property of being *odd*, (*c*) is true. For $\phi(n)$ is odd if n is odd and even if n is even, and either the odd or the even values of n form an infinite class.

Example. Consider, in each of the following cases, whether (*a*), (*b*), or (*c*) is true:—

(i) $\phi(n)=n$, P being the property of being a perfect square,

(ii) $\phi(n)=$ the nth prime number, P being the property of being odd,

(iii) $\phi(n)=$ the nth prime number, P being the property of being even,

(iv) $\phi(n)=$ the nth prime number, P being the property $\phi(n)>n$,

(v) $\phi(n)=1-(-1)^n(1/n)$, P being the property $\phi(n)<1$,

(vi) $\phi(n)=1-(-1)^n(1/n)$, P being the property $\phi(n)<2$,

(vii) $\phi(n)=1000\{1+(-1)^n\}/n$, P being the property $\phi(n)<1$,

(viii) $\phi(n)=1/n$, P being the property $\phi(n)<\cdot 001$,

(ix) $\phi(n)=(-1)^n/n$, P being the property $|\phi(n)|<\cdot 001$,

(x) $\phi(n)=10000/n$, or $(-1)^n 10000/n$, P being either of the properties $\phi(n)<\cdot 001$ or $|\phi(n)|<\cdot 001$,

(xi) $\phi(n)=(n-1)/(n+1)$, P being the property $1-\phi(n)<\cdot 0001$.

47. Let us now suppose that $\phi(n)$ and P are such that the assertion (*a*) is true, i.e. that $\phi(n)$ has the property P, if not for all values of n, at any rate for all values of n except a definite number N of such values. We may denote these exceptional values by

$$n_1,\ n_2,\ \ldots,\ n_N.$$

There is of course no reason why these N values should be the *first* N values 1, 2, ..., N, though, as the preceding examples show, this is frequently the case in practice.

But whether this is so or not we know that $\phi(n)$ has the property P if $n > n_N$. Thus the nth prime is odd if $n > 2$ ($n = 2$ being the only exception to the statement), and $1/n < \cdot 001$ if $n > 1000$ (the first 1000 values of n being the exceptions), and

$$1000\{1 + (-1)^n\}/n < 1,$$

if $n > 2000$, the exceptional values being 2, 4, 6, ..., 2000. That is to say, in each of these cases the property is possessed *for all values of n from a definite value onwards.*

We shall frequently express this by saying that $\phi(n)$ has the property for **large**, or *very large*, or *all sufficiently large* values of n. Thus when we say that $\phi(n)$ *has the property* P (which will as a rule be a property expressed by some relation of inequality) *for large values of* n, what we mean is that we can determine some definite number, n_0 say, such that $\phi(n)$ has the property for all values of n greater than or equal to n_0. This number n_0, in the examples considered above, may be taken to be any number greater than n_N, the greatest of the exceptional numbers: it is most natural to take it to be $n_N + 1$.

Thus we may say that 'all large primes are odd,' or that '$1/n$ is less than ·001 for large values of n.' And the reader must make himself familiar with the use of the word *large* in statements of this kind. *Large* is in fact a word which, standing by itself, has no more absolute meaning in mathematics than in the language of common sense. It is a truism that in common life a number which is large in one connection is small in another; 6 goals is a large score in a football match, but 6 runs is not a large score in a cricket match; and 300 runs is a large score, but £300 is not a large income—and so of course in mathematics *large* generally means *large enough*, and what is large enough for one purpose may not be large enough for another.

We know now what is meant by the assertion '$\phi(n)$ has the property P for large values of n.' It is with assertions of this kind that we shall be concerned throughout this chapter. Given a function $\phi(n)$, are there any properties of which such an assertion is true?

48. The phrase 'n tends to infinity.' There is a somewhat different way of looking at the matter which it is natural to adopt. Suppose that n assumes successively the values 1, 2, 3, ... The word 'successively' naturally suggests succession *in time*, and we may suppose n, if we like, to assume these values at successive moments of time (e.g. at the beginnings of successive seconds). Then as the seconds pass n gets larger and larger and there is no limit to the extent of its increase. However large a number we may think of (e.g. 969372855) a time will come when n has become larger than this number.

It is convenient to have a short phrase to express this unending growth of n, and we shall say that n **tends to infinity**, or $n \to \infty$, this last symbol being usually employed as an abbreviation for 'infinity.' The phrase 'tends to' like the word 'successively' naturally suggests the idea of change *in time*, and it is convenient to think of the variation of n as accomplished in time in the manner described above. This however is a mere matter of convenience. The variable n is a purely logical entity which has in itself nothing to do with time.

The reader cannot too strongly impress upon himself that when we say that n 'tends to ∞' we mean *simply* that n is supposed to assume a series of values which increase continually and without limit. **There is no number 'infinity':** such an equation as

$$n = \infty$$

is as it stands *absolutely meaningless*: n cannot be equal to ∞, because 'equal to ∞' means nothing. So far in fact the symbol ∞ means nothing at all except in the one phrase 'tends to ∞,' the meaning of which we have explained above. Later on we shall learn how to attach a meaning to other phrases involving the symbol ∞, but the reader will always have to bear in mind

(1) that ∞ *by itself* means nothing, although *phrases containing it* sometimes mean something,

(2) that in every case in which a phrase containing the symbol ∞ means something it will do so simply because we have previously attached a meaning to it by means of a special definition.

Now it is clear that if $\phi(n)$ has the property P for large values of n, and if n 'tends to ∞,' in the sense which we have just explained, n will ultimately assume values large enough to ensure that $\phi(n)$ has the property P. And so another way of putting the question 'what properties has $\phi(n)$ for sufficiently large values of n?' is 'how does $\phi(n)$ behave as n tends to ∞?'

49. The behaviour of a function of n as n tends to infinity. We shall now proceed, in the light of the remarks made in the preceding sections, to consider the meaning of some kinds of statements which are perpetually occurring in higher mathematics. Let us consider for example, the two following statements—(*a*) *$1/n$ is small for large values of n*, (*b*) *$1-(1/n)$ is nearly equal to* 1 *for large values of n*,—neither of which, we imagine, anyone will be inclined to dispute. Yet, obvious as they may seem, there is a good deal in them which will repay the reader's attention. Let us take (*a*) first, as being slightly the simpler.

We have already considered the statement '*$1/n$ is less than* ·01 *for large values of n*.' This, we saw, means that the inequality $1/n < {\cdot}01$ is true for all values of n greater than some definite value, in fact greater than 100. Similarly it is true that '*$1/n$ is less than* ·0001 *for large values of n*': in fact $1/n < {\cdot}0001$ if $n > 10000$. And instead of ·01 or ·0001 we might take ·000001 or ·00000001, or indeed any positive number we like.

It is obviously convenient to have some way of expressing the fact that *any* such statement as '*$1/n$ is less than* ·01 *for large values of n*' is true, when we substitute for ·01 some smaller number, such as ·0001 or ·000001 or any other number we care to choose. And clearly we can do this by saying that '*however small δ may be* (provided of course it is positive) *$1/n < \delta$ for sufficiently large values of n*.' That this is true is obvious. For $1/n < \delta$ if $n > 1/\delta$; so that our 'sufficiently large' values of n need only all be greater than $1/\delta$. The assertion is however a complex one, in that it really stands for the whole class of assertions which we obtain by giving to δ special values such as ·01. And of course the smaller is δ and the larger $1/\delta$ the larger must the least of the 'sufficiently large' values of n be, values which are sufficiently large when δ has one value being inadequate when it has another.

The last statement italicised is what is really meant by the statement (a), that $1/n$ is small when n is large. Similarly for (b), which really means "*if* $\phi(n) = 1 - (1/n)$, *then the statement* '$1 - \phi(n) < \delta$ *for sufficiently large values of* n' *is true whatever positive value* (*such as* ·01 *or* ·0001) *we attribute to* δ." That the statement (b) is true is obvious from the fact that $1 - \phi(n) = 1/n$.

There is another way in which it is common to state the facts expressed by the assertions (a) and (b). This is suggested at once by § 48. Instead of saying '$1/n$ is small for large values of n' we say '$1/n$ tends to 0 as n tends to ∞.' Similarly we say that '$1-(1/n)$ tends to 1 as n tends to ∞': and these statements are to be regarded as precisely equivalent to (a) and (b). Thus the statements

'$1/n$ is small when n is large,'

'$1/n$ tends to 0 as n tends to ∞,'

are equivalent to one another and to the more formal statement

> 'if δ is any positive number, however small, $1/n < \delta$ for sufficiently large values of n,'

or to the still more formal statement

> 'if δ is any positive number, however small, we can find a number n_0 such that $1/n < \delta$ for all values of n greater than or equal to n_0.'

The reader should imagine himself confronted by an opponent who questions the truth of the statement. He would name a series of numbers growing smaller and smaller. He might begin with ·001. The reader would reply that $1/n < ·001$ as soon as $n > 1000$. The opponent would be bound to admit this, but would try again with some smaller number, such as ·0000001. The reader would reply that $1/n < ·0000001$ as soon as $n > 10000000$: and so on. In this simple case it is evident that the reader would always have the better of the argument.

We shall now introduce yet another way of expressing this property of the function $1/n$. We shall say that '*the* **limit** *of* $1/n$ *as* n *tends to* ∞ *is* 0,' a statement which we may express symbolically in the form

$$\lim_{n \to \infty} (1/n) = 0,$$

or simply $\lim (1/n) = 0$. We shall also sometimes use the notation

$$\underset{(n \to \infty)}{1/n \to 0},$$

or simply $1/n \rightarrow 0$, which may be read '$1/n$ tends to 0 as n tends to ∞.' In the same way we shall write

$$\lim_{(n\rightarrow\infty)} \{1-(1/n)\} = 1, \quad \lim \{1-(1/n)\} = 1, \quad 1-(1/n) \underset{(n\rightarrow\infty)}{\rightarrow} 1,$$

or $1-(1/n) \rightarrow 1$.

50. Now let us consider a different example: let $\phi(n) = n^2$. Then '*n^2 is large when n is large.*' This statement is equivalent to the more formal statements

> 'if G is any positive number, however large, $n^2 > G$ for sufficiently large values of n,'
>
> 'we can find a number n_0 such that $n^2 > G$ for all values of n greater than or equal to n_0.'

And it is natural in this case to say that 'n^2 tends to ∞ as n tends to ∞,' or 'n^2 tends to ∞ with n' and to write

$$n^2 \underset{(n\rightarrow\infty)}{\rightarrow} \infty,$$

or simply $n^2 \rightarrow \infty$.

Finally consider the function $\phi(n) = -n^2$. In this case $\phi(n)$ is large, but negative, when n is large, and we naturally say that '$-n^2$ tends to $-\infty$ as n tends to ∞' and write

$$-n^2 \rightarrow -\infty.$$

And the use of the symbol $-\infty$ in this sense suggests that it will sometimes be convenient to write $n^2 \rightarrow +\infty$ for $n^2 \rightarrow \infty$ and generally to use $+\infty$ instead of ∞, in order to secure greater uniformity of notation.

But we must once more repeat that in all these statements the symbols ∞, $+\infty$, $-\infty$ mean nothing whatever in themselves, and only acquire a meaning when they occur in certain special connections in virtue of the explanations which we have just given.

51. Definition of a limit. After the discussion which precedes the reader should be in a position to appreciate the general notion of a *limit*. Roughly we may say that $\phi(n)$ ***tends to a limit l as n tends to ∞ if $\phi(n)$ is nearly equal to l when n is large.*** But although the meaning of this statement should be clear enough after the preceding explanations, it is not, as it stands, precise enough to serve as a strict mathematical definition.

It is, in fact, equivalent to a whole class of statements of the type '*for sufficiently large values of* n, $\phi(n)$ *differs from* l *by less than* δ.' This statement has to be true for $\delta = \cdot 01$ or $\cdot 0001$ or *any* positive number; and for any such value of δ it has to be true for *any* value of n after a certain definite value n_0, though, the smaller δ, the larger (as a rule) will be this value n_0.

We accordingly frame the following formal definition:

DEFINITION I. *The function* $\phi(n)$ *is said to tend to the limit* l *as* n *tends to* ∞, *if, however small be the positive number* δ, $\phi(n)$ *differs from* l *by less than* δ *for sufficiently large values of* n; *or if, however small be the positive number* δ, *we can determine a value* n_0 *corresponding to* δ *and such that* $\phi(n)$ *differs from* l *by less than* δ *for all values of* n *greater than or equal to* n_0.

It is usual to denote the difference $\phi(n) \sim l$, taken positively, by $|\phi(n) - l|$. It is equal to $\phi(n) - l$ or to $l - \phi(n)$, whichever is positive, and agrees with the definition of the *modulus* of $\phi(n) - l$, as given in Ch. III, though at present we are only considering real values, positive or negative.

With this notation the definition may be stated more shortly as follows: '*if, given any positive number*, δ, *however small, we can find* n_0 *so that* $|\phi(n) - l| < \delta$ *for* $n \geqq n_0$, *then we say that* $\phi(n)$ *tends to the limit* l *as* n *tends to* ∞, *and write*

$$\lim_{n \to \infty} \phi(n) = l.\text{'}$$

Sometimes we may omit the '$n \to \infty$'; and sometimes it is convenient, for brevity, to write $\phi(n) \to l$.

It should be observed that n_0 is a function of δ. Thus if $\phi(x) = 1/n$, $l = 0$, and the condition reduces to $1/n < \delta$ $(n \geqq n_0)$, which is satisfied if $n_0 = 1 + [1/\delta]$ (the integer larger by one than the greatest integer contained in $1/\delta$). There is one and only one case in which *the same* n_0 will do for *all* values of δ. If, from a certain value N of n onwards $\phi(n)$ is constant, say equal to C, it is evident that $\phi(n) - C = 0$ for $n \geqq N$, so that the inequality $|\phi(n) - C| < \delta$ is satisfied for $n \geqq N$ and *all* positive values of δ. And if $|\phi(n) - l| < \delta$ for $n \geqq N$ and *all* positive values of δ it is evident that $\phi(n) = l$ for $n \geqq N$, so that $\phi(n)$ is constant for all such values of n.

52. The definition of a limit may be illustrated geometrically as follows. The graph of $\phi(n)$ consists of a number of points corresponding to the values $n = 1, 2, 3, \ldots$.

Draw the line $y = l$, and the parallel lines $y = l - \delta$, $y = l + \delta$ at distance δ from it. Then

$$\lim_{n \to \infty} \phi(n) = l,$$

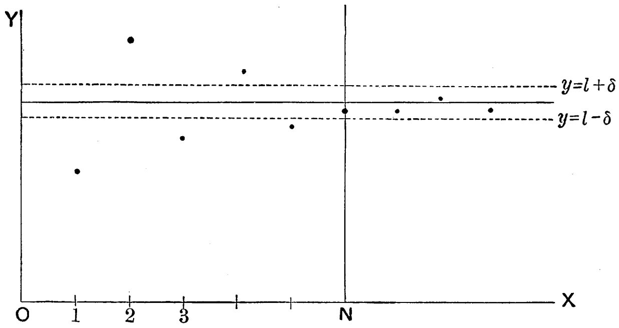

FIG. 35.

if, when once these lines have been drawn, no matter how close they may be together, we can always draw a line $x = N$ (as in the figure) in such a way that the point of the graph on this line, and all points to the right of it, lie between them. We shall find this geometrical way of looking at our definition particularly useful when we come to deal with functions defined for all values of a real variable and not merely for positive integral values.

53. So much for functions of n which tend to a limit as n tends to ∞. We must now frame corresponding definitions for functions which, like the functions n^2 or $-n^2$, tend to positive or negative infinity. The reader should by now find no difficulty in appreciating the point of

DEFINITION II. *The function* $\phi(n)$ *is said to tend to* $+\infty$ *(positive infinity) with* n, *if, when any number* G, *however large, is assigned, we can determine* n_0 *so that* $\phi(n) > G$ *for* $n \geqq n_0$; *or if, however large* G *may be,* $\phi(n) > G$ *for sufficiently large values of* n.

Another, less precise, form of statement is '*if we can make* $\phi(n)$ *as large as we please by sufficiently increasing* n.' This is open to the objection that it obscures a fundamental point, viz. that $\phi(n)$ must be greater than G for *all* values of $n \geqq n_0$, and not merely for *some* such values. But there is no harm in using this form of expression if we are clear what it means.

When $\phi(n)$ tends to $+\infty$ we write

$$\underset{(n \to \infty)}{\phi(n) \to +\infty}.$$

We may leave it to the reader to frame the corresponding definition for functions which tend to negative infinity.

54. Some points concerning the definitions. The reader should carefully observe the following points.

(1) We may obviously alter the values of $\phi(n)$ for any finite number of values of n, in any way we please, without in the least affecting the behaviour of $\phi(n)$ as n tends to ∞. For example $1/n$ tends to 0 as n tends to ∞. We may deduce any number of new functions from $1/n$ by altering a finite number of its values. For instance we may consider the function $\phi(n)$ which is equal to 3 for $n = 1, 2, 7, 9, 106, 107, 108, 237$ and equal to $1/n$ for all other values of n. For this function, just as for the original function $1/n$, $\lim \phi(n) = 0$. Similarly, for the function $\phi(n)$ which is equal to 3 if $n = 1, 2, 7, 9, 106, 107, 108, 237$, and to n^2 otherwise, it is true that $\phi(n) \to +\infty$.

(2) On the other hand we cannot as a rule alter an *infinite* number of the values of $\phi(n)$ without fundamentally affecting its behaviour as n tends to ∞. If for example we altered the function $1/n$ by changing its value to 1 whenever n is a multiple of 100 it would no longer be true that $\lim \phi(n) = 0$. So long as a finite number of values only were affected we could always choose the number n_0 of the definition so as to be greater than the greatest of the values of n for which $\phi(n)$ was altered. In the examples above, for instance, we could always take $n_0 > 237$, and indeed we should be compelled to do so as soon as our imaginary opponent of § 49 had assigned a value of δ as small as 3 (in the first example) or a value of G as great as 3 (in the second). But now *however* large n_0 may be there will be greater values of n for which $\phi(n)$ has been altered.

(3) In applying the test of Definition I. it is of course absolutely essential that we should have $|\phi(n) - l| < \delta$ not merely for $n = n_0$ but for $n \geqq n_0$, i.e. *for* n_0 *and for all larger values of* n. In the last example, given δ we can obviously choose n_0 so that $\phi(n) < \delta$ for $n = n_0$: we have only to choose a sufficiently large value of n which is not a multiple of 100. But when n_0 is thus

chosen it is not true that $\phi(n) < \delta$ for $n \geqq n_0$: all the multiples of 100 which are greater than n_0 are exceptions to this statement.

(4) If $\phi(n)$ is always greater than l we can replace $|\phi(n) - l|$ by $\phi(n) - l$. Thus the test whether $1/n$ tends to the limit 0 as n tends to ∞ is simply whether $1/n < \delta$ for $n \geqq n_0$. If however $\phi(n) = (-1)^n/n$, l is again 0, but $\phi(n) - l$ is sometimes positive and sometimes negative. In such a case we must state the condition in the form $|\phi(n) - l| < \delta$, in this particular case in the form $|\phi(n)| < \delta$.

(5) *The limit l may itself be one of the actual values of* $\phi(n)$. Thus if $\phi(n) = 0$ for all values of n it is obvious that $\lim \phi(n) = 0$. Again if we had (in (2) and (3) above) altered the value of the function (when n is a multiple of 100) to 0 instead of to 1 we should have obtained a function $\phi(n) = 0$ (n a multiple of 100), $\phi(n) = 1/n$ (otherwise). The limit of this function as n tends to ∞ is still obviously 0. This limit is itself the value of the function for an infinite number of values of n, viz. all multiples of 100.

On the other hand *the limit itself need not (and in general will not) be the value of the function for any value of n*. This is sufficiently obvious in the case of $\phi(n) = 1/n$. The limit is 0; but the function is never zero for any value of n.

The reader cannot impress these facts too strongly on his mind. **A limit is not a value of the function:** it is something quite distinct from these values, though defined by its relations to them. The limits may possibly be equal to some of the values of the function—whether this be so or not has absolutely nothing to do with the notion of the limit: it is, so to say, a mere accident.

For the functions $\phi(n) = 0, \ 1,$
the limit is equal to *all* the values of $\phi(n)$: for

$$\phi(n) = 1/n, \quad (-1)^n/n, \quad 1 + (1/n), \quad 1 + \{(-1)^n/n\}$$

it is not equal to *any* value of $\phi(n)$: for

$$\phi(n) = (\sin \tfrac{1}{2}n\pi)/n, \quad 1 + \{(\sin \tfrac{1}{2}n\pi)/n\}$$

(whose limits as n tends to ∞ are easily seen to be 0 and 1, since $\sin \frac{1}{2}n\pi$ is never numerically greater than 1) the limit is equal to the value which $\phi(n)$ assumes for all even values of n, but the values assumed for odd values of n are all different from the limit and from one another.

(6) A function may be always numerically very large when n is very large without tending either to $+\infty$ or to $-\infty$. A sufficient illustration of this is given by $\phi(n) = (-1)^n n$ or $(-1)^n n^2$. A function can only tend to $+\infty$ or to $-\infty$ if, after a certain value of n, it maintains a constant sign.

Examples XXVI. Consider the behaviour of the following functions of x as n tends to ∞:

1. $\phi(n) = n^k$, where k is a positive or negative integer or rational fraction. If k is positive n^k tends to $+\infty$ with n. If k is negative $\lim n^k = 0$. If $k=0$, $n^k = 1$ for all values of n (by the definition of n^0). Hence $\lim n^k = 1$.

The reader will find it instructive, even in an almost obvious case like this, to write down a formal proof that the conditions of our definitions are satisfied. Take for instance the case of $k > 0$. Let G be any assigned number, however large. We wish to choose n_0 so that $n^k > G$ for $n \geqq n_0$. We have in fact only to take for n_0 any number greater than $\sqrt[k]{G}$. If e.g. $k=4$, $n^4 > 10000$ for $n \geqq 11$, $n^4 > 100000000$ for $n \geqq 101$, and so on.

From a geometrical point of view the matter may be stated as follows. If $k > 0$ the graph of $y = x^k$ is of the general form of A in Fig. 36; if $k < 0$ of the form of B; if $k=0$ it is a line C parallel to the axis of x. At present we are only concerned with the series of points marked on these curves.

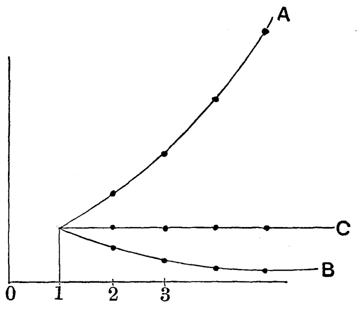

Fig. 36.

2. $\phi(n) =$ the nth prime number. If there were only a finite number of primes $\phi(n)$ would be defined only for a finite number of values of n. There are however, as was first shown by Euclid, infinitely many primes. Euclid's proof is as follows. If there are only a finite number of primes let them be $1, 2, 3, 5, 7, 11, \ldots N$. Consider the number $1 + (1 . 2 . 3 . 5 . 7 . 11 \ldots N)$. This number is evidently not divisible by any of $2, 3, 5, \ldots N$, since the remainder when it is divided by any of these numbers is 1. It is therefore

not divisible by any prime save 1, and is therefore itself prime, which is contrary to our hypothesis.

Again it is obvious that for all values of n (save $n=1, 2, 3$) $\phi(n) > n$. Hence $\phi(n) \to +\infty$.

3. $\phi(n)=$ the number of primes less than n. Here again $\phi(n) \to +\infty$.

4. $\phi(n)=[an]$, where a is any positive number. Here

$$\phi(n)=0 \quad (0 \leqq n < 1/a), \quad \phi(n)=1 \quad (1/a \leqq n < 2/a),$$

and so on; and $\phi(n) \to +\infty$.

5. If $\phi(n)=1000000/n$, $\lim \phi(n)=0$: if $\psi(n)=n/1000000$, $\psi(n) \to +\infty$. These conclusions are in no way affected by the fact that at first $\phi(n)$ is much larger than $\psi(n)$ (being, in fact, larger until $n=1000$).

6. $\phi(n)=1/\{n-(-1)^n\}$, $n-(-1)^n$, $n\{1-(-1)^n\}$. The first function tends to 0, the second to $+\infty$, the third does not tend either to a limit or to $+\infty$.

7. $\phi(n)=(\sin n\theta)/n$, where θ is any real number. Since $|\sin n\theta| \leqq 1$, $|\phi(n)| < 1/n$, and $\lim \phi(n)=0$.

8. $\phi(n)=(\sin n\theta)/\sqrt{n}$, $(\sin n\theta)/n^2$, $(a \cos^2 n\theta + b \sin^2 n\theta)/n$, where a and b are any real numbers.

9. $\phi(n)=\sin n\theta\pi$. If θ is integral $\phi(n)=0$ for all values of n, and therefore $\lim \phi(n)=0$.

Next let θ be rational, e.g. $\theta=p/q$, where p and q are positive integers. Let $n=aq+b$ where a is the quotient and b the remainder when n is divided by q. Then $\sin(np\pi/q)=(-1)^{ap}\sin(bp\pi/q)$. Suppose, for example, p even; then as n increases from 0 to $q-1$, $\phi(n)$ takes the values

$$0, \quad \sin(p\pi/q), \quad \sin(2p\pi/q), \ \ldots \ \sin\{(q-1)p\pi/q\}.$$

When n increases from q to $2q-1$ these values are repeated; and so also as n goes from $2q$ to $3q-1$, $3q$ to $4q-1$, and so on. Thus the values of $\phi(n)$ form *a perpetual cyclic repetition of a finite series of different values.* It is evident that when this is the case $\phi(n)$ cannot tend to a limit or to $+\infty$ or to $-\infty$ as n tends to infinity.

The case in which θ is irrational is a little more difficult. It is discussed in the next set of examples.

55. Oscillating Functions.

DEFINITION. *When $\phi(n)$ does not tend to a limit, nor to $+\infty$, nor to $-\infty$, as n tends to ∞, we say that $\phi(n)$* **oscillates** *as n tends to ∞.*

A function $\phi(n)$ certainly oscillates if its values form, as in the case considered in the last example above, a continual repetition of a cycle of values. But of course it may oscillate without possessing this peculiarity. Oscillation is, according to its definition, a **purely negative** quality—a function oscillates when it does not do certain other things.

The simplest example of an oscillatory function is given by

$$\phi(n) = (-1)^n,$$

which is equal to $+1$ when n is even and to -1 when n is odd. In this case the values recur cyclically. But consider

$$\phi(n) = (-1)^n + (1/n),$$

the values of which are

$$-1+1,\quad 1+(1/2),\quad -1+(1/3),\quad 1+(1/4),\quad -1+(1/5),\ \ldots$$

When n is large every value is nearly equal to $+1$ or -1, and obviously $\phi(n)$ does not tend to a limit or to $+\infty$ or to $-\infty$, and therefore it oscillates: but the values do not recur. It is to be observed that in this case every value of $\phi(n)$ is numerically less than or equal to $3/2$. Similarly

$$\phi(n) = (-1)^n\, 100 + (1000/n)$$

oscillates. When n is large enough every value is nearly equal to 100 or -100. The numerically greatest value is 900 (for $n=1$). But now consider $\phi(n) = (-1)^n n$, the values of which are -1, 2, -3, 4, -5, ... This function oscillates, for it does not tend to a limit, nor to $+\infty$, nor to $-\infty$. But in this case we cannot assign any limit beyond which the numerical value of the terms does not rise. The distinction between these two examples suggests a further definition.

DEFINITION. *If $\phi(n)$ oscillates as n tends to ∞ it will be said to* **oscillate finitely** *or* **infinitely** *according as it is or is not possible to assign a number K such that all the values of $\phi(n)$ are numerically less than K, i.e. $|\phi(n)| < K$ for all values of n.*

These definitions, as well as those of § 54, are further illustrated in the following examples.

Examples XXVII. Consider the behaviour as n tends to ∞ of the following functions:

1. $(-1)^n$, $5+3(-1)^n$, $(1000000/n)+(-1)^n$, $1000000(-1)^n+(1/n)$.

2. $(-1)^n n$, $1000000+(-1)^n n$. 3. $1000000-n$, $(-1)^n(1000000-n)$.

4. $n\{1+(-1)^n\}$. In this case the values of $\phi(n)$ are

$$0,\ 4,\ 0,\ 8,\ 0,\ 12,\ 0,\ 16,\ \ldots.$$

The odd terms are all zero and the even terms tend to $+\infty$: $\phi(n)$ oscillates infinitely.

5. $n^2+(-1)^n n$. The second term oscillates infinitely, but the first is very much larger than the second when n is large. In fact $\phi(n) \geqq n^2-n$ and $n^2-n=(n-\frac{1}{2})^2-\frac{1}{4}$ is greater than any assigned value G if $n>\frac{1}{2}+\sqrt{\{G+\frac{1}{4}\}}$. Thus $\phi(n) \rightarrow +\infty$.

6. $n^3+(-1)n^2$, $n^2\{1+(-1)^n\}$, $(-1)^n n^2+n$.

7. $11+3(-1)^n$, $(11/n)+3(-1)^n$, $11n+3(-1)^n$, $11+\{3(-1)^n/n\}$, $11+3n(-1)^n$, $\{11+3(-1)^n\}/n$, $\{11+3(-1)^n\}n$, $(11/n)+3(-1)^n n$, $11n+\{3(-1)^n/n\}$.

8. $\sin n\theta\pi$. We have already seen (Exs. XXVI. 9) that when θ is rational $\phi(n)$ oscillates finitely—unless θ is an integer, when $\phi(n)=0$, $\phi(n)\rightarrow 0$.

The case in which θ is irrational is a little more difficult. But it is not difficult to see that $\phi(n)$ still oscillates finitely. We can without loss of generality suppose $0<\theta<1$. In the first place $|\phi(n)|<1$. Hence $\phi(n)$ must oscillate finitely or tend to a limit. We shall consider whether the second alternative is really possible. Let us suppose then that

$$\lim \sin n\theta\pi = l.$$

Then however small ϵ is we can choose n_0 so that $\sin n\theta\pi$ lies between $l-\epsilon$ and $l+\epsilon$ for all values of n greater than or equal to n_0. Hence $\sin(n+1)\theta\pi-\sin n\theta\pi$ is numerically less than 2ϵ for all such values of n, and so $|\sin\frac{1}{2}\theta\pi\cos(n+\frac{1}{2})\theta\pi|<\epsilon$.

Hence $$\cos(n+\tfrac{1}{2})\theta\pi=\cos n\theta\pi\cos\tfrac{1}{2}\theta\pi-\sin n\theta\pi\sin\tfrac{1}{2}\theta\pi$$
must be numerically less than $\epsilon/|\sin\frac{1}{2}\theta\pi|$. Similarly
$$\cos(n-\tfrac{1}{2})\theta\pi=\cos n\theta\pi\cos\tfrac{1}{2}\theta\pi+\sin n\theta\pi\sin\tfrac{1}{2}\theta\pi$$
must be numerically less than $\epsilon/|\sin\frac{1}{2}\theta\pi|$; and so each of $\cos n\theta\pi\cos\frac{1}{2}\theta\pi$, $\sin n\theta\pi\sin\frac{1}{2}\theta\pi$ must be numerically less than $\epsilon/|\sin\frac{1}{2}\theta\pi|$. That is to say, if n is large $\cos n\theta\pi\cos\frac{1}{2}\theta\pi$ is very small, and this can only be the case if $\cos n\theta\pi$ is very small. Similarly $\sin n\theta\pi$ must be very small (so that l must be zero). But it is impossible that $\cos n\theta\pi$ and $\sin n\theta\pi$ can *both* be very small, as the sum of their squares is unity. Thus the hypothesis that $\sin n\theta\pi$ tends to a limit l is impossible, and therefore $\sin n\theta\pi$ oscillates as n tends to ∞.

The reader should consider with particular care the argument '$\cos n\theta\pi\cos\frac{1}{2}\theta\pi$ is very small, and this can only be the case if $\cos n\theta\pi$ is very small.' Why, he may ask, should it not be the other factor $\cos\frac{1}{2}\theta\pi$ which is 'very small'? The answer is to be found, of course, in the meaning of the phrase 'very small' as used in this connection. When we say '$\phi(n)$ is very small' for large values of n, we mean that we can choose n_0 so that $\phi(n)$ is numerically smaller than *any* assigned number, if n is sufficiently large. Such an assertion is palpably absurd when made of a *fixed* number such as $\cos\frac{1}{2}\theta\pi$, which is not zero.

9. $\sin n\theta\pi+(1/n)$, $\sin n\theta\pi+(1000000/n)$, $\sin n\theta\pi+1$, $\sin n\theta\pi+n$, $(-1)^n\sin n\theta\pi$, $\sin n\theta\pi+(-1)^n$, $\sin n\theta\pi+\{(-1)^n/n\}$, $\sin n\theta\pi+(-1)^n n$.

10. $a\cos n\theta\pi+b\sin n\theta\pi$, $\sin^2 n\theta\pi$, $\cos^2 n\theta\pi$, $a\cos^2 n\theta\pi+b\sin^2 n\theta\pi$.

11. $a+bn+(-1)^n(c+dn)+e\cos n\theta\pi+f\sin n\theta\pi$.

12. $n \sin n\theta\pi$. If n is integral, $\phi(n)=0$, $\phi(n) \to 0$. If n is rational but not integral, or irrational, $\phi(n)$ oscillates infinitely.

13. $n(a\cos^2 n\theta\pi + b\sin^2 n\theta\pi)$. In this case $\phi(n) \to +\infty$ if a and b are both positive, $\to -\infty$ if both are negative. Consider the special cases in which $a=0$, $b>0$, or $a>0$, $b=0$, or $a=0$, $b=0$. If a and b have opposite signs $\phi(n)$ generally oscillates infinitely. Consider any exceptional cases.

14. $\sin(n^2\theta\pi)$. If θ is integral, $\phi(n) \to 0$. Otherwise $\phi(n)$ oscillates finitely, as may be shown by arguments similar to though more complex than those used in Exs. XXVI. 9 and XXVII. 8*.

15. $\sin(n!\,\theta\pi)$. If θ has any rational value p/q, $n!\,\theta$ is certainly integral for all values of n greater than or equal to q. Hence $\phi(n) \to 0$. The case in which θ is irrational cannot be settled without the aid of considerations of a much more difficult character.

16. $\cos(n!\,\theta\pi)$, $a\cos^2(n!\,\theta\pi) + b\sin^2(n!\,\theta\pi)$, where θ is rational.

17. $an - [bn]$, $(-1)^n(an - [bn])$. 18. $[\sqrt{n}]$, $(-1)^n[\sqrt{n}]$, $\sqrt{n} - [\sqrt{n}]$.

19. *The smallest prime factor of n.* When n is a prime, $\phi(n)=n$. When n is even, $\phi(n)=2$. Thus $\phi(n)$ oscillates infinitely.

20. *The largest prime factor of n.*

21. *The number of days in the year n* A.D.

Examples XXVIII. 1. If $\phi(n) \to +\infty$ and $\psi(n) \geqq \phi(n)$ for all values of n, then $\psi(n) \to +\infty$.

2. If $\phi(n) \to 0$, and $|\psi(n)| \leqq |\phi(n)|$ for all values of n, then $\psi(n) \to 0$.

3. If $\lim |\phi(n)|=0$, then $\lim \phi(n)=0$.

4. If $\phi(n)$ tends to a limit or oscillates finitely, and $|\psi(n)| \leqq |\phi(n)|$ for $n \geqq n_0$, then $\psi(n)$ tends to a limit or oscillates finitely.

5. If $\phi(n) \to +\infty$, or $-\infty$, or oscillates infinitely, and $|\psi(n)| \geqq |\phi(n)|$ for $n \geqq n_0$, then $\psi(n) \to +\infty$ or $-\infty$ or oscillates infinitely.

6. 'If $\phi(n)$ oscillates and, however great be n_0 we can find values of n greater than n_0 and for which $\psi(n)$ is either greater than or less than $\phi(n)$, then $\psi(n)$ oscillates.' Is this true? If not give an example to the contrary [$\phi(n)=(-1)^n$, $\psi(n)=0$].

7. If $\phi(n) \to l$ as $n \to \infty$, then also $\phi(n+p) \to l$, p being any fixed integer. [This follows at once from the definition. Similarly we see that if $\phi(n)$ tends to $+\infty$ or $-\infty$ or oscillates so also does $\phi(n+p)$.]

8. The same conclusions hold (except in the case of oscillation) if p varies with n but is always numerically less than a fixed positive integer N; or if p varies with n in any way, so long as it is always positive.

9. Determine the least value of n_0 for which it is true that

(a) $n^2+n>1000 \quad (n \geqq n_0)$, (b) $n^2+n>1000000 \quad (n \geqq n_0)$.

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* See Bromwich's *Infinite Series*, p. 485.

10. Determine the least value of n_0 for which it is true that

(*a*) $n+(-1)^n>1000 \quad (n \geqq n_0)$, (*b*) $n+(-1)^n>1000000 \quad (n \geqq n_0)$.

11. Determine the least value of n_0 for which it is true that

(*a*) $n^2+n>G \quad (n \geqq n_0)$, (*b*) $n+(-1)^n>G \quad (n \geqq n_0)$,

G being any positive number.

[(*a*) $n_0=[\frac{1}{2}+\sqrt{(G+\frac{1}{4})}]$: (*b*) $n_0=1+[G]$ or $2+[G]$, according as $[G]$ is odd or even: i.e. $n_0=1+[G]+\frac{1}{2}\{1+(-1)^{[G]}\}$.]

12. Determine the least value of n_0 such that

(*a*) $n/(n^2+1)<\cdot 0001$. (*b*) $(1/n)+\{(-1)^n/n^2\}<\cdot 000001$, for $n \geqq n_0$.

[Let us take the latter case. In the first place

$$(1/n)+\{(-1)^n/n^2\} \leqq (n+1)/n^2,$$

and it is easy to see that the least value of n_0 such that $(n+1)/n^2<\cdot 000001$, for $n \geqq n_0$, is 1000002. But the inequality given is satisfied by $n=1000001$, and this is the value of n_0 required.]

56. Some general theorems with regard to limits.

A. The behaviour of the sum of two functions whose behaviour is known.

THEOREM I. *If $\phi(n)$ and $\psi(n)$ tend to limits a, b, then $\phi(n)+\psi(n)$ tends to the limit $a+b$.*

This is almost obvious. The argument which the reader will at once form in his mind is roughly this: 'when n is large $\phi(n)$ is nearly equal to a and $\psi(n)$ to b and therefore their sum is nearly equal to $a+b$.' It is well to state the argument quite formally, however.

Let δ be any assigned positive number (e.g. $\cdot 001$, $\cdot 0000001$, ...). We require to show that a number n_0 can be found such that

$$|\phi(n)+\psi(n)-a-b|<\delta \;\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

for $n \geqq n_0$. Now by a proposition proved in Chapter III. (more generally indeed than we need here) the modulus of the sum of two numbers is less than or equal to the sum of their moduli. Thus

$$|\phi(n)+\psi(n)-a-b| \leqq |\phi(n)-a|+|\psi(n)-b|.$$

It follows that the desired condition will certainly be satisfied if n_0 can be so chosen that

$$|\phi(n)-a|+|\psi(n)-b|<\delta \;\ldots\ldots\ldots\ldots\ldots (2),$$

for $n \geqq n_0$. But this is certainly the case. For since $\lim \phi(n)=a$ we can, by the definition of a limit, find n_1 so that $|\phi(n)-a|<\delta'$, for $n \geqq n_1$, and this however small may be δ'. Nothing prevents our taking $\delta' = \frac{1}{2}\delta$, so that $|\phi(n) - a| < \frac{1}{2}\delta$, for $n \geqq n_1$. Similarly we can find n_2 so that $|\psi(n) - b| < \frac{1}{2}\delta$ for $n \geqq n_2$. Now take n_0 to be *the greater of the two numbers* n_1, n_2. Then if $n \geqq n_0$ $|\phi(n) - a| < \frac{1}{2}\delta$ *and* $|\psi(n) - b| < \frac{1}{2}\delta$, and therefore (2) is satisfied and the theorem is proved.

The argument may be concisely stated thus: since $\lim \phi(n) = a$ and $\lim \psi(n) = b$ we can choose n_1, n_2 so that

$$|\phi(n) - a| < \tfrac{1}{2}\delta \ (n \geqq n_1), \quad |\psi(n) - b| < \tfrac{1}{2}\delta \ (n \geqq n_2),$$

and then, if n is greater than either n_1 or n_2,

$$|\phi(n) + \psi(n) - a - b| \leqq |\phi(n) - a| + |\phi(n) - b| < \delta,$$

and therefore $$\lim \{\phi(n) + \psi(n)\} = a + b.$$

Even when stated thus the argument may possibly appear to the reader to be merely a piece of useless pedantry, or an attempt to manufacture difficulties out of what is really obvious. We do not assert that such an opinion is, in this case, entirely groundless. The result really *is* very obvious: nor would any mathematician think it worth while as a rule to state arguments for what is so obvious at such length.

But the reader must remember that the theorem, obvious though it may be, is one of the most fundamental and important theorems in all mathematics. It is one which every mathematician uses, consciously or unconsciously, twenty times a day. The proof of such a theorem must be made absolutely clear, explicit, and rigorous: no room must be left for any possible misapprehension or confusion. And this is not all. The great majority of theorems concerning limits are, as the reader will discover before long, far from being so simple and so obvious as this one. In this case the result obviously indicated by common sense was true. In more difficult cases common sense as often indicates an untrue result as a true one: sometimes it fails to give any indication at all. In such cases vague general arguments are worse than useless: they lead to mistakes not only gross in themselves but entirely confusing in their consequences. And unless the reader is prepared to take the trouble to try and understand the way in which rigorous methods apply to simple and obvious cases, where their application is easy, he will find that when he comes to difficult questions, which cannot be settled without them, he has not the capacity to use them.

57. Results subsidiary to Theorem I. The reader should have no difficulty in verifying the following subsidiary results.

1. *If* $\phi(n)$ *tends to a limit, but* $\psi(n)$ *tends to* $+\infty$ *or to* $-\infty$ *or oscillates finitely or infinitely, then* $\phi(n) + \psi(n)$ *behaves like* $\psi(n)$.

2. *If* $\phi(n) \to +\infty$, *and* $\psi(n) \to +\infty$ *or oscillates finitely, then* $\phi(n)+\psi(n) \to +\infty$.

In this statement we may obviously change $+\infty$ into $-\infty$ throughout.

3. *But if* $\phi(n) \to \infty$ *and* $\psi(n) \to -\infty$, *then* $\phi(n)+\psi(n)$ *may either tend to a limit or to* $+\infty$ *or to* $-\infty$ *or may oscillate either finitely or infinitely.*

These five possibilities are illustrated in order by (i) $\phi(n)=n$, $\psi(n)=-n$, (ii) $\phi(n)=n^2$, $\psi(n)=-n$, (iii) $\phi(n)=n$, $\psi(n)=-n^2$, (iv) $\phi(n)=n+(-1)^n$, $\psi(n)=-n$, (v) $\phi(n)=n^2+(-1)^n n$, $\psi(n)=-n^2$. The reader should construct additional examples of each case.

4. *If* $\phi(n) \to +\infty$ *and* $\psi(n)$ *oscillates infinitely,* $\phi(n)+\psi(n)$ *may tend to* $+\infty$ *or oscillate infinitely, but cannot tend to a limit, or to* $-\infty$, *or oscillate finitely.*

For $\psi(n)=\{\phi(n)+\psi(n)\}-\phi(n)$; and, if $\phi(n)+\psi(n)$ behaved in any of the three last ways, it would follow, from the previous results, that $\psi(n) \to -\infty$, which is not the case. As examples of the two cases which are possible, consider (i) $\phi(n)=n^2$, $\psi(n)=(-1)^n n$, (ii) $\phi(n)=n$, $\psi(n)=(-1)^n n^2$. Here again the signs of $+\infty$ and $-\infty$ may be permuted throughout.

5. *If* $\phi(n)$ *and* $\psi(n)$ *both oscillate finitely,* $\phi(n)+\psi(n)$ *must tend to a limit or oscillate finitely.*

As examples take

(i) $\phi(n)=\psi(n)=(-1)^n$, (ii) $\phi(n)=\cos\frac{1}{3}n\pi$, $\psi(n)=\sin\frac{1}{3}n\pi$.

6. *If* $\phi(n)$ *oscillates finitely, and* $\psi(n)$ *infinitely, then* $\phi(n)+\psi(n)$ *oscillates infinitely.*

For $\phi(n)$ is in absolute value always less than a certain constant, say G. On the other hand $\psi(n)$, since it oscillates *infinitely* must assume values numerically greater than any assignable number (e.g. $10G$, $100G$, ...). Hence $\phi(n)+\psi(n)$ must assume values numerically greater than any assignable number (e.g. $9G$, $99G$, ...). Hence $\phi(n)+\psi(n)$ must either tend to $+\infty$ or $-\infty$ or oscillate infinitely. But if it tended to $+\infty$, for instance,

$$\psi(n)=\{\phi(n)+\psi(n)\}-\phi(n)$$

would also tend to $+\infty$, by the preceding results. Thus $\phi(n)+\psi(n)$ cannot tend to $+\infty$, nor, for similar reasons, to $-\infty$: hence it oscillates infinitely.

7. *If both* $\phi(n)$ *and* $\psi(n)$ *oscillate infinitely,* $\phi(n)+\psi(n)$ *may tend to a limit, or to* $+\infty$, *or to* $-\infty$, *or oscillate either finitely or infinitely.*

Suppose, for instance, that $\phi(n)=(-1)^n n$, while $\psi(n)$ is in turn each of the functions $(-1)^{n+1}n$, $\{1+(-1)^{n+1}\}n$, $-\{1+(-1)^n\}n$, $(-1)^{n+1}(n+1)$, $(-1)^n n$. We thus obtain examples of all five possibilities.

This exhausts all the possibilities which are really distinct. The results may be conveniently summarised in the following tabular form, in which 1 stands for 'tends to a limit,' 2 for 'tends to $+\infty$,' 3 for 'tends to $-\infty$,' 4 for 'oscillates finitely,' and 5 for 'oscillates infinitely.'

$\phi(n)$	$\psi(n)$	$\phi(n)+\psi(n)$
1	1	1
1	2	2
1	3	3
1	4	4
1	5	5
2	2	2
2	3	1, 2, 3, 4, or 5
2	4	2
2	5	2 or 5
4	4	1 or 4
4	5	5
5	5	1, 2, 3, 4, or 5

Before passing on to consider the *product* of two functions we may point out that the result of Theorem I. may be immediately extended to the sum of three or more functions which tend to limits as $n \rightarrow \infty$.

58. B. The behaviour of the product of two functions whose behaviour is known. We can now prove a similar set of theorems concerning the product of two functions. The principal result is the following.

THEOREM II. *If* $\lim \phi(n) = a$ *and* $\lim \psi(n) = b$, *then*

$$\lim \phi(n)\psi(n) = ab.$$

Let $\phi(n) = a + \phi_1(n), \quad \psi(n) = b + \psi_1(n),$

so that $\lim \phi_1(n) = 0$ and $\lim \psi_1(n) = 0$. Then

$$\phi(n)\psi(n) = ab + a\psi_1(n) + b\phi_1(n) + \phi_1(n)\psi_1(n).$$

Hence the numerical value of the difference $\phi(n)\psi(n) - ab$ is certainly not greater than the sum of the numerical values of $a\psi_1(n)$, $b\phi_1(n)$, $\phi_1(n)\psi_1(n)$. From this it is obvious that

$$\lim \{\phi(n)\psi(n) - ab\} = 0,$$

which proves the theorem.

The following is a strictly formal proof. We have

$$|\phi(n)\psi(n)-ab| \leqq |a\psi_1(n)|+|b\phi_1(n)|+|\phi_1(n)||\psi_1(n)|.$$

Choose n_0 so that for $n \geqq n_0$

$$|\phi_1(n)|<\tfrac{1}{3}\delta/|b|, \quad |\psi_1(n)|<\tfrac{1}{3}\delta/|a|.$$

Then $$|\phi(n)\psi(n)-ab|<\tfrac{1}{3}\delta+\tfrac{1}{3}\delta+\{\tfrac{1}{9}\delta^2/(|a||b|)\},$$

which is certainly less than δ, if $\delta<\frac{1}{3}|a||b|$. That is to say we can choose n_0 so that $|\phi(n)\psi(n)-ab|<\delta\ (n \geqq n_0)$, and so the theorem follows. The reader should study the details of this proof attentively; it is an elementary specimen of a type of proof perpetually occurring in higher analysis.

We need hardly point out that this theorem, like Theorem I., may be immediately extended to the product of any number of functions of n.

59. Results subsidiary to Theorem II. There is of course a series of theorems concerning products analogous to those stated in § 57 for sums. It will be convenient to present the results in tabular form. We must distinguish now *six* different ways in which $\phi(n)$ may behave as n tends to ∞. It may (1) tend to a limit *other than zero*, (2) tend to zero, (3) tend to $+\infty$, (3′) tend to $-\infty$, (4) oscillate finitely, (5) oscillate infinitely. We need not, as a rule, take account separately of (3) and (3′), as the results for one may be deduced from those for the other by a change of sign.

Case	$\phi(n)$	$\psi(n)$	$\phi(n)\psi(n)$
1	1	1	1
2	1	2	2
3	1	3	3 or 3′
4	1	4	4
5	1	5	5
6	2	2	2
7	2	3	any way
8	2	4	2
9	2	5	any way
10	3	3	3
11	3	4	3, 3′, or 5
12	3	5	3, 3′, or 5
13	4	4	1, 2, or 4
14	4	5	any way
15	5	5	any way

We leave the verification of this table as an exercise to the reader. The more difficult cases 7, 9, 11, 13, 14, 15 may be illustrated by the following examples.

7

$\phi(n)$	$\psi(n)$	$\phi(n)\psi(n)$
$1/n$	n	1
$1/n^2$	n	2
$1/n$	n^2	3
$-1/n$	n^2	3′
$(-1)^n/n$	n	4
$(-1)^n/n$	n^2	5

9

$\phi(n)$	$\psi(n)$	$\phi(n)\psi(n)$
$(-1)^n/n$	$(-1)^n n$	1
$(-1)^n/n^2$	$(-1)^n n$	2
$(-1)^n/n$	$(-1)^n n^2$	3
$(-1)^n/n$	$(-1)^{n+1} n^2$	3′
$1/n$	$(-1)^n n$	4
$1/n$	$(-1)^n n^2$	5

11

$\phi(n)$	$\psi(n)$	$\phi(n)\psi(n)$
n	$2+(-1)^n$	3
n	$-2-(-1)^n$	3′
n	$(-1)^n$	5

13

$\phi(n)$	$\psi(n)$	$\phi(n)\psi(n)$
$(-1)^n$	$(-1)^n$	1
$1+(-1)^n$	$1-(-1)^n$	2
$\cos\frac{1}{3}n\pi$	$\sin\frac{1}{3}n\pi$	4

14

$\phi(n)$	$\psi(n)$	$\phi(n)\psi(n)$
$\{1-(-1)^n\}-[\{1+(-1)^n\}/n]$	$\{1-(-1)^n\}-n\{1+(-1)^n\}$	1
$1+(-1)^n$	$\{1-(-1)^n\}n$	2
$(-1)^n$	$(-1)^n n$	3
$(-1)^n$	$(-1)^{n+1} n$	3′
$1+(-1)^n$	$1+\{1-(-1)^n\}n$	4
$\cos\frac{1}{3}n\pi$	$n\sin\frac{1}{3}n\pi$	5

15

$\phi(n)$	$\psi(n)$	$\phi(n)\psi(n)$
$n\{1-(-1)^n\}-[\{1+(-1)^n\}/n]$	$[\{1-(-1)^n\}/n]-n\{1+(-1)^n\}$	1
$\{1+(-1)^n\}n$	$\{1-(-1)^n\}n$	2
$(-1)^n n$	$(-1)^n n$	3
$(-1)^n n$	$(-1)^{n+1} n$	3′
$\{1+(-1)^n\}n+[\{1-(-1)^n\}/n]$	$\{1-(-1)^n\}n$	4
$n\cos\frac{1}{3}n\pi$	$n\sin\frac{1}{3}n\pi$	5

As an illustration of how to verify these examples we may take the first example under Case 14. Since $1-(-1)^n=0$ or 2 according as n is even or odd, while $1+(-1)^n=0$ or 2 according as n is odd or even, the values of $\phi(n)$ are

$$2,\quad -2/2,\quad 2,\quad -2/4,\quad 2,\quad -2/6,\quad \dots,$$

and so $\phi(n)$ oscillates finitely; while the values of $\psi(n)$ are

$$2,\quad -2\times 2,\quad 2,\quad -2\times 4,\quad 2,\quad -2\times 6,\quad \dots,$$

and $\psi(n)$ oscillates infinitely. But $\phi(n)\psi(n)=4$ for all values of n.

60. A particular case of Theorem II which is important is that in which $\psi(n)$ is constant. The theorem then asserts simply that $\lim k\phi(n) = ka$ if $\lim \phi(n) = a$.

To this we may join the subsidiary theorem that if $\phi(n) \rightarrow +\infty$, then $a\phi(n) \rightarrow +\infty$ or $-\infty$, according as a is positive or negative, unless $a = 0$, when of course $a\phi(n) = 0$ for all values of n and $\lim a\phi(n) = 0$. And if $\phi(n)$ oscillates finitely or infinitely so does $a\phi(n)$, unless $a = 0$.

61. C. The behaviour of the difference or quotient of two functions whose behaviour is known. There is, of course, a similar set of theorems for the *difference* of two given functions: but they are such obvious corollaries from what precedes that it would be waste of time to state them at length. In order to deal with the quotient

$$\psi(n)/\phi(n),$$

we begin with the following theorem.

THEOREM III. *If* $\lim \phi(n) = a$, *and* a *is not zero, then*

$$\lim \{1/\phi(n)\} = 1/a.$$

Let

$$\phi(n) = a + \phi_1(n),$$

so that $\lim \phi_1(n) = 0$. Then

$$|\{1/\phi(n)\} - (1/a)| = |\phi_1(n)|/\{|a||a + \phi_1(n)|\},$$

and it is plain, since $\lim \phi_1(n) = 0$, that we can choose n_0 so that for $n \geqq n_0$ this is smaller than any assigned number δ.

The theorems subsidiary to this may again be stated concisely by means of a table.

Case	$\phi(n)$	$1/\phi(n)$
1	1	1
2	2	3, 3′, or 5
3	3	2
4	4	4 or 5
5	5	2, 4, or 5

The three more complicated cases may be illustrated by the following examples: the number indicates the behaviour of $1/\phi(n)$.

Case 2.	$\phi(n)=1/n$	3
	$\phi(n)=-1/n$	3′
	$\phi(n)=(-1)^n/n$	5
Case 4.	$\phi(n)=(-1)^n$	4
	$\phi(n)=\{1+(-1)^n\}+[\{1-(-1)^n\}/n]$	5
Case 5.	$\phi(n)=(-1)^n n$	2
	$\phi(n)=\{1+(-1)^n\}+\{1-(-1)^n\}n$	4
	$\phi(n)=\{1+(-1)^n\}n+[\{1-(-1)^n\}/n]$	5

It will not be necessary now to attempt to state an exhaustive series of theorems for the quotient $\psi(n)/\phi(n)$, such as may be deduced from the results above and those of § 59. The principal theorem is

THEOREM IV. *If* $\lim \phi(n) = a$ *and* $\lim \psi(n) = b$, *and* $a \neq 0$, *then*

$$\lim \{\psi(n)/\phi(n)\} = b/a.$$

This requires no proof, being an immediate consequence of Theorems II and III.

The reader will however find it very instructive to draw up, at any rate partially, a table for the quotient similar to those we have given for the product and reciprocal, and to illustrate some of the possible cases with examples.

62. THEOREM V. *If* $R\{\phi(n), \psi(n), \chi(n), \ldots\}$ *is any rational function of* $\phi(n)$, $\psi(n)$, $\chi(n)$, *etc., i.e. any function of the form*

$$P\{\phi(n), \psi(n), \chi(n), \ldots\}/Q\{\phi(n), \psi(n), \chi(n), \ldots\},$$

where P *and* Q *denote polynomials in* $\phi(n)$, $\psi(n)$, $\chi(n)$, ...: *and if*

$$\lim \phi(n) = a, \quad \lim \psi(n) = b, \quad \lim \chi(n) = c, \ldots,$$

and

$$Q(a, b, c, \ldots) \neq 0;$$

then

$$\lim R\{\phi(n), \psi(n), \chi(n), \ldots\} = R(a, b, c, \ldots).$$

For P is a sum of a finite number of terms of the type

$$A\{\phi(n)\}^p\{\psi(n)\}^q \ldots,$$

where A is a constant and p, q positive integers. This term, by Theorem II (or rather by its obvious extension to the product of any number of functions) tends to the limit $Aa^pb^q\ldots$, and so P tends to the limit $P(a, b, c, \ldots)$, by the similar extension of Theorem I. Similarly for Q: and the result then follows from Theorem IV.

63. The preceding general theorem may be applied to the following very important particular problem: *what is the behaviour of the most general rational function of* n, viz.

$$S(n)=\frac{a_0n^p+a_1n^{p-1}+\dots+a_p}{b_0n^q+b_1n^{q-1}+\dots+b_q},$$

as n *tends to* ∞.

In order to apply the theorem we transform $S(n)$ by writing it in the form

$$n^{p-q}\left(a_0+\frac{a_1}{n}+\dots+\frac{a_p}{n^p}\right)\Big/\left(b_0+\frac{b_1}{n}+\dots+\frac{b_q}{n^q}\right).$$

The term in brackets is of the form $R\{\phi(n)\}$, where $\phi(n)=1/n$, and so, by Theorem V, it tends, as n tends to ∞, to the limit $R(0)=a_0/b_0$. Now, if $p<q$, $\lim n^{p-q}=0$; if $p=q$, $n^{p-q}=1$ and $\lim n^{p-q}=1$; if $p>q$, $n^{p-q}\to+\infty$. Hence, by Theorem II,

$$\lim S(n)=0\quad(p<q),\qquad \lim S(n)=a_0/b_0\quad(p=q),$$

$$S(n)\to+\infty\ \ (p>q,\ a_0/b_0\ \textit{positive}),\qquad S(n)\to-\infty\ \ (p>q,\ a_0/b_0\ \textit{negative}).$$

Examples XXIX. 1. Determine the behaviour, as $n\to\infty$, of each of the following functions of n, and of their sums, differences, products and quotients, taken in pairs: $1+\{(-1)^n/n\}$, $(-1)^n+(1/n)$, $1+(-1)^n n$, $(-1)^n+n$, $n+\{(-1)^n/n\}$, $(-1)^n n+(1/n)$, $(-1)^n(1+n)$, $(-1)^n\{1+(1/n)\}$, $(-1)^n\{n+(1/n)\}$.

2. Do the same for the functions

$$\cos^2\tfrac{1}{2}n\pi+(\sin^2\tfrac{1}{2}n\pi)/n,\quad \cos^2\tfrac{1}{2}n\pi+n\sin^2\tfrac{1}{2}n\pi,\quad n\cos^2\tfrac{1}{2}n\pi+(\sin^2\tfrac{1}{2}n\pi)/n.$$

3. Which (if any) of the functions

$$1/(\cos^2\tfrac{1}{2}n\pi+n\sin^2\tfrac{1}{2}n\pi),\quad 1/\{n(\cos^2\tfrac{1}{2}n\pi+n\sin^2\tfrac{1}{2}n\pi)\},$$

$$(n\cos^2\tfrac{1}{2}n\pi+\sin^2\tfrac{1}{2}n\pi)/\{n(\cos^2\tfrac{1}{2}n\pi+n\sin^2\tfrac{1}{2}n\pi)\}$$

tend to a limit as $n\to\infty$?

4. Denoting by $S(n)$ the general rational function of n, considered in § 63, show that in all cases

$$\lim\{S(n+1)/S(n)\}=1,\quad \lim[S\{n+(1/n)\}/S(n)]=1.$$

64. Functions of n which increase steadily with n. A special but particularly important class of functions of n is formed of those whose variation as n tends to ∞ is always in the same direction, that is to say those which always increase (or always decrease) as n increases. Since $-\phi(n)$ always increases if $\phi(n)$ always decreases, it is not necessary to consider the two kinds of functions separately; for theorems proved for one kind can at once be extended to the other.

DEFINITION. *The function* $\phi(n)$ *will be said to increase steadily with* n *if* $\phi(n+1) \geqq \phi(n)$ *for all values of* n.

It is to be observed that we do not exclude the case in which $\phi(n)$ has the *same* value for several values of n; all we exclude is possible *decrease.* Thus the function

$$\phi(n) = 2n + (-1)^n,$$

whose values for $n = 0, 1, 2, 3, 4, \dots$ are

$$1, 1, 5, 5, 9, 9, \dots$$

is said to increase steadily with n. Our definition would indeed include even functions which, from some value of n, remain constant; thus $\phi(n) = 1$ steadily increases according to our definition. However, as these functions are extremely special ones, and as there can be no doubt as to their behaviour as n tends to ∞, this apparent incongruity in the definition is not a serious defect.

There is one exceedingly important theorem concerning functions of this class.

THEOREM. *If* $\phi(n)$ *steadily increases with* n, *then either* (i) $\phi(n)$ *tends to a limit as* n *tends to* ∞, *or* (ii) $\phi(n) \to +\infty$.

That is to say, while there are in general *five* alternatives as to the behaviour of a function, there are *two* only for this special kind of function.

The proof is very simple. Imagine the various values of $\phi(n)$ represented by points along the line L of Chap. I. Each point lies to the *right* of the preceding point (or coincides with it).

Let P_n be the point corresponding to $\phi(n)$. Let Q be any other point whatever on the line. Then *either*

(1) there are values of n such that P_n lies to the right of Q (or coincides with it), *or*

(2) there are no such values.

In the first case we say that Q is a point which is *reached* for some value of n, in the second case that it is a point which is *not reached.* Every point is a reached point or a not reached point. If any point Q is reached, so obviously are all points to the left of Q.

There are two alternatives: (1) *every* point may be reached. Then it is clear that if G is any number, however large, it will correspond to a point Q, and, for sufficiently large values of n, P_n

will lie to the right of Q, and so will $P_{n+1}, \dots$. In other words $\phi(n) > G$ *for all values of n from a certain value.* That is

$$\phi(n) \rightarrow +\infty.$$

Or (2) *not every* point may be reached. Then we can divide L into two segments, L_1, L_2, of which the first includes all reached points, the second all not reached points. The only doubt is as to whether the point R which divides the two segments is reached or not.

If R is reached, then, since no point P_n can lie to the right of R (as in that case other points in L_2 would be reached), all the points P_n must coincide with R from some value of n, the first for which R is reached. Thus if $OR = l$, we have $\phi(n) = l$ from a certain value of n onwards; so that, of course, $\lim \phi(n) = l$.

On the other hand, if R is not reached all points to the left of R, however close to R, are reached. Thus we can choose n_0 so that $\phi(n)$ is as nearly equal to l as we please when $n = n_0$. Since, as n increases beyond n_0, $\phi(n)$ approaches even more nearly to the value l, it is clear that

$$\lim \phi(n) = l.$$

The theorem is thus proved.

For example, if $\phi(n) = 3 - (1/n)$, $l = 3$: the point R ($OR = 3$) is not reached. From a common-sense point of view the theorem may be stated thus.

Let the point P move along the line L in a series of jumps, *its motion always being from left to right.* Then either P will pass over the whole line, or its position will gradually approximate to a definite position R on the line L. The theorem is almost intuitive: the proof which precedes is merely a careful analysis of the process of argument implied in but suppressed by our intuition of its truth.

COR. 1. *If $\phi(n)$ increases steadily with n it will tend to a limit or to $+\infty$ according as it is or is not possible to find a fixed number G such that $\phi(n) < G$.*

We shall find this corollary exceedingly useful later on.

COR. 2. *If $\phi(n)$ increases steadily with n and $\phi(n) < G$ for all values of n, $\phi(n)$ tends to a limit and this limit is less than or equal to G.*

It should be noticed that the limit may be equal to G: if e.g. $\phi(n) < 3 - (1/n)$, every value of $\phi(n)$ is less than 3, but the limit is equal to 3.

The reader should write out for himself the corresponding theorems and corollaries for the case in which $\phi(n)$ *decreases* as n increases.

65. The great importance of these theorems lies in the fact that they give us (what we have so far been without) a means of deciding (in a great many cases) whether a given function of n does or does not tend to a limit as $n \to \infty$, *without requiring us to be able to guess or otherwise infer beforehand what the limit is.* If we know what the limit must be (if there is one) we can use the test

$$|\phi(n) - l| < \epsilon \qquad (n \geqq n_0):$$

as for example in the case of $\phi(n) = 1/n$, where it is obvious that the limit can only be zero.

But suppose we have to determine whether

$$\phi(n) = \left(1 + \frac{1}{n}\right)^n$$

tends to a limit. In this case it is not obvious what the limit, if there is one, will be: and it is evident that the test above, which involves l, cannot, at any rate directly, be used to decide whether l exists or not.

Of course the test can sometimes be used indirectly, to prove that l *cannot* exist by means of a *reductio ad absurdum.* If e.g. $\phi(n) = (-1)^n$, it is clear that l would have to be equal to 1 and also equal to -1, which is obviously impossible.

66. The limit of x^n as n tends to ∞. Let us apply some of the preceding results to the particularly important case in which $\phi(n) = x^n$.

First, suppose x positive. Then since $\phi(n+1) = x\phi(n)$, $\phi(n)$ increases with n if $x > 1$, decreases as n increases if $x < 1$. If $x = 1$, $\phi(n) = 1$, $\lim \phi(n) = 1$, so that this special case need not detain us.

Thus, if $x > 1$, x^n must tend either to a limit (which must obviously be greater than 1), or to $+\infty$. Suppose it tends to a limit l. Then (Ex. XXVIII. 7) $\lim \phi(n+1) = \lim \phi(n) = l$; but

$$\lim \phi(n+1) = \lim x\phi(n) = x \lim \phi(n) = xl,$$

and therefore $l = xl$: and as x and l are both greater than 1, this is impossible. Hence

$$x^n \to +\infty \qquad (x > 1).$$

Ex. The reader may give an alternative proof, showing by the binomial theorem that, if $x = 1 + \delta$ $(\delta > 0)$, $x^n > 1 + n\delta$, and so that $x^n \to +\infty$.

On the other hand, if $x < 1$, x^n is a decreasing function and must therefore tend to a limit or to $-\infty$. Since x^n is positive the second alternative may be ignored. Thus $\lim x^n = l$, say, and as above $l = xl$, so that l must be zero. Hence

$$\lim x^n = 0 \qquad (0 < x < 1).$$

In the special cases of $x = 0$, 1, we clearly have $\lim x^n = 0$, $\lim x^n = 1$ respectively.

Ex. Prove as in the preceding example that, if $0 < x < 1$, $(1/x)^n$ tends to $+\infty$, and deduce that x^n tends to 0.

We have finally to consider the case in which x is negative. If $-1 < x < 0$ and $x = -y$, we have $\lim y^n = 0$ by what precedes and therefore $\lim x^n = 0$.

If $x = -1$ it is obvious that x^n oscillates, taking the values -1, 1 alternatively.

Finally if $x < -1$, $y > 1$, y^n tends to $+\infty$ and therefore x^n takes values, both positive and negative, numerically greater than any assigned number. Hence x^n oscillates infinitely.

To sum up:

$$\begin{aligned} \phi(n) = x^n \to +\infty, &\qquad (x > 1), \\ \lim \phi(n) = 1, &\qquad (x = 1), \\ \lim \phi(n) = 0, &\qquad (-1 < x < 1), \\ \phi(n) \text{ oscillates finitely}, &\qquad (x = -1), \\ \phi(n) \text{ oscillates infinitely}, &\qquad (x < -1). \end{aligned}$$

Examples XXX. 1. If $\phi(n)$ is positive and $\phi(n+1) > K\phi(n)$, where $K > 1$, for all values of n, then $\phi(n) \to +\infty$.

[For $\phi(n) > K\phi(n-1) > K^2\phi(n-2) \ldots > K^{n-1}\phi(1)$, from which the conclusion follows at once.]

2. The same result is true if the conditions above stated are satisfied only for $n \geqq n_0$.

3. If $\phi(n)$ is positive and $\phi(n+1) < K\phi(n)$, where $0 < K < 1$, then $\lim \phi(n) = 0$. This result also is true if the conditions are satisfied only for $n \geqq n_0$.

4. If $|\phi(n+1)| < K|\phi(n)|$ for $n \geqq n_0$, where $0 < K < 1$, then $\lim \phi(n) = 0$.

5. If $\phi(n)$ is positive and $\lim \{\phi(n+1)\}/\{\phi(n)\} = l > 1$, then $\phi(n) \rightarrow +\infty$.

[For we can determine n_0 so that $\{\phi(n+1)\}/\{\phi(n)\} > K > 1$, for $n \geqq n_0$: we may, e.g., take K half-way between 1 and l. Now apply Ex. 1.]

6. If $\lim \{\phi(n+1)\}/\{\phi(n)\} = l$, where l is numerically less than unity, then $\lim \phi(n) = 0$. [This follows from Ex. 4 as Ex. 5 follows from Ex. 1.]

7. Determine the behaviour, as $n \rightarrow \infty$, of $\phi(n) = n^r x^n$, where r is any positive integer.

[Here
$$\{\phi(n+1)\}/\{\phi(n)\} = \{(n+1)/n\}^r x \rightarrow x$$
as $n \rightarrow \infty$. If x is positive and greater than 1, $\phi(n) \rightarrow +\infty$. If x is positive and $0 < x < 1$, $\phi(n) \rightarrow 0$. If x is negative and equal to $-y$, $\phi(n) = (-1)^n n^r y^n$, and it is easy to see that $\phi(n) \rightarrow 0$ $(-1 < x < 0)$ and $\phi(n)$ oscillates infinitely $(x \leqq -1)$. Finally if $x = 1$, $\phi(n) = n^r$, and $\phi(n) \rightarrow +\infty$; and if $x = 0$, $\phi(n) = 0$ for all values of n.]

8. Discuss $n^{-r} x^n$ in the same way. [The results are the same, except that when $x = 1$ or -1, $\phi(n) \rightarrow 0$.]

9. Draw up a table to show how $n^k x^n$ behaves as $n \rightarrow \infty$, for all real values of x, and all (positive and negative) integral values of k.

[The reader will observe that *the value of k is immaterial* except in the special cases when $x = 1$ or -1. In other words, *it is the factor x^n which is the most important factor*: the second factor only asserts itself in the special cases when, owing to the fact that $x = \pm 1$, the first factor loses all or most of its importance. The fact is that since $\lim \{(n+1)/n\}^k = 1$ for all values of k, positive or negative, the limit of the ratio $\phi(n+1)/\phi(n)$ depends only upon x.]

10. Prove that if x is positive $\sqrt[n]{x} \rightarrow 1$, as $n \rightarrow \infty$. [Suppose, e.g., $x > 1$. Then $x, \sqrt{x}, \sqrt[3]{x}, \dots$ is a *decreasing* sequence, and $\sqrt[n]{x} > 1$ for all values of n. Thus $\sqrt[n]{x} \rightarrow l$, where $l \geqq 1$. But if $l > 1$ we could find values of n, as large as we please, for which $\sqrt[n]{x} > l$ or $x > l^n$: and as $l^n \rightarrow +\infty$ as $n \rightarrow \infty$ this is impossible.]

11. $\sqrt[n]{n} \rightarrow 1$. [For $\sqrt[n+1]{(n+1)} < \sqrt[n]{n}$ if $(n+1)^n < n^{n+1}$ or $\{1 + (1/n)\}^n < n$, which is certainly satisfied if $n \geqq 3$ (see § 67 for a proof). Thus $\sqrt[n]{n}$ decreases as n increases from 3 onwards, and as it is always greater than unity it tends to a limit which is greater than or equal to unity. But if $\sqrt[n]{n} \rightarrow l$ $(l > 1)$, $n > l^n$, which is certainly untrue for sufficiently large values of n, since $l^n/n \rightarrow +\infty$ with n (Exs. 7, 8).]

12. $\sqrt[n]{(n!)} \rightarrow +\infty$. [However large G may be, $n! > G^n$ if n is large enough. For if $u_n = G^n/n!$, $u_{n+1}/u_n = G/n+1$, which tends to zero as $n \rightarrow \infty$, so that u_n does the same (Ex. 6).]

67. The limit of $\left(1 + \dfrac{1}{n}\right)^n$. A more difficult case which can be settled by the help of § 64 is given by $\phi(n) = \{1 + (1/n)\}^n$.

We shall prove first that

$$\left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{n-1}\right)^{n-1} \quad \text{..................(1)},$$

i.e. that

$$\left(\frac{n+1}{n}\right)^{n/(n-1)} - 1 > \frac{1}{n-1} \quad \text{..................(2)}.$$

Let

$$\left(\frac{n+1}{n}\right)^{1/(n-1)} = \alpha,$$

so that $\alpha > 1$. Then the inequality (2) may be written in the form

$$\alpha^n - 1 > \frac{n}{n-1}\left(\frac{n+1}{n} - 1\right),$$

or

$$(\alpha^n - 1)/n > (\alpha^{n-1} - 1)/(n-1) \quad \text{..............(3)},$$

or, dividing by the positive factor $\alpha - 1$,

$$(\alpha^{n-1} + \alpha^{n-2} + \dots + 1)/n > (\alpha^{n-2} + \alpha^{n-3} + \dots + 1)/(n-1) \dots(4).$$

Multiplying up and subtracting we obtain

$$(n-1)\,\alpha^{n-1} - \alpha^{n-2} - \alpha^{n-3} - \dots - 1 > 0 \quad \text{.........(5)},$$

and this inequality is evidently true, since $\alpha > 1$. Thus the inequality (1) is established. Hence, by the theorem of § 65, $\{1 + (1/n)\}^n$ tends to a limit, or to $+\infty$, as $n \to \infty$.

But

$$\left(1+\frac{1}{n}\right)^n = 1 + n\,.\,\frac{1}{n} + \frac{n(n-1)}{1\,.\,2}\frac{1}{n^2} + \dots + \frac{n(n-1)\dots(n-n+1)}{1\,.\,2\dots n}\frac{1}{n^n},$$

by the binomial theorem; and so

$$\left(1+\frac{1}{n}\right)^n < 1 + 1 + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \dots + \frac{1}{1\,.\,2\,.\,3\dots n}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} < 3.$$

Thus $\{1 + (1/n)\}^n$ cannot tend to $+\infty$, and so

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,$$

where e is a number such that $2 < e \leqq 3$. We shall have a great deal to do with this number e later on.

68. The limit of $n(\sqrt[n]{\beta} - 1)$. We proved above that if $a > 1$

$$(a^n - 1)/n > (a^{n-1} - 1)/(n-1).$$

Let $a^{n(n-1)} = \beta$. Then $\beta > 1$, and the inequality may be written in the form

$$(n-1)(\sqrt[n-1]{\beta} - 1) > n(\sqrt[n]{\beta} - 1).$$

Thus, if $\phi(n)=n(\sqrt[n]{\beta}-1)$, $\phi(n)$ *decreases* steadily as n increases. Also $\phi(n)$ is obviously always positive. Hence $\phi(n)$ tends to a limit l as $n\to\infty$, and $l\geqq 0$.

Moreover $\phi(n)>1-(1/\beta)$ for all values of n. For the inequality $n(\sqrt[n]{\beta}-1)>1-(1/\beta)$ becomes, if we put γ^n for β, $n\gamma^n(\gamma-1)>\gamma^n-1$, or

$$n\gamma^n>\gamma^{n-1}+\gamma^{n-2}+\dots+1,$$

which is obviously true, since $\gamma>1$. *Hence*

$$\lim_{n\to\infty} n(\sqrt[n]{\beta}-1)=f(\beta),$$

where $f(\beta)$ is a function of β, and $f(\beta)\geqq 1-(1/\beta)$ for all values of $\beta>1$.

Next suppose $\beta<1$, and let $\beta=1/\gamma$; then $n(\sqrt[n]{\beta}-1)=-n(\sqrt[n]{\gamma}-1)/\sqrt[n]{\gamma}$. As $n\to\infty$, $n(\sqrt[n]{\gamma}-1)\to f(\gamma)$, by what precedes. Also (Ex. XXX. 10)

$$\sqrt[n]{\gamma}\to 1.$$

Hence if $\beta=(1/\gamma)<1$,

$$n(\sqrt[n]{\beta}-1)\to -f(\gamma).$$

Finally, if $\beta=1$, $n(\sqrt[n]{\beta}-1)=0$ for all values of n.

Thus we arrive at the result: *the limit*

$$\lim n(\sqrt[n]{\beta}-1)$$

defines a function of β for all positive values of β. This function $f(\beta)$ possesses the properties

$$f(1/\beta)=-f(\beta),\quad f(1)=0,$$

and is positive or negative according as $\beta \gtrless 1$.

Later on we shall be able to identify this function as the *Napierian logarithm* of β.

Example. Prove that $f(\alpha\beta)=f(\alpha)+f(\beta)$. [Use the equations

$$f(\alpha\beta)=\lim n(\sqrt[n]{\alpha\beta}-1)=\lim\{n(\sqrt[n]{\alpha}-1)\sqrt[n]{\beta}+n(\sqrt[n]{\beta}-1)\}.]$$

69. Infinite Series. Suppose that $u(n)$ is any function of n defined for all values of n. If we add up the values of $u(\nu)$ for $\nu=1, 2, \dots n$ we obtain another function of n, viz.

$$s(n)=u(1)+u(2)+\dots+u(n),$$

also defined for all values of n. It is generally most convenient to alter our notation slightly and write this equation in the form

$$s_n=u_1+u_2+\dots+u_n,$$

or, more shortly,

$$s_n=\sum_{\nu=1}^{n} u_\nu.$$

Now suppose that s_n tends to a limit s when n tends to ∞, i.e. that

$$\lim_{n\to\infty}\sum_{\nu=1}^{n} u_\nu=s.$$

This equation is usually written in one of the forms

$$\sum_{\nu=1}^{\infty} u_\nu = s, \quad u_1 + u_2 + u_3 + \dots = s,$$

the dots denoting the indefinite continuance of the series of *u*'s.

The meaning of the above equations, expressed roughly, is that by adding more and more of the *u*'s together we get nearer and nearer to the limit s. More precisely, if any small positive number ϵ is chosen, we can choose n_0 so that the sum of the first n_0 or any greater number of terms lies between $s - \epsilon$ and $s + \epsilon$; or in symbols

$$s - \epsilon < s_n < s + \epsilon,$$

if $n \geqq n_0$.

In these circumstances we shall call the series

$$u_1 + u_2 + \dots,$$

a **convergent infinite series,** and we shall call s the *sum* of the series, or the sum of *all* the terms of the series.

Thus to say that the series $u_0 + u_1 + \dots$ *converges and has the sum s,* or *converges to the sum s* or simply *converges to s,* is merely another way of stating that the sum $s_n = u_0 + u_1 + \dots + u_n$ of the first n terms tends to the limit s as $n \rightarrow \infty$, and the consideration of such infinite series introduces no new ideas beyond those with which the early part of this chapter should already have made the reader familiar. In fact the sum s_n is merely a function $\phi(n)$, such as we have been considering, expressed in a particular form. And any function $\phi(n)$ may be expressed in this form, by writing

$$\phi(n) = \phi(0) + [\phi(1) - \phi(0)] + \dots + [\phi(n) - \phi(n-1)].$$

It is sometimes convenient to say that $\phi(n)$ *converges to the limit l,* say, as $n \rightarrow \infty$. The use of the phrase 'converges' instead of 'tends to' is of course suggested by the phraseology usually employed in speaking of infinite series.

If $s_n \rightarrow +\infty$ or to $-\infty$ we shall say that the series $u_0 + u_1 + \dots$ is **divergent** or, *diverges to* $+\infty$, or $-\infty$, as the case may be. These phrases too may be applied to any function $\phi(n)$—e.g. if $\phi(n) \rightarrow +\infty$ we may say that $\phi(n)$ *diverges to* $+\infty$. If s_n does not tend to a limit or to $+\infty$ or to $-\infty$ it oscillates finitely or infinitely: in this case we say that the series $u_0 + u_1 + \dots$ oscillates finitely or infinitely.

70. General theorems concerning infinite series. When we are dealing with infinite series we shall constantly have occasion to use the following general theorems.

(1) If $u_1 + u_2 + \dots$ is convergent, and has the sum s, then $a + u_1 + u_2 + \dots$ is convergent and has the sum $a + s$. Similarly $a + b + c + \dots + k + u_1 + u_2 + \dots$ is convergent and has the sum $a + b + c + \dots + k + s$.

(2) If $u_1 + u_2 + \dots$ is convergent and has the sum s, then $u_{m+1} + u_{m+2} + \dots$ is convergent and has the sum

$$s - u_1 - u_2 - \dots - u_m.$$

(3) If any series considered in (1) or (2) diverges or oscillates so do the others.

(4) If $u_1 + u_2 + \dots$ is convergent and has the sum s, then $au_1 + au_2 + \dots$ is convergent and has the sum as.

(5) If the first series considered in (4) diverges or oscillates so does the second, unless $a = 0$.

(6) If $u_1 + u_2 + \dots$ and $v_1 + v_2 + \dots$ are both convergent the series $(u_1 + v_1) + (u_2 + v_2) + \dots$ is convergent and its sum is the sum of the first two series.

All these theorems are almost obvious and may be proved at once from the definitions or by applying the results of §§ 56–60 to the sum $s_n = u_1 + u_2 + \dots + u_n$.

(7) *If* $u_1 + u_2 + \dots$ *is convergent, then* $\lim u_n = 0$.

For $u_n = s_n - s_{n-1}$, and s_n and s_{n-1} have the same limit s. Hence $\lim u_n = s - s = 0$.

The reader may be tempted to think that the converse of the theorem is true and that if $\lim u_n = 0$ the series Σu_n must be convergent. That this is not the case is easily seen from an example. Let the series be

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \dots$$

so that $u_n = 1/n$. The sum of the first four terms is

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} > 1 + \tfrac{1}{2} + \tfrac{2}{4} = 1 + \tfrac{1}{2} + \tfrac{1}{2}.$$

The sum of the next four terms is $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$; the sum of the next eight terms $> \frac{8}{16} = \frac{1}{2}$, and so on. The sum of the first

$$4 + 4 + 8 + 16 + \dots + 2^n = 2^{n+1}$$

terms is greater than

$$2 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \dots + \tfrac{1}{2} = \tfrac{1}{2}(n + 3),$$

and this increases beyond all limit with n: hence the series diverges to $+\infty$.

(8) If $u_1 + u_2 + u_3 + \ldots$ is convergent, so is any series formed by grouping the terms in brackets in any way to form new single terms, as e.g. in $(u_1 + u_2 + u_3) + u_4 + (u_5 + u_6) + \ldots$, and the sums of the two series are the same.

Here again the converse is not true. Thus, e.g. $1 - 1 + 1 - 1 + \ldots$ oscillates, while $(1 - 1) + (1 - 1) + \ldots$ or $0 + 0 + 0 + \ldots$ converges to 0.

(9) *If every term u_n is positive* (*or zero*) *the series Σu_n must either converge or diverge to $+\infty$. If it converges its sum must be positive* (unless all the terms are zero, when of course its sum is zero).

For s_n is an increasing function of n, according to the definition of § 64, and we can apply the results of that section to s_n.

(10) *If every term u_n is positive* (*or zero*) *the necessary and sufficient condition that the series Σu_n should be convergent is that it should be possible to find a number G such that the sum of any number of terms is less than G, and if G can be so found the sum of the series is not greater than G.*

This also follows at once from § 64. It is perhaps hardly necessary to point out that the theorem is not true if the condition that every u_n is positive is not fulfilled. For example

$$1 - 1 + 1 - 1 + \ldots$$

obviously oscillates, s_n being alternately equal to $+1$ and to 0.

(11) *If $u_1 + u_2 + \ldots$, $v_1 + v_2 + \ldots$ are two series of positive* (*or zero*) *terms, and the second series is convergent, and if $u_n \leqq v_n$ for all values of n, then the first series is also convergent, and its sum is less than or equal to that of the second.*

For, if $v_1 + v_2 + \ldots = t$, $v_1 + v_2 + \ldots + v_n \leqq t$, for all values of n, and so $u_1 + u_2 + \ldots + u_n \leqq t$; which proves the theorem.

Conversely, if Σu_n is divergent, and $v_n \geqq u_n$, then Σv_n is divergent.

71. The infinite geometrical series. We shall now consider the 'geometrical' series, whose general term is $u_n = r^{n-1}$. In this case

$$s_n = 1 + r + r^2 + \ldots + r^{n-1} = (1 - r^n)/(1 - r),$$

except in the special case in which $r = 1$, when

$$s_n = 1 + 1 + \ldots + 1 = n.$$

In the last case $s_n \rightarrow +\infty$. In the general case s_n will tend to a limit if and only if r^n does so. Referring to the results of § 66 we see that *the series* $1+r+r^2+\dots$ *is convergent and has the sum* $1/(1-r)$ *if and only if* $-1<r<1$.

If $r \geqq 1$, $s_n \geqq n$, and so $s_n \rightarrow +\infty$; i.e. the series diverges to $+\infty$.

If $r=-1$, $s_n=1$ or 0 as n is odd or even: i.e. s_n oscillates finitely. If $r<-1$, s_n oscillates infinitely. Thus, to sum up, *the series* $1+r+r^2+\dots$ *diverges to* $+\infty$ *if* $r \geqq 1$, *converges to* $1/(1-r)$ *if* $-1<r<1$, *oscillates finitely if* $r=-1$, *and oscillates infinitely if* $r<-1$.

Examples XXXI. 1. **Recurring decimals.** The commonest example of an infinite geometric series is given by an ordinary recurring decimal. Consider for example the decimal $\cdot 217\dot{1}\dot{3}$. This stands, according to the ordinary rules of arithmetic, for

$$\frac{2}{10}+\frac{1}{10^2}+\frac{7}{10^3}+\frac{1}{10^4}+\frac{3}{10^5}+\frac{1}{10^6}+\frac{3}{10^7}+\dots=\frac{217}{1000}+\frac{13}{10^5}\Big/\left(1-\frac{1}{10^2}\right)=\frac{2687}{12375}.$$

The reader should consider where and how any of the general theorems of § 70 have been used in this reduction.

2. Show that in general

$$\cdot a_1 a_2 \dots a_m \dot{a}_1 a_2 \dots \dot{a}_n = \frac{a_1 a_2 \dots a_m a_1 \dots a_n - a_1 a_2 \dots a_n}{99\dots 900\dots 0},$$

the denominator containing n 9's and m 0's.

3. Show that a pure recurring decimal is always equal to a proper fraction whose denominator does not contain 2 or 5 as a factor.

4. A decimal with m non-recurring and n recurring decimal figures is equal to a proper fraction whose denominator is divisible by 2^m or 5^m but by no higher power of either. [For the decimal is converted into the sum of an integer and a pure recurring decimal by multiplication by 10^m, but not by multiplication by any lower power of 10.]

5. The converses of Exs. 3, 4 are also true, but their proof depends on Fermat's Theorem in the Theory of Numbers. If $r=p/q$, and q is prime to 10, it is known that we can find n so that 10^n-1 is divisible by q. Hence r may be expressed in the form $P/(10^n-1)$ or in the form

$$\frac{P}{10^n}+\frac{P}{10^{2n}}+\dots$$

i.e. as a pure recurring decimal with n figures. But if $q=2^\alpha 5^\beta Q$, where Q is prime to 10, and m is the greater of α and β, $10^m r$ has a denominator prime to 10, and is therefore expressible as the sum of an integer and a pure recurring decimal. But this is not true of $10^\mu r$, for any value of μ less than m; hence the decimal for r has exactly m non-recurring figures.

6. To the results of Exs. 2—5 we must add that of Ex. I. 4. Finally, if we observe that

$$\cdot\dot{9} = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \dots = 1,$$

we see that every terminating decimal can also be expressed as a mixed recurring decimal whose recurring part is composed entirely of 9's. For example, $\cdot 217 = \cdot 216\dot{9}$. Thus every proper fraction can be expressed as a recurring decimal, and conversely.

7. **Decimals in general. The expression of irrational numbers as non-recurring decimals.** Any decimal, whether recurring or not, corresponds to a definite number between 0 and 1. For the decimal $\cdot a_1 a_2 a_3 a_4 \dots$ stands for the series

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \dots .$$

Since all the digits a_r are positive the sum s_n of the first n terms of this series increases with n: also it is certainly less than $\cdot\dot{9}$ or 1. Hence s_n tends to a limit between 0 and 1.

Moreover no two decimals can correspond to the same number (except in the special case noticed in Ex. 6). For suppose that $\cdot a_1 a_2 a_3 \dots$, $\cdot b_1 b_2 b_3 \dots$ are two decimals which agree as far as the figures a_{r-1}, b_{r-1}, while $a_r > b_r$. Then $a_r \geqq b_r + 1 > b_r . b_{r+1} b_{r+2} \dots$ (unless b_{r+1}, b_{r+2}, ... are all 9's), and so

$$\cdot a_1 a_2 \dots a_r a_{r+1} \dots > \cdot b_1 b_2 \dots b_r b_{r+1} \dots .$$

It follows that the expression of a rational fraction as a recurring decimal (Exs. 2—6) is unique. It also follows that every decimal which does *not* recur represents some *irrational* number between 0 and 1. Conversely, any such number can be expressed as such a decimal. For it must lie in one of the intervals

$$0,\ 1/10;\quad 1/10,\ 2/10;\quad \dots;\quad 9/10;\ 1.$$

If it lies in $r/10$, $(r+1)/10$ the first figure is r: by subdividing this interval into 10 parts we can determine the second figure; and so on.

Thus we see that the decimal $1 \cdot 414 \dots$, obtained by the ordinary process for the extraction of $\sqrt{2}$, cannot recur.

8. The decimals $\cdot 1010010001000010 \dots$ and $\cdot 20200200020000020 \dots$, in which the number of zeros between two 1's or 2's increases by one at each stage, represent irrational numbers.

9. The decimal $\cdot 11101010001010 \dots$, in which the nth figure is 1 if n is prime, and zero otherwise, represents an irrational number. [Since the number of primes is infinite the decimal does not terminate. Nor can it recur: for if it did we could determine m and p so that m, $m+p$, $m+2p$, $m+3p$, ... are all prime numbers; and this is absurd, since the series includes $m+mp$.]*

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* All the results of Exs. XXXI. may be extended, with suitable modifications, to decimals in any scale of notation. For a fuller discussion see Bromwich, *Infinite Series*, Appendix I.

Examples XXXII. 1. If $-1<r<1$, the series $r^m+r^{m+1}+\dots$ is convergent and its sum is $1/(1-r)-1-r-\dots-r^{m-1}$ (§ 70, (2)).

2. The series $r^m+r^{m+1}+\dots$ is convergent and its sum is $r^m/(1-r)$ (§ 70, (4)). Verify that the results of Exs. 1 and 2 are in agreement.

3. Prove that the series $1+2r+2r^2+\dots$ is convergent, and that its sum is $(1+r)/(1-r)$, (α) by writing it in the form $-1+2(1+r+r^2+\dots)$, (β) by writing it in the form $1+2(r+r^2+\dots)$, (γ) by adding the two series $1+r+r^2+\dots$, $r+r^2+\dots$. In each case mention which of the theorems of § 70 are used in your proof.

4. Prove that the arithmetic series

$$a+(a+b)+(a+2b)+\dots$$

is always divergent, unless both a and b are zero. Show that if $b \neq 0$ it diverges to $+\infty$ or to $-\infty$ according to the sign of b, while if $b=0$ it diverges to $+\infty$ or $-\infty$ according to the sign of a.

5. What is the sum of the series

$$(1-r)+(r-r^2)+(r^2-r^3)+\dots$$

when the series is convergent? [The series converges only if $-1<r\leqq 1$. Its sum is 1, except when $r=1$, when its sum is 0.]

6. Sum the series $r^2+\dfrac{r^2}{1+r^2}+\dfrac{r^2}{(1+r^2)^2}+\dots$. [The series is always convergent. Its sum is $1+r^2$, except when $r=0$, when its sum is 0.]

7. If we assume that $1+r+r^2+\dots$ is convergent we can prove that its sum is $1/(1-r)$ by means of § 70, (1) and (4). For if $1+r+r^2+\dots=s$,

$$s=1+r(1+r^2+\dots)=1+rs.$$

8. Sum the series $$r+\frac{r}{1+r}+\frac{r}{(1+r)^2}+\dots$$

when it is convergent. [The series is convergent if $-1<1/(1+r)<1$, i.e. if $r<-2$ or if $r>0$, and its sum is $1+r$. It is also convergent for $r=0$, when its sum is 0.]

9. Answer the same question for the series

$$r-\frac{r}{1+r}+\frac{r}{(1+r)^2}-\dots,\qquad r\pm\frac{r}{1-r}+\frac{r}{(1-r)^2}\pm\dots,$$

$$1\pm\frac{r}{1+r}+\left(\frac{r}{1+r}\right)^2\pm\dots,\qquad 1\pm\frac{r}{1-r}+\left(\frac{r}{1-r}\right)^2\pm\dots.$$

10. Consider the convergence of

$$(1+r)+(r^2+r^3)+\dots,\quad (1+r+r^2)+(r^3+r^4+r^5)+\dots,$$

$$1-2r+r^2+r^3-2r^4+r^5+\dots,\quad (1-2r+r^2)+(r^3-2r^4+r^5)+\dots,$$

and find their sums when they are convergent.

11. If a_n is positive and not greater than 1, the series $a_0+a_1r+a_2r^2+\dots$ is convergent for $0\leqq r<1$, and the sum of the series is not greater than $1/(1-r)$.

12. If in addition the series $a_0+a_1+a_2+\dots$ is convergent, the series $a_0+a_1r+a_2r^2+\dots$ is convergent for $0\leqq r\leqq 1$, and its sum is not greater than the lesser of $a_0+a_1+a_2+\dots$ and $1/(1-r)$.

13. The series $$1+\frac{1}{1.2}+\frac{1}{1.2.3}+\dots$$ is convergent. [For $1/(1.2\dots n)<1/2^{n-1}$.]

14. The series
$$1+\frac{1}{1.2}+\frac{1}{1.2.3.4}+\dots,\quad 1+\frac{1}{1.2.3}+\frac{1}{1.2.3.4.5}+\dots$$
are convergent.

15. The general harmonic series
$$\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\dots,$$
where a and b are positive, diverges to $+\infty$.

[For $u_n=1/(a+nb)>1/\{n(a+b)\}$. Now compare with $1+(1/2)+(1/3)+\dots$.]

16. Show that the series
$$(u_0-u_1)+(u_1-u_2)+(u_2-u_3)+\dots$$
is convergent if and only if u_n tends to a limit as $n\to\infty$.

17. If $u_1+u_2+u_3+\dots$ is divergent, so is any series formed by grouping the terms in brackets in any way to form new single terms.

18. Any series, formed by taking a selection of the terms of a convergent series of positive terms, is itself convergent.

72. The representation of functions of a continuous real variable by means of limits. In the preceding sections we have frequently been concerned with limits such as
$$\lim_{n\to\infty}\phi_n(x),$$
and series such as
$$u_1(x)+u_2(x)+\dots=\lim_{n\to\infty}\{u_1(x)+u_2(x)+\dots+u_n(x)\},$$
in which the function of n whose limit we are seeking involves, besides n, another variable x. In such cases the limit is of course a function of x. Thus in § 69 we came across the function
$$f(x)=\lim_{n\to\infty} n(\sqrt[n]{x}-1):$$
and the sum of the geometrical series $1+x+x^2+\dots$ is a function of x, viz. the function which is equal to $1/(1-x)$ if $-1<x<1$ and is undefined for all other values of x.

Many of the apparently 'arbitrary' or 'unnatural' functions considered in Ch. II are capable of a simple representation of this kind, as will appear from the following examples.

Examples XXXIII. 1. $\phi_n(x)=x$. Here n does not appear at all in the expression of $\phi_n(x)$, and $\phi(x)=\lim\phi_n(x)=x$ for all values of x.

2. $\phi_n(x)=x/n$. Here $\phi(x)=\lim\phi_n(x)=0$ for all values of x.

3. $\phi_n(x)=nx$. If $x>0$, $\phi_n(x)\to+\infty$; if $x<0$, $\phi_n(x)\to-\infty$: only for $x=0$ has $\phi_n(x)$ a limit (viz. 0) as $n\to\infty$. Thus $\phi(x)=0$ when $x=0$ and is not defined for any other value of x.

4. $\phi_n(x)=1/nx$, $nx/(nx+1)$.

5. $\phi_n(x)=x^n$. Here $\phi(x)=0$, $(-1<x<1)$; $\phi(x)=1$, $(x=1)$; and $\phi(x)$ is not defined for any other value of x.

6. $\phi_n(x)=x^n(1-x)$. Here $\phi(x)$ differs from the $\phi(x)$ of Ex. 5 in that it is defined and has the value 0 for $x=1$.

7. $\phi_n(x)=x^n/n$. Here $\phi(x)$ differs from the $\phi(x)$ of Ex. 6 in that it is defined and has the value 0 for $x=-1$ as well as $+1$.

8. $\phi_n(x)=x^n/(x^n+1)$. [$\phi(x)=0$, $(-1<x<1)$; $\phi(x)=\frac{1}{2}$, $(x=1)$; $\phi(x)=1$, $(x<-1$ or $x>1)$; and $\phi(x)$ is not defined for $x=-1$.]

9. $\phi_n(x)=x^n/(x^n-1)$, $1/(x^n+1)$, $1/(x^n-1)$, $1/(x^n+x^{-n})$, $1/(x^n-x^{-n})$.

10. $\phi_n(x)=(x^n-1)/(x^n+1)$, $(nx^n-1)/(nx^n+1)$, $(x^n-n)/(x^n+n)$. [In the first case $\phi(x)=1$ if $|x|>1$, $\phi(x)=-1$ if $|x|<1$, $\phi(x)=0$ if $x=1$ and $\phi(x)$ is not defined for $x=-1$. The second and third functions differ from the first in that they are defined both for $x=1$ and $x=-1$: the second has the value 1 and the third the value -1 for both these values of x.]

11. Construct an example in which $\phi(x)=1$, $(|x|>1)$; $\phi(x)=-1$, $(|x|<1)$; and $\phi(x)=0$, $(x=\pm1)$.

12. $\phi_n(x)=x\{(x^{2n}-1)/(x^{2n}+1)\}^2$, $n/(x^n+x^{-n}+n)$.

13. $\phi_n(x)=\{x^n f(x)+g(x)\}/(x^n+1)$. [Here $\phi(x)=f(x)$, $(|x|>1)$; $\phi(x)=g(x)$, $(|x|<1)$; $\phi(x)=\frac{1}{2}\{f(x)+g(x)\}$, $(x=1)$; and $\phi(x)$ is undefined for $x=-1$.]

14. $\phi_n(x)=(2/\pi)\arctan(nx)$. [$\phi(x)=1$, $(x>0)$; $\phi(x)=0$, $(x=0)$; $\phi(x)=-1$, $(x<0)$. This function is important in the Theory of Numbers, and is usually denoted by *sgn* x.]

15. $\phi_n(x)=(1/n)\sin nx\pi$. [$\phi(x)=0$ for all values of x.]

16. $\phi_n(x)=\sin nx\pi$. [$\phi(x)=0$ when x is an integer, and is otherwise undefined.]

17. $\phi_n(x)=(1/n)\cos nx\pi$, $\cos nx\pi$, $a\cos^2 nx\pi+b\sin^2 nx\pi$.

18. If $\phi_n(x)=\sin(n!x\pi)$, $\phi(x)=0$ for all *rational* values of x (Exs. XXVI. 9, XXVII. 8). The consideration of irrational values presents greater difficulties.

19. $\phi_n(x)=(\cos^2 x\pi)^n$. [$\phi(x)=0$ except when x is integral, when $\phi(x)=1$.]

20. $\phi_n(x)=(\sin^2 x\pi)^n$, $(\cos x\pi)^n$, $(\sin x\pi)^n$.

21. $\phi_n(x)=(a\cos^2 x\pi+b\sin^2 x\pi)^n$. [Here $\phi(x)=0$ if $|a\cos^2 x\pi+b\sin^2 x\pi|<1$, $\phi(x)=1$ if $a\cos^2 x\pi+b\sin^2 x\pi=1$, and $\phi(x)$ is otherwise undefined. For what values of x these respective conditions are satisfied depends on the values of a and b. Thus if a and b are both numerically less than unity, $\phi(x)=0$ for *all* values of x. Consider, e.g., the cases $a=b=1$; $a=b=\frac{1}{2}$; $a=b=2$; $a=1$, $b=2$; $a=2$, $b=1$; $a=2$, $b=\frac{1}{2}$.]

22. If $N\geqq 1752$, the number of days in the year N A.D. is

$$\lim\{365+(\cos^2\tfrac{1}{4}N\pi)^n-(\cos^2\tfrac{1}{100}N\pi)^n+(\cos^2\tfrac{1}{400}N\pi)^n\}.$$

73. Limits of Complex functions and series of Complex terms. In this chapter we have, up to the present, concerned ourselves only with real functions of n and series all of whose terms are real. There is however no difficulty in extending our ideas and definitions to the case in which the functions or the terms of the series are complex.

Suppose that $\phi(n)$ is complex and equal to

$$R(n)+iS(n),$$

where $R(n)$, $S(n)$ are real functions of n. Then *if, as $n\to\infty$, $R(n)$ and $S(n)$ converge respectively to limits r and s, we shall say that $\phi(n)$ converges to the limit $r+is$, and write*

$$\lim\phi(n)=r+is.$$

Similarly if u_n is complex and equal to v_n+iw_n we shall say that *the series*

$$u_1+u_2+u_3+\dots$$

is convergent and has the sum $r+is$, if the series

$$v_1+v_2+v_3+\dots,\quad w_1+w_2+w_3+\dots$$

are convergent and have the sums r, s respectively.

To say that $u_1+u_2+u_3+\dots$ is convergent and has the sum $r+is$ is of course the same as to say that the sum

$$s_n=u_1+u_2+\dots+u_n=(v_1+v_2+\dots+v_n)+i(w_1+w_2+\dots+w_n)$$

converges to the limit $r+is$ as $n\to\infty$.

In the case of real functions and series we also gave definitions of *divergence* and *oscillation* (finite or infinite). But in the case of complex functions and series there are so many possibilities—e.g. $R(n)$ may tend to $+\infty$ and $S(n)$ oscillate—that this is hardly worth while. When it is necessary to make further distinctions of this kind, we shall make them by stating the way in which the real or imaginary parts behave when taken separately.

74. The reader will find no difficulty in proving such theorems as the following, which are obvious extensions of theorems already proved for real functions and series.

(1) If $\lim \phi(n) = r + is$, then $\lim \phi(n+p) = r + is$, for any fixed value of p.

(2) If $u_1 + u_2 + \dots$ is convergent and has the sum $r + is$, then $a + b + c + \dots + k + u_1 + u_2 + \dots$ is convergent and has the sum $a + b + c + \dots + k + r + is$, and $u_{p+1} + u_{p+2} + \dots$ is convergent and has the sum $r + is - u_1 - u_2 - \dots - u_p$.

(3) If $\lim \phi(n) = a$ and $\lim \psi(n) = b$, then $\lim \{\phi(n) + \psi(n)\} = a + b$.

(4) If $\lim \phi(n) = a$, $\lim k\phi(n) = ka$.

(5) If $\lim \phi(n) = a$ and $\lim \psi(n) = b$, then $\lim \phi(n)\psi(n) = ab$.

(6) If $u_1 + u_2 + \dots$ converges to the sum a, and $v_1 + v_2 + \dots$ to the sum b, then $(u_1 + v_1) + (u_2 + v_2) + \dots$ converges to the sum $a + b$.

(7) If $u_1 + u_2 + \dots$ converges to the sum a, $ku_1 + ku_2 + \dots$ converges to the sum ka.

(8) If $u_1 + u_2 + u_3 + \dots$ is convergent, then $\lim u_n = 0$.

(9) If $u_1 + u_2 + u_3 + \dots$ is convergent, so is any series formed by grouping the terms in brackets, and the sums of the two series are the same.

As an example, let us prove theorem (5). Let

$$\phi(n) = R(n) + iS(n), \quad \psi(n) = R'(n) + iS'(n), \quad a = r + is, \quad b = r' + is'.$$

Then $$R(n) \to r, \quad S(n) \to s, \quad R'(n) \to r', \quad S'(n) \to s'.$$

But $$\phi(n)\,\psi(n) = RR' - SS' + i(RS' + R'S)$$

and $$RR' - SS' \to rr' - ss', \quad RS' + R'S \to rs' + r's,$$

so that $$\phi(n)\,\psi(n) \to rr' - ss' + i(rs' + r's),$$

i.e. $$\phi(n)\,\psi(n) \to (r + is)(r' + is') = ab.$$

The following theorems are of a somewhat different character.

(10) In order that $\phi(n) = R(n) + iS(n)$ should converge to zero as $n \to \infty$ it is necessary and sufficient that

$$|\phi(n)| = \sqrt{\{R(n)\}^2 + \{S(n)\}^2}$$

should converge to zero.

If $R(n)$ and $S(n)$ both converge to zero it is plain that $\sqrt{(R^2 + S^2)}$ does so. The converse follows from the fact that the numerical value of R or S cannot be greater than $\sqrt{(R^2 + S^2)}$.

(11) More generally, in order that $\phi(n)$ should converge to a limit l it is necessary and sufficient that

$$|\phi(n)-l|$$

should converge to zero.

For $\phi(n)-l$ converges to zero, and we can apply (10).

75. The limit of x^n as $n\to\infty$, x being any complex number. Let us consider the important case in which $\phi(n)=x^n$. This problem has already been discussed for real values of x in § 66.

If $x^n\to l$, $x^{n+1}\to l$, by (1) above. But since $x^{n+1}=x\,.\,x^n$, $x^{n+1}\to xl$, by (4) above; and therefore $l=xl$, which is only possible if (*a*) $l=0$ or (*b*) $x=1$. If $x=1$, $\lim x^n=1$. Apart from this special case the limit, if it exists, can only be zero.

Now if
$$x=r(\cos\theta+i\sin\theta),$$
where r is positive, we know that
$$x^n=r^n(\cos n\theta+i\sin n\theta),$$
so that $|x^n|=r^n$. Thus $|x^n|$ tends to zero if and only if $r<1$; and it follows from (10) of the last paragraph that
$$\lim x^n=0,$$
if and only if $r<1$. In no other case does x^n converge to a limit, except when $x=1$ and $x^n\to 1$.

76. The geometric series $1+x+x^2+\dots$, when x is complex. Since
$$s_n=1+x+x^2+\dots+x^{n-1}=(1-x^n)/(1-x),$$
unless $n=1$, when the value of s_n is n, it follows that *the series $1+x+x^2+\dots$ is convergent if and only if $r=|x|<1$. And its sum when convergent is $1/(1-x)$.*

Thus if $x=r(\cos\theta+i\sin\theta)=r\operatorname{Cis}\theta$, and $r<1$,
$$1+x+x^2+\dots=1/(1-r\operatorname{Cis}\theta),$$
or
$$1+r\operatorname{Cis}\theta+r^2\operatorname{Cis}2\theta+\dots=1/(1-r\operatorname{Cis}\theta)$$
$$=(1-r\cos\theta+ir\sin\theta)/(1-2r\cos\theta+r^2),$$
or, separating the real and imaginary parts,
$$1+r\cos\theta+r^2\cos 2\theta+\dots=(1-r\cos\theta)/(1-2r\cos\theta+r^2),$$
$$r\sin\theta+r^2\sin 2\theta+\dots=r\sin\theta/(1-2r\cos\theta+r^2),$$
provided $r<1$. If we change θ into $\theta+\pi$ we see that these results hold also for negative values of r numerically less than 1. Thus they hold for $-1<r<1$.

Examples XXXIV. 1. Prove directly that $\phi(n)=r^n \cos n\theta$ converges to 0 if $r<1$ and to 1 if $r=1$, $\theta=0$. Prove further that if $r=1$, $\theta\neq 0$ it oscillates finitely, if $r>1$, $\theta=0$ it diverges to $+\infty$ and if $r>1$, $\theta\neq 0$ it oscillates infinitely.

2. Establish a similar series of results for $\phi(n)=r^n \sin n\theta$.

3. Prove (as for the case of a real x in Ex. XXXII. 7) that if $1+x+x^2+\dots$ converges its sum can only be $1/(1-x)$.

4. Prove that

$$\begin{aligned} x^m+x^{m+1}+\dots &= x^m/(1-x), \\ x^m-x^{m+1}+\dots &= x^m/(1+x), \\ x^m+2x^{m+1}+2x^{m+2}+\dots &= x^m(1+x)/(1-x), \\ x^m-2x^{m+1}+2x^{m+2}-\dots &= x^m(1-x)/(1+x), \end{aligned}$$

if and only if $|x|<1$. Which of the theorems of § 74 do you use?

5. Let (in the notation of Chap. III, §§ 25 *et seq.*) $\overline{P_0P_1}=1$, $\overline{P_1P_2}=x$, $\overline{P_2P_3}=x^2\dots$ where $x=r\operatorname{Cis}\theta$. Plot the points P_0, P_1, P_2, ..., and show how the figure obtained indicates the result of § 76. Prove that, if $r<1$, the point P_n, where n is large, is very near to the point

$$(1-r\cos\theta)/(1-2r\cos\theta+r^2), \quad r\sin\theta/(1-2r\cos\theta+r^2).$$

6. Prove that, if $-1<r<1$,

$$1+2r\cos\theta+2r^2\cos 2\theta+\dots=(1-r^2)/(1-2r\cos\theta+r^2).$$

7. The series $\quad 1+\{x/(1+x)\}+\{x/(1+x)\}^2+\dots$

converges to the sum $1\Big/\left(1-\dfrac{x}{1+x}\right)=1+x$ if $|x/(1+x)|<1$. Show that this is equivalent to the assertion that x has a real part greater than $-\frac{1}{2}$.

8. Determine similarly the regions of values of x for which the series, obtained by writing x for r in Ex. XXXII. 9, are convergent, and find their sums when they are convergent.

MISCELLANEOUS EXAMPLES ON CHAPTER IV.

1. The function $\phi(n)$ takes for $n=0, 1, 2, \dots$ the values 1, 0, 0, 0, 1, 0, 0, 0, 1, ... Express $\phi(n)$ in terms of n by a formula which does not involve trigonometrical functions. [$\phi(n)=\frac{1}{4}\{1+(-1)^n+i^n+(-i)^n\}$.]

2. If $\phi(n)$ steadily increases, and $\psi(n)$ steadily decreases, as n tends to ∞, and if $\psi(n)>\phi(n)$ for all values of n, then both $\phi(n)$ and $\psi(n)$ tend to limits, and $\lim\phi(n)\leqq\lim\psi(n)$. [This is an intermediate corollary from § 64.]

3. Prove that if

$$\phi(n)=\left(1+\frac{1}{n}\right)^n, \quad \psi(n)=\left(1-\frac{1}{n}\right)^{-n},$$

then $\phi(n+1)>\phi(n)$ and $\psi(n+1)<\psi(n)$.

[The first inequality has already been proved in § 67; the second may be proved similarly.]

4. Prove also that $\psi(n) > \phi(n)$ for all values of n: and deduce (by means of the preceding examples) that both $\phi(n)$ and $\psi(n)$ tend to limits as n tends to ∞ *.

5. If $\binom{m}{n} = \dfrac{m(m-1)\dots(m-n+1)}{1.2\dots n}$, m not being a positive integer, and $-1 < x < 1$, then $u_n = \binom{m}{n} x^n \rightarrow 0$ as $n \rightarrow \infty$.

[For $u_{n+1}/u_n = \{(m-n)/(n+1)\}\, x \rightarrow -x$. Now apply Ex. XXX. 6.]

6. The arithmetic mean of the products of all distinct pairs of positive integers, whose sum is n, is denoted by S_n. Show that $\lim (S_n/n^2) = 1/6$. (*Math. Trip.* 1903.)

7. If $x_1 = \frac{1}{2}\{x + (A/x)\}$, $x_2 = \frac{1}{2}\{x_1 + (A/x_1)\}$, and so on, x and A being positive, prove that $\lim x_n = \sqrt{A}$.

$$\left[\text{Prove first that } \frac{x_n - \sqrt{A}}{x_n + \sqrt{A}} = \left(\frac{x - \sqrt{A}}{x + \sqrt{A}}\right)^{2^n}.\right]$$

8. If $\phi(n)$ is a positive integer for all values of n, and tends to ∞ with n, then $x^{\phi(n)} \rightarrow 0$ or $+\infty$ according as $0 < x < 1$ or $x > 1$. Discuss the behaviour of $x^{\phi(n)}$, as $n \rightarrow \infty$, for other values of x.

9†. If a_n increases (decreases) steadily as n increases, the same is true of $(a_1 + a_2 + \dots + a_n)/n$.

10. If $x_{n+1} = \sqrt{(k + x_n)}$, and k and x_1 are positive, the sequence $x_1, x_2, x_3, \dots$ is an increasing or decreasing sequence according as x_1 is less than or greater than a, the positive root of the equation $x^2 = x + k$; and in either case $x_n \rightarrow a$ as $n \rightarrow \infty$.

11. If $x_{n+1} = k/(1 + x_n)$, and k and x_1 are positive, the sequence $x_1, x_2, x_3, \dots$ is an increasing or decreasing sequence, according as x_1 is less than or greater than a, the positive root of the equation $x^2 + x = k$; and in either case $x_n \rightarrow a$ as $n \rightarrow \infty$.

12. Suppose that $f(x)$ is a positive and increasing function of x such that the equation $x = f(x)$ has just one positive root a. Show, graphically or otherwise, that if $x_1 > 0$ and $x_{n+1} = f(x_n)$ then the sequence $x_1, x_2, \dots$ has the limit a as $n \rightarrow \infty$.

Discuss the case in which the equation $x = f(x)$ has several positive roots.

13. If x_1, x_2 are positive and $x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$, the sequences $x_1, x_3, x_5, \dots$ and $x_2, x_4, x_6, \dots$ are one a decreasing and one an increasing sequence, and their common limit is $\frac{1}{3}(a_1 + 2a_2)$.

14. Draw a graph of the function y defined by the equation

$$y = \lim_{n \rightarrow \infty} \frac{x^{2n} \sin \frac{1}{2}\pi x + x^2}{x^{2n} + 1}.$$

(*Math. Trip.* 1901.)

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* A proof that $\lim \{\psi(n) - \phi(n)\} = 0$, and that therefore each function tends to the limit e, will be found in Chrystal's *Algebra*, vol. ii, p. 78. We shall however prove this in Ch. IX by a different method.

† Exs. 9—13 are taken from Bromwich's *Infinite Series*.

15. The function $$y = \lim_{n\to\infty} \frac{1}{1+n\sin^2 \pi x}$$ is equal to 0 except when x is an integer, and then equal to 1. The function
$$y = \lim_{n\to\infty} \frac{\psi(x) + n\phi(x)\sin^2 \pi x}{1+n\sin^2 \pi x}$$
is equal to $\phi(x)$ unless x is an integer, and then equal to $\psi(x)$.

16. Show that the graph of the function
$$y = \lim_{n\to\infty} \frac{x^n \phi(x) + x^{-n}\psi(x)}{x^n + x^{-n}}$$
is composed of parts of the graphs of $\phi(x)$ and $\psi(x)$, together with (as a rule) one isolated point. Is y defined for (*a*) $x=1$, (*b*) $x=-1$, (*c*) $x=0$?

17. Prove that the function y, which is equal to 0 when x is rational and to 1 when x is irrational, may be represented in the form
$$y = \lim_{m\to\infty} sgn \{\sin^2(m!\,\pi x)\}$$
where, as in Ex. XXXIII. 14, $sgn\, z = \lim_{n\to\infty} (2/\pi) \arctan (nz)$.

[If x is rational, $\sin^2(m!\,\pi x)$, and therefore $sgn\{\sin^2(m!\,\pi x)\}$ is equal to zero from a certain value of m onwards: if x is irrational, $\sin^2(m!\,\pi x)$ is always positive, and so $sgn\{\sin^2(m!\,\pi x)\}$ is always equal to 1.]

Prove that y may also be represented in the form
$$1 - \lim_{m\to\infty} [\lim_{n\to\infty} \{\cos(m!\,\pi x)\}^{2n}].$$

18. Sum the series
$$\sum_1^\infty \frac{1}{\nu(\nu+1)}, \quad \sum_1^\infty \frac{1}{\nu(\nu+1)\dots(\nu+k)}.$$
[Since
$$\frac{1}{\nu(\nu+1)\dots(\nu+k)} = \frac{1}{k}\left\{\frac{1}{\nu(\nu+1)\dots(\nu+k-1)} - \frac{1}{(\nu+1)(\nu+2)\dots(\nu+k)}\right\},$$
we find $$\sum_1^n \frac{1}{\nu(\nu+1)\dots(\nu+k)} = \frac{1}{k}\left\{\frac{1}{1.2\dots k} - \frac{1}{(n+1)(n+2)\dots(n+k)}\right\}$$
and so $$\sum_1^\infty \frac{1}{\nu(\nu+1)\dots(\nu+k)} = \frac{1}{k(k!)}.]$$

19. If $|x|<|a|$, $$\frac{L}{x-a} = -\frac{L}{a}\left(1+\frac{x}{a}+\frac{x^2}{a^2}+\dots\right);$$
while if $|x|>|a|$, $$\frac{L}{x-a} = \frac{L}{x}\left(1+\frac{a}{x}+\frac{a^2}{x^2}+\dots\right).$$

20. **Expansion of $(Ax+B)/(ax^2+2bx+c)$ in powers of x.** Let α, β be the roots of $ax^2+2bx+c=0$, so that $ax^2+2bx+c \equiv a(x-\alpha)(x-\beta)$. It is easy to verify that (unless $\alpha=\beta$)
$$\frac{Ax+B}{ax^2+2bx+c} \equiv \frac{1}{a(\alpha-\beta)}\left(\frac{A\alpha+B}{x-\alpha} - \frac{A\beta+B}{x-\beta}\right).$$

We shall suppose A, B, a, b, c all real. Then there are two cases, according as $b^2 \gtrless ac$.

(1) If $b^2>ac$, the roots α, β are real and distinct. If $|x|$ is less than either $|\alpha|$ or $|\beta|$ we can expand $1/(x-\alpha)$ and $1/(x-\beta)$ in ascending powers of x (Ex. 19). If $|x|$ is greater than either $|\alpha|$ or $|\beta|$ we must expand in descending powers of x; while if $|x|$ lies between $|\alpha|$ and $|\beta|$ one fraction must be expanded in ascending and one in descending powers of x. The reader should write down the actual results. If $|x|$ is equal to $|\alpha|$ or $|\beta|$ no such expansion is possible.

(2) If $b^2<ac$ the roots are conjugate complex numbers (Chap. III, § 34) and we can write

$$\alpha=\rho \operatorname{Cis} \phi, \qquad \beta=\rho \operatorname{Cis}(-\phi),$$

where $\rho^2=\alpha\beta=c/a$, $\rho \cos \phi=\frac{1}{2}(\alpha+\beta)=-b/a$, so that $\cos \phi=-\sqrt{(b^2/ac)}$, $\sin \phi=\sqrt{\{1-(b^2/ac)\}}$.

If $|x|<\rho$ each fraction may be expanded in ascending powers of x. The coefficient of x^n will be found to be

$$\{A\rho \sin n\phi+B \sin(n+1)\phi\}/a\rho^{n+1} \sin \phi.$$

If $|x|>\rho$ we obtain a similar expansion in descending powers, while if $|x|=\rho$ no such expansion is possible.

21. Show that, if $|x|<1$,

$$1+2x+3x^2+\dots+(n+1)x^n+\dots=1/(1-x)^2.$$

$\left[\text{The sum to } n \text{ terms is } \dfrac{1-x^n}{(1-x)^2}-\dfrac{nx^n}{1-x}.\right]$

22. Expand $L/(x-a)^2$ in powers of x, ascending or descending according as $|x|<|a|$ or $|x|>|a|$.

23. Show that if $b^2=ac$ and $|ax|<|b|$

$$(Ax+B)/(ax^2+2bx+c)=\sum_1^\infty p_n x^n,$$

where $p_n=\{(-a)^n/b^{n+2}\}\{(n+1)aB-nbA\}$, and find the corresponding expansion, in descending powers of x, which holds when $|ax|>|b|$.

24. Verify the result of Ex. 20 in the case of the fraction $1/(1+x^2)$. [We have $1/(1+x^2)=\Sigma x^n \sin\{\frac{1}{2}(n+1)\pi\}=1-x^2+x^4-\dots$.]

25. Prove that, if $|x|<1$, $1/(1+x+x^2)=(2/\sqrt{3})\sum_0^\infty x^n \sin\{\frac{2}{3}(n+1)\pi\}$.

26. Expand $(1+x)/(1+x^2)$, $(1+x^2)/(1+x^3)$ and $(1+x+x^2)/(1+x^4)$ in ascending powers of x. For what values of x do your results hold?

27. If $a/(a+bx+cx^2)=1+p_1x+p_2x^2+\dots$ then

$$1+p_1^2x+p_2^2x^2+\dots=\frac{a+cx}{a-cx}\,\frac{a^2}{a^2-(b^2-2ac)x+c^2x^2}.$$

(*Math. Trip.* 1900.)

28. If $\lim\limits_{x\to\infty} s_n = l$, then

$$\lim_{n\to\infty} (s_1+s_2+\dots+s_n)/n = l.$$

[Let $s_n = l + t_n$. Then we have to prove that $(t_1+t_2+\dots+t_n)/n$ tends to zero if t_n does so.

We divide the numbers $t_1, t_2, \dots t_n$ into two sets $t_1, t_2, \dots, t_p$ and $t_{p+1}, t_{p+2}, \dots, t_n$. Here we suppose that p is a function of n which tends to ∞ with n, but *more slowly than* n, so that $p\to\infty$ but $p/n\to 0$: e.g. we might suppose p to be the integral part of $\sqrt{n}$.

Let ϵ be any positive number. However small ϵ may be, we can choose n_0 so that $t_{p+1}, t_{p+2}, \dots, t_n$ are all numerically less than $\frac{1}{2}\epsilon$ when $n \geqq n_0$, and so

$$|(t_{p+1}+t_{p+2}+\dots+t_n)/n| < \tfrac{1}{2}\epsilon\,(n-p)/n < \tfrac{1}{2}\epsilon.$$

But, if A is the greatest of the moduli of all the numbers $t_1, t_2, \dots$, we have also

$$|(t_1+t_2+\dots+t_p)/n| < pA/n,$$

and, if n_0 is large enough, this will also be less than $\frac{1}{2}\epsilon$ when $n \geqq n_0$, since $p/n\to 0$ as $n\to\infty$. Thus, if n_0 is large enough,

$$|(t_1+t_2+\dots+t_n)/n| \leqq |(t_1+t_2+\dots+t_p)/n| + |(t_{p+1}+\dots+t_n)/n| < \epsilon,$$

when $n \geqq n_0$: which proves the theorem.

The reader, if he desires to become expert in dealing with questions about limits, should study the argument here given with great care. It is very often necessary, in proving the limit of some given expression to be zero, to split it into two parts which have to be proved to have the limit zero in slightly different ways. When this is the case the proof is never very easy.

The point of the proof is this: we have to prove that $(t_1+t_2+\dots+t_n)/n$ is small when n is large, the t's being small when their suffixes are large. We split up the terms in the bracket into two groups. The terms in the first group are not all small, but their *number* is small compared with n. The number in the second group is *not* small compared with n, but the terms are all small, and their number at any rate less than n, so that their sum is small compared with n. Hence each of the parts into which $(t_1+t_2+\dots+t_n)/n$ has been divided is small when n is large.]

29. If $\phi(n)-\phi(n-1)\to l$ as $n\to\infty$ then also $\phi(n)/n\to l$.

[If we put $\phi(n) = s_1+s_2+\dots+s_n$, we have $\phi(n)-\phi(n-1) = s_n$, and the theorem reduces to that proved in the last example.]

30. If $s_n = \frac{1}{2}\{1-(-1)^n\}$, so that s_n is equal to 1 or 0 according as n is odd or even, then $(s_1+s_2+\dots+s_n)/n\to\frac{1}{2}$ as $n\to\infty$.

[This example proves that the converse of 28 is *not* true: for s_n oscillates as $n\to\infty$.]

31. Let c_n, s_n denote the sums of the first n terms of the series

$$\tfrac{1}{2}+\cos\theta+\cos 2\theta+\dots,\quad \sin\theta+\sin 2\theta+\dots.$$

Prove that

$$\lim\,(c_1+c_2+\dots+c_n)/n = 0,\quad \lim\,(s_1+s_2+\dots+s_n)/n = \tfrac{1}{2}\cot\tfrac{1}{2}\theta.$$

CHAPTER V.

LIMITS OF FUNCTIONS OF A CONTINUOUS VARIABLE. CONTINUOUS AND DISCONTINUOUS FUNCTIONS.

77. Limits as x tends to ∞. We shall now return to functions of a continuous real variable. We shall denote the typical such function by $\phi(x)$. We suppose x to assume successively all values corresponding to points on our fundamental straight line L, starting from some definite point on the line and progressing always to the right. This variation of x, like the corresponding variation of n (Chap. IV, § 48), is often conveniently thought of as taking place in time. In these circumstances we say that x *tends to* ∞, and write $x \to \infty$. The only difference between the 'tending of n to ∞' discussed in the last chapter, and this 'tending of x to ∞,' is that x varies through all values as it tends to ∞, i.e. that the point P which corresponds to x coincides in turn with every point of L to the right of its initial position, whereas n tends to ∞ by a series of jumps. We can express this distinction by saying that x tends *continuously* to ∞.

As we explained at the beginning of the last chapter, there is a very close correspondence between functions of x and functions of n. Every function of n may be regarded as a selection from the values of a function of x. In the last chapter we discussed the peculiarities which may characterise the behaviour of a function $\phi(n)$ as n tends to ∞. Now we are concerned with the same problem for a function $\phi(x)$: and the definitions and theorems to which we are led are practically repetitions of those of the last chapter. Thus corresponding to Def. 1 of § 51 we have:

DEFINITION 1. *The function $\phi(x)$ is said to tend to the limit l as x tends to ∞ if, when any positive number ϵ, however small, is* *assigned, a value X can be chosen such that, for all values of x equal to or greater than X, $\phi(x)$ differs from l by less than ϵ, i.e. if*

$$|\phi(x) - l| < \epsilon \qquad (x \geqq X).$$

When this is the case we may write

$$\lim_{x \to \infty} \phi(x) = l, \quad \underset{(x \to \infty)}{\phi(x) \to l},$$

or, when there is no risk of ambiguity, simply $\lim \phi(x) = l$, or $\phi(x) \to l$. Similarly we have:

DEFINITION 2. *The function $\phi(x)$ is said to tend to $+\infty$ with x if when any number G, however large, is assigned, we can choose X so that*

$$\phi(x) > G \qquad (x \geqq X).$$

We then write $\qquad \phi(x) \to +\infty$.

Similarly we define $\phi(x) \to -\infty$. Finally we have:

DEFINITION 3. *If the conditions of none of the two preceding definitions are satisfied $\phi(x)$ is said to oscillate as x tends to ∞. If, for all values of x, $|\phi(x)|$ is less than some constant K, $\phi(x)$ is said to oscillate finitely; otherwise infinitely.*

The reader will remember that in the last chapter we considered very carefully various less formal ways of expressing the facts represented by the equations $\phi(n) \to l$, $\phi(n) \to +\infty$. Similar modes of expression may of course be used in the present case. Thus we may say that $\phi(x)$ is small or nearly equal to l or large when n is large, using the words 'small,' 'nearly,' 'large' in a sense precisely similar to that in which they were used in Ch. IV.

Examples XXXV. 1. Consider the behaviour of the following functions as $x \to \infty$: $(1/x)$, $1+(1/x)$, x^2, x^k, $[x]$, $x-[x]$, $[x]+\sqrt{\{x-[x]\}}$.

The first four functions correspond exactly to functions of n fully discussed in Ch. IV. The graphs of the last three were constructed in Ch. II. (Exs. XVII.), and the reader will see at once that $[x] \to +\infty$, $x-[x]$ oscillates finitely, and $[x]+\sqrt{\{x-[x]\}} \to +\infty$.

One simple remark may be inserted here. The function $\phi(x) = x - [x]$ oscillates (between 0 and 1) as is obvious from the form of its graph. It is equal to zero whenever x is an integer, so that the function $\phi(n)$ derived from it is always zero and so tends to the limit zero. The same is true of

$$\phi(x) = \sin x\pi, \quad \phi(n) = \sin n\pi = 0.$$

In such cases as these, it is evident that $\phi(x) \to l$ or $\phi(x) \to +\infty$ or $-\infty$ involves the corresponding property for $\phi(n)$, but that the converse is by no means true.

2. Consider in the same way the functions:

$$\cos x\pi,\quad \tan x\pi,\quad (\cos x\pi)/x,\quad (\tan x\pi)/x,\quad (1/x)+\cos x\pi,\quad x\cos x\pi,\quad x^2\cos x\pi,$$
$$x\cos^2 x\pi,\quad (x\cos x\pi)^2,\quad a\cos^2 x\pi + b\sin^2 x\pi,\quad (a\cos^2 x\pi + b\sin^2 x\pi)/x,$$

illustrating your remarks by means of the graphs of the functions*.

3. Give a geometrical explanation of Def. 1, analogous to the geometrical explanation of Ch. IV, § 52.

4. If $\phi(x) \to l$, $\phi(x)\cos x\pi$ and $\phi(x)\sin x\pi$ oscillate finitely. If $\phi(x) \to +\infty$ (or $-\infty$) they oscillate infinitely. The graph of either function is a wavy curve oscillating between the curves $y=\phi(x)$, $y=-\phi(x)$.

5. Discuss the behaviour, as $x \to \infty$, of the function

$$y = f(x)\cos^2 x\pi + F(x)\sin^2 x\pi.$$

The graph of y is a curve oscillating between the curves $y=f(x)$, $y=F(x)$. Consider in particular the cases

(i) $f(x)=1+(1/x)$, $F(x)=1-(1/x)$; (ii) $f(x)=a+(a/x)$, $F(x)=b+(\beta/x)$, where $a \neq b$; (iii) $f(x)=1$, $F(x)=x$; (iv) $f(x)=-x$, $F(x)=x$; (v) $f(x)=\sin x\pi$, $F(x)=\cos x\pi$; (vi) $f(x)=\cos^4 x\pi + 3\sin^4 x\pi$, $F(x)=3\cos^4 x\pi + \sin^4 x\pi$.

78. Limits as x tends to $-\infty$. The reader will have no difficulty in finding for himself definitions of the meaning of the assertions 'x tends to $-\infty$' ($x \to -\infty$) and

$$\lim_{x \to -\infty} \phi(x) = l, \qquad \underset{(x \to -\infty)}{\phi(x)} \to \infty \text{ (or } -\infty\text{)}.$$

In fact if $x=-y$ and $\phi(x)=\phi(-y)=\psi(y)$, then x tends to $-\infty$ as y tends to ∞, and the question of the behaviour of $\phi(x)$ as x tends to $-\infty$ is the same as that of the behaviour of $\psi(y)$ as y tends to ∞.

79. Theorems corresponding to those of Ch. IV, §§ 56—63. The theorems concerning the sums, products, and quotients of functions, proved in Ch. IV, are all true (with the obvious verbal alterations which the reader will have no difficulty in supplying) for functions of the continuous variable x. Not only the enunciations but the proofs remain substantially the same.

Ex. Draw up a table, with examples of each case, similar to the table on p. 130 in Ch. IV, i.e. to illustrate the different possibilities with regard to the behaviour of $\phi(x)+\psi(x)$ when the behaviour of $\phi(x)$ and $\psi(x)$ is known. The other tables of Ch. IV suggest similar examples.

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* The reader has probably already drawn graphs of some of the functions considered in Exs. 2, 4, 5, while engaged on Ch. II and in particular Exs. XVI.

80. Steadily increasing or decreasing functions. The definition which corresponds to that of § 64 is as follows: *the function $\phi(x)$ will be said to increase steadily with x if $\phi(x_1) \geqq \phi(x_0)$ whenever $x_1 > x_0$.* In many cases, of course, this condition is only satisfied from a definite value $x = X$ onwards, i.e. when $x_1 > x_0 \geqq X$.

The theorem which follows requires no alteration but that of n into x: and the proof is also practically the same.

The reader should consider whether or no the following functions increase steadily with x (or at any rate increase steadily from a certain value of x onwards): $x^2 - x$, $x + \sin x$, $x + 2\sin x$, $x^2 + 2\sin x$, $[x]$, $[x] + \sin x$, $[x] + \sqrt{\{x - [x]\}}$. All these functions tend to $+\infty$ with x.

Ex. Show that if $\phi(x)$ steadily increases (or decreases) as $x \to \infty$, then the behaviour of $\phi(x)$ as $x \to \infty$ is the same as that of $\phi(n)$ as $n \to \infty$.

81. Limits as x tends to 0. Let $\phi(x)$ be such a function of x that $\lim\limits_{x \to \infty} \phi(x) = l$, and let $y = 1/x$. Then

$$\phi(x) = \phi(1/y) = \psi(y)$$

say. As x tends to ∞, y tends to the limit 0, and $\psi(y)$ tends to the limit l.

Let us now dismiss x and consider $\psi(y)$ simply as a function of y. We are for the moment concerned only with those values of y which correspond to *large* positive values of x, that is to say with *small* positive values of y. And $\psi(y)$ has the property that by making y sufficiently small we can make $\psi(y)$ differ by as little as we please from l. To put the matter more precisely, the statement expressed by $\lim \phi(x) = l$ meant that, when any positive number ϵ, however small, was assigned, we could choose X so that $|\phi(x) - l| < \epsilon$ for all values of x greater than or equal to X. But this is the same thing as saying that we can choose $\eta = 1/X$ so that $|\psi(y) - l| < \epsilon$ for all positive values of y less than or equal to η.

We are thus led to the following definitions.

A. *If when any positive number ϵ, however small, is assigned we can choose η so that*

$$|\phi(y) - l| < \epsilon,$$

for $0 < y \leqq \eta$, we say that $\phi(y)$ tends to the limit l when y tends to 0 by positive values, and we write

$$\lim_{y \to +0} \phi(y) = l.$$

B. *If when any number G, however large, is assigned we can choose η so that*

$$\phi(y) > G$$

for $0 < y \leqq \eta$, we say that $\phi(y)$ tends to $+\infty$ as y tends to 0 by positive values, and we write

$$\underset{(y \to +0)}{\phi(y)} \to +\infty .$$

We define in a similar way the meaning of '$\phi(y)$ tends to the limit l as y tends to 0 by negative values, or $\lim\limits_{y \to -0} \phi(y) = l$.' We have in fact only to alter $0 < y \leqq \eta$ to $-\eta \leqq y < 0$ in A. The reader will find it a useful exercise to write out formal definitions of the statements expressed by

$$`\underset{(y \to -0)}{\phi(y)} \to +\infty ,' \quad `\underset{(y \to +0)}{\phi(y)} \to -\infty ,' \quad `\underset{(y \to -0)}{\phi(y)} \to -\infty .'$$

If $\lim\limits_{(y \to +0)} \phi(y) = l$ **and** $\lim\limits_{(y \to -0)} \phi(y) = l$, we write simply

$$\lim_{(y \to 0)} \phi(y) = l.$$

This case is so important that it is worth while to give a formal definition.

If when any positive number ϵ, however small, is assigned we can choose η so that, for all values of y different from zero but **numerically** *less than or equal to η, $\phi(y)$ differs from l by less than ϵ, we say that $\phi(y)$ tends to the limit l as y tends to 0, and write*

$$\lim_{y \to 0} \phi(y) = l.$$

So also if $\underset{(y \to +0)}{\phi(y)} \to +\infty$ and $\underset{(y \to -0)}{\phi(y)} \to +\infty$ we write $\underset{y \to 0}{\phi(y)} \to +\infty$. Similarly we define the statement $\underset{(y \to 0)}{\phi(y)} \to -\infty$. Finally, if $\phi(y)$ does not tend to a limit, or to $+\infty$, or to $-\infty$, as $y \to 0$, we say that $\phi(y)$ oscillates as $y \to 0$, finitely or infinitely as the case may be.

The preceding definitions have been stated in terms of a variable denoted by y: what letter is used is of course immaterial, and we may suppose x written instead of y throughout them.

82. Limits as x tends to a. Suppose that $\lim\limits_{y \to 0} \phi(y) = l$ and write

$$y = x - a, \quad \phi(y) = \phi(x - a) = \psi(x).$$

As $y \to 0$, $x \to a$ and $\psi(x) \to l$, and we are naturally led to write

$$\lim_{x \to a} \psi(x) = l, \quad \psi(x) \underset{x \to a}{\to} l$$

or simply $\lim \psi(x) = l$ or $\psi(x) \to l$, and to say that $\psi(x)$ *tends to the limit* l *as* x *tends to* a. The meaning of this equation may be directly and formally defined as follows: *if, given* ϵ, *we can always determine* η *so that*

$$|\phi(x) - l| < \epsilon$$

for all values of x *such that* $0 < |x - a| \leqq \eta$, *then*

$$\lim_{x \to a} \phi(x) = l.$$

In other words, given any positive number ϵ we can find another η such that if x is different from a, but its difference from a is less than η, $\phi(x)$ will differ from l by less than ϵ.

By restricting ourselves to values of x greater than a, i.e. by replacing $0 < |x - a| \leqq \eta$ by $a < x \leqq a + \eta$, we define '$\phi(x)$ tends to l when x approaches a from the right'; which we may write as

$$\lim_{(x \to a+0)} \phi(x) = l.$$

Similarly we define

$$\lim_{x \to a-0} \phi(x) = l.$$

Thus $\lim\limits_{x \to a} \phi(x) = l$ is equivalent to the two assertions

$$\lim_{x \to a+0} \phi(x) = l = \lim_{x \to a-0} \phi(x).$$

And we can give similar definitions referring to the cases in which $\phi(x) \to +\infty$ (or $-\infty$) as $x \to a$ through values greater (or less) than a; but it is probably unnecessary to dwell further on these definitions, since they are exactly similar to those stated above in the special case when $a = 0$, and since we can always discuss the behaviour of $\phi(x)$ as $x \to a$ by putting $x = y + a$ and supposing that $y \to 0$.

Examples XXXVI. 1. If

$$\phi(x) \underset{x \to a}{\to} b, \quad \psi(x) \underset{x \to a}{\to} c,$$

then $\phi(x) \pm \psi(x) \to b \pm c$, $\phi(x)\psi(x) \to bc$, and $\phi(x)/\psi(x) \to b/c$, unless in the last case $c = 0$.

[We saw in § 79 that the theorems of Ch. IV, §§ 56 *et seq.* held also for functions of x when $x \to \infty$ (or $-\infty$). By putting $x = 1/y$ we may extend them to functions of y, when $y \to 0$, and by putting $y = x + a$ to functions of x, when $x \to a$.

The reader should however try to prove them directly from the formal definition given above. Thus, in order to obtain a strict direct proof of the first result he need only take the proof of Theorem I. in Ch. IV. and write throughout x for n, a for ∞ and $0 < |x - a| \leqq \eta$ instead of $n \geqq n_0$.]

2. If m is a positive integer $x^m \to 0$ as $x \to 0$.

3. If m is a negative integer $x^m \to +\infty$ as $x \to +0$, while $x^m \to -\infty$ or $+\infty$ as $x \to -0$, according as m is odd or even. If $m = 0$, $x^m = 1$ and $x^m \to 1$.

4. $\lim\limits_{x \to 0} (a + bx + cx^2 + \dots + kx^m) = a$.

5. $\lim\limits_{x \to 0} \left\{(a + bx + \dots + kx^m)/(\alpha + \beta x + \dots + \kappa x^\mu)\right\} = a/\alpha$ unless $\alpha = 0$. If $\alpha = 0$ the function tends to $+\infty$ or $-\infty$, as $x \to +0$, according as a, β have like or unlike signs; the case is reversed if $x \to -0$.

6. $\lim\limits_{x \to a} x^m = a^m$, if m is any positive or negative integer.

[If $m > 0$, put $x = y + a$ and apply Ex. 4. When $m < 0$ the result follows from the theorem concerning $1/\phi(x)$. There is one exceptional case, viz. when $a = 0$ and m is negative.

It follows at once that if $P(x)$ is any polynomial, $\lim P(x) = P(a)$.]

7. $\lim\limits_{x \to a} R(x) = R(a)$, if R denotes any rational function and a is not one of the roots of its denominator.

8. Prove that if x and a are positive and unequal, and m is any rational number greater than 1,

$$mx^{m-1}(x - a) > x^m - a^m > ma^{m-1}(x - a);$$

while if $0 < m < 1$ signs of the inequalities must be reversed.

[Suppose first that $a = 1$ and let $m = p/q$. It follows from the inequality (3) of § 67 that, if ξ is any number greater than unity,

$$\xi^p - 1 \gtrless (p/q)(\xi^q - 1)$$

according as $p \gtrless q$, and it is easy to see, by similar reasoning, that the result remains true if $0 < \xi < 1$, though both sides of the inequality are then negative. Writing $x^{1/q}$ for ξ and m for p/q we obtain

$$x^m - 1 \gtrless m(x - 1) \quad \text{..................................(1)},$$

according as $m \gtrless 1$. If now we replace x by $1/x$, and multiply by $-x^m$, we obtain

$$mx^{m-1}(x - 1) \gtrless x^m - 1 \quad \text{..............................(2)}.$$

From (1) and (2) the result follows in the case of $a = 1$. The proof may now be completed by writing x/a for x.]

9. Show that the inequality stated in Ex. 8 holds also if m is negative. Obtain corresponding inequalities when x and a are both negative. [See Chrystal's *Algebra*, vol. ii, pp. 43–45.]

10. Deduce from Ex. 8 that $\lim_{x \to a} x^m = a^m$ for all rational values of m, except when $a=0$ and m is negative.

[This follows at once, when a is positive, from the inequalities of Ex. 8. For

$$|x^m - a^m| < H|x-a|$$

where H is the greater of the absolute values of mx^{m-1} and ma^{m-1}. If a is negative we write $x=-y$ and $a=-b$. Then $\lim x^m = \lim (-1)^m y^m = (-1)^m b^m = a^m$.]

83. The reader will probably fail to see at first that any proof of such results as those of Exs. 4, 5, 6, 7, 10 above is necessary. He may ask 'why not simply put $x=0$ (or $x=a$)? Of course then we get a, a/α, a^m, $P(a)$, $R(a)$.' It is very important that he should see exactly where he is wrong. We shall therefore consider this point carefully before passing on to any further examples.

The statement $$\lim_{x \to 0} \phi(x) = l$$

is a statement about the values of $\phi(x)$ when x has any value distinct from *but differing by little from* 0[*]. It is *not* a statement about the *value of* $\phi(x)$ *when* $x=0$. When we make the statement we assert that, when x is *nearly* equal to 0, $\phi(x)$ is nearly equal to l. We assert nothing whatever about what happens when x is *actually* equal to 0. So far as we know or care $\phi(x)$ may not be defined at all for $x=0$; or it may have some value other than l. For example, consider the function defined for all values of x by the equation $\phi(x)=0$. It is obvious that

$$\lim \phi(x) = 0 \quad \text{........................(1).}$$

Now consider the function $\psi(x)$ which differs from $\phi(x)$ only in that, when $x=0$, $\psi(x)=1$. In this case too

$$\lim \psi(x) = 0 \text{..........................(2),}$$

for, when x is nearly equal to 0, $\psi(x)$ is not only nearly but exactly equal to 0. But $\psi(0)=1$. The graph of this function consists of the axis of x, with the point $x=0$ left out, and one isolated point, viz. the point $(0, 1)$. The equation (2) expresses the fact that if we move along the graph towards the axis of y, from either side, the ordinate of the curve tends to the limiting value 0: this is obvious from the fact that the ordinate is equal to 0 all the time. This fact is in no way affected by the position of the isolated point (0, 1).

---

[*] Thus in Def. A (§ 81) we make a statement about values of y such that $0<|y|\leqq\eta$; the first of these inequalities being inserted *expressly in order to exclude the value* $y=0$.

If the reader objects to this example on the score of artificiality it is easy to give him simple formulae representing functions which behave precisely like this near $x=0$. One is

$$\psi(x)=[1-x^2]$$

where $[1-x^2]$ denotes as usual the algebraically greatest integer contained in $1-x^2$.

For if $x=0$, $\psi(x)=[1]=1$; while if $0<x<1$ or $-1<x<0$, $0<1-x^2<1$ and so $\psi(x)=[1-x^2]=0$.

Or again, let us consider the function

$$y=x/x$$

already discussed in Ch. II (§ 14, (2)). This function is equal to 1 for all values of x save $x=0$. It is *not* equal to 1 when $x=0$: it is in fact not defined at all for $x=0$. For when we say that $\phi(x)$ is defined for $x=0$ we mean (as we explained in Ch. II, *l.c.*) that we can calculate its value for $x=0$ by putting $x=0$ in the actual expression of $\phi(x)$. In this case we cannot. When we put $x=0$ in $\phi(x)$ we obtain $0/0$, which is a meaningless expression. The reader may object 'divide numerator and denominator by x.' But he must admit that when $x=0$ this is impossible.

Thus $y=x/x$ is a function which differs from $y=1$ solely in that it is not defined for $x=0$. None the less

$$\lim(x/x)=1$$

for x/x is equal to 1 so long as x differs from 0, *however small the difference may be.*

Similarly $\phi(x)=\{(x+1)^2-1\}/x=x+2$ so long as x is not equal to 0, but is undefined when $x=0$. None the less $\lim\phi(x)=2$.

On the other hand there is of course nothing to prevent the limit of $\phi(x)$ as x tends to 0 from being equal to $\phi(0)$, the value of $\phi(x)$ for $x=0$. Thus if $\phi(x)=x$, $\phi(0)=0$ and $\lim\phi(x)=0$. This is in fact, from a *practical* point of view, i.e. from the point of view of what most frequently occurs in applications, the ordinary case.

Examples XXXVII. 1. $\lim\limits_{x\to a}(x^2-a^2)/(x-a)=2a$.

2. $\lim\limits_{x\to a}(x^m-a^m)/(x-a)=ma^{m-1}$, if m is any integer (zero excluded).

3. Consider the case in which $a>0$ and $m=\frac{1}{2}$. [Use the equation $(\sqrt{x}-\sqrt{a})/(x-a)=1/(\sqrt{x}+\sqrt{a})$: the same method may be applied to extend the result of Ex. 2 to the case in which m is the reciprocal of any positive integer.]

4. Show that the result of Ex. 2 remains true for all rational values of m, provided a is positive.

[If $x=y^q$, $a=b^q$, then, by Ex. 3, $y\to b$ as $x\to a$, and

$$\lim_{x\to a}\frac{x^{p/q}-a^{p/q}}{x-a}=\lim_{y\to b}\frac{y^p-b^p}{y^q-b^q}=\lim_{y\to b}\left\{\left(\frac{y^p-b^p}{y-b}\right)\bigg/\left(\frac{y^q-b^q}{y-b}\right)\right\}$$
$$=pb^{p-1}/qb^{q-1}=(p/q)\,a^{(p/q)-1}.$$

If q is odd the restriction that a is positive is unnecessary. The reader should also deduce the result directly from the inequalities given in Ex. XXXVI. 8.]

5. $\lim\limits_{x\to 1}(x^7-2x^5+1)/(x^3-3x^2+2)=1$. [Divide numerator and denominator by $x-1$.]

6. Discuss the behaviour of

$$\phi(x)=(a_0x^m+a_1x^{m+1}+\dots+a_kx^{m+k})/(b_0x^n+b_1x^{n+1}+\dots+b_lx^{n+l})$$

as x tends to 0 by positive or negative values.

[If $m>n$, $\lim\phi(x)=0$. If $m=n$, $\lim\phi(x)=a_0/b_0$. If $m<n$ and $n-m$ is even $\phi(x)\to\infty$ or $\phi(x)\to-\infty$ according as $a_0/b_0>$ or <0. If $m<n$ and $n-m$ is odd $\underset{(x\to+0)}{\phi(x)}\to+\infty$, $\underset{(x\to-0)}{\phi(x)}\to-\infty$ or $\underset{(x\to+0)}{\phi(x)}\to-\infty$, $\underset{(x\to-0)}{\phi(x)}\to+\infty$ according as $a_0/b_0>$ or <0.]

7. **Orders of smallness.** When x is small x^2 is very much smaller, x^3 much smaller still, and so on: in other words

$$\lim_{x\to 0}(x^2/x)=0,\quad \lim_{x\to 0}(x^3/x^2)=0,\quad \dots.$$

Another way of stating the matter is to say that when x tends to 0, x^2, x^3, ... all also tend to 0, but x^2 tends to 0 much more rapidly than x, x^3 than x^2, and so on. It is convenient to have some *scale* by which to measure the rapidity with which a function, whose limit, as x tends to 0, is 0, diminishes with x, and it is natural to take the simple functions x, x^2, x^3, ... as the measures of our scale.

We say, therefore, that $\phi(x)$ *is of the first order of smallness* if $\phi(x)/x$ tends to a limit other than zero as x tends to 0. Thus $2x+3x^2+x^7$ is of the first order of smallness, since $\lim(2x+3x^2+x^7)/x=2$.

Similarly we define the second, third, fourth, ... orders of smallness. It must not be imagined that this scale of orders of smallness is in any way complete. If it were complete every function $\phi(x)$ which tends to zero with x would be of either the first or second or some higher order of smallness.

This is obviously not the case. For example $\phi(x)=x^{7/3}$ tends to zero more rapidly than x^2 and less rapidly than x^3.

The reader may not unnaturally think that our scale might be made complete by including in it *fractional* orders of smallness. Thus we might say that $x^{7/3}$ was of the $\frac{7}{3}$th order of smallness. We shall however see later on that such a scale of orders would still be altogether incomplete. And as a matter of fact the *integral* orders of smallness defined above are so much more important in applications than any others that it is hardly necessary to attempt to make our definitions more precise.

Orders of greatness. Similar definitions are at once suggested to meet the case in which $\phi(x)$ is large (positively or negatively) when x is small. We shall say that $\phi(x)$ is of the kth order of greatness when x is small if $\phi(x)/x^{-k}=x^k\phi(x)$ tends to a limit different from zero as x tends to 0.

These definitions have reference to the case in which $x\rightarrow 0$. There are of course corresponding definitions relating to the cases in which $x\rightarrow\infty$ or a. Thus if $x^k\phi(x)$ tends to a limit other than zero, as $x\rightarrow\infty$, we say that $\phi(x)$ is of the kth order of smallness when x is large: while if $(x-a)^k\phi(x)$ tends to a limit other than zero as $x\rightarrow a$, we say that $\phi(x)$ is of the kth order of largeness when x is nearly equal to a.

*8. $\lim\sqrt{(1+x)}=\lim\sqrt{(1-x)}=1$. [Put $1+x$ (or $1-x$) $=y$, and use Ex. XXXVI. 10.]

9. $\lim\{\sqrt{(1+x)}-\sqrt{(1-x)}\}/x=1$. [Multiply numerator and denominator by $\sqrt{(1+x)}+\sqrt{(1-x)}$.]

10. Consider the behaviour of $\{\sqrt{(1+x^m)}-\sqrt{(1-x^m)}\}/x^n$ as $x\rightarrow 0$, m and n being positive integers.

11. $\lim\{\sqrt{(1+x+x^2)}-1\}/x=\frac{1}{2}$.

12. $\lim\{\sqrt{(1+x)}-\sqrt{(1+x^2)}\}/\{\sqrt{(1-x^2)}-\sqrt{(1-x)}\}=1$.

13. Draw a graph of the function

$$y=\left\{\frac{1}{x-1}+\frac{1}{x-\frac{1}{2}}+\frac{1}{x-\frac{1}{3}}+\frac{1}{x-\frac{1}{4}}\right\}\Bigg/\left\{\frac{1}{x-1}+\frac{1}{x-\frac{1}{2}}+\frac{1}{x-\frac{1}{3}}+\frac{1}{x-\frac{1}{4}}\right\}.$$

Has it a limit as $x\rightarrow 0$?

[$y=1$ except for $x=1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, when y is not defined, and $y\rightarrow 1$ as $x\rightarrow 0$.]

14. $\lim(\sin x)/x=1$.

[It is proved in books on elementary Trigonometry (it follows in fact from the definitions of the trigonometrical ratios and of circular measure)† that if x is positive and less than $\frac{1}{2}\pi$

$$\sin x<x<\tan x$$

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* In the examples which follow it is to be assumed that limits as $x\rightarrow 0$ are required, unless (as in Exs. 23, 27) the contrary is explicitly stated.

† The definition of circular measure presupposes that any arc of a circle has associated with it a definite number called its *length*, and the proof of the inequalities which follow depends on certain properties of the 'length' which are taken to be geometrically intuitive, as that the length of the arc is greater than that of the chord of the arc. These questions will be discussed more fully in Ch. VII.

or $$\cos x < (\sin x)/x < 1$$

or $$0 < 1 - (\sin x)/x < 1 - \cos x = 2\sin^2 \tfrac{1}{2}x.$$

But $2\sin^2\frac{1}{2}x < 2(\frac{1}{2}x)^2 < \frac{1}{2}x^2$. It follows at once that $\lim\limits_{x\to+0}\{1-(\sin x)/x\}=0$, and $\lim\limits_{x\to+0}(\sin x)/x=0$. As $(\sin x)/x$ is an even function the result follows.

We note that $1-(\sin x)/x$ tends to 0 with x *at least* as rapidly as $\frac{1}{2}x^2$.]

15. $\lim(1-\cos x)/x^2=\frac{1}{2}$. 16. $\lim(\sin ax)/x=a$. Is this true if $a=0$?

17. $\lim(1-\cos ax)/x^2=\frac{1}{2}a^2$.

18. In the figure $OP=1$, $A\hat{O}P=\text{arc } AP=x$, $PN=\sin x$, $ON=\cos x$, $PT=\tan x$, $OT=\sec x$. Show that when x is small PN, PA, PT are of the first order of smallness, NA and AT of the second, and the triangle PNT of the third.

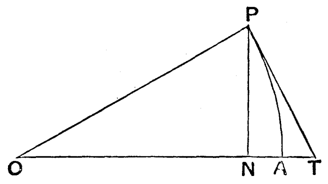

Fig. 37.

19. $\lim(\arcsin x)/x=1$. [Put $x=\sin y$.]

20. $\lim(\tan ax)/x=a$, $\lim(\arctan ax)/x=a$.

21. Show that $(\sin mx)/(\sin nx)$, $(\tan mx)/(\tan nx)$, $(\arcsin mx)/(\arcsin nx)$, $(\arctan mx)/(\arctan nx)$, $(\sin mx)/(\tan nx)$ and $(\arctan mx)/(\arcsin nx)$ all tend to the limit m/n as $x\to 0$.

22. $\lim(\operatorname{cosec} x-\cot x)/x=\frac{1}{2}$. 23. $\lim(1+\cos\pi x)/\tan^2\pi x=\frac{1}{2}$ (as $x\to 1$).

24. How do the functions $\sin(1/x)$, $(1/x)\sin(1/x)$, $x\sin(1/x)$ behave as $x\to 0$? [The first oscillates finitely, the second infinitely, the third tends to the limit 0. None is defined for $x=0$. See Exs. XVI.]

25. Does the function
$$y=\left(\sin\frac{1}{x}\right)\bigg/\left(\sin\frac{1}{x}\right)$$
tend to a limit as x tends to 0?

[*No.* The function is equal to 1 *except when* $\sin(1/x)=0$; i.e. when $x=1/\pi,\ 1/2\pi,\ \dots,\ -1/\pi,\ -1/2\pi,\ \dots$. For these values the expression of y assumes the meaningless form 0/0, and y is therefore not defined for an infinity of values of x near $x=0$.]

26. Prove that $\lim\limits_{x\to+0}[x]=0$, $\lim\limits_{x\to-0}[x]=-1$.

27. Prove that, if m is any integer, $[x]\to m$ and $x-[x]\to 0$ as $x\to m+0$, and $[x]\to m-1$, $x-[x]\to 1$ as $x\to m-0$. What are the corresponding limits for the functions $\sqrt{\{x-[x]\}}$, $(x-[x])^2$, $[x]+\sqrt{\{x-[x]\}}$, $[x]+(x-[x])^2$?

28. Find the limits of $[x^2]$ and $[1-x^2]$ as $x \to +0$ and $x \to -0$.

29. **Functions which steadily increase or decrease as** $x \to a+0$. If there is a segment to the right of a, say $(a, a+\delta)$, such that $\phi(x'') \geqq \phi(x')$ if $a < a+x'' < a+x' \leqq a+\delta$, $\phi(x)$ is said to increase steadily as $x \to a+0$. Similarly we define what is meant by saying that $\phi(x)$ increases steadily as $x \to a-0$, and there are corresponding definitions for decreasing functions.

Show (cf. §§ 64, 80) that if $\phi(x)$ steadily increases as $x \to a+0$, then $\phi(x)$ tends to a limit or tends to $+\infty$ as $x \to a+0$. State the corresponding theorems for the other cases.

Do the functions considered in Exs. 26–28 steadily increase or decrease as $x \to a+0$, or $x \to a-0$ (where $a=0$ or m as the case may be)?

30. Show that $(\sin x)/x$ steadily increases as $x \to +0$ or $x \to -0$. [Use the equation

$$\frac{\sin x}{x} - \frac{\sin(x+h)}{x+h} = \frac{x \sin x (1-\cos h) + \cos x (h \tan x - x \sin h)}{x(x+h)}.]$$

84. Continuous functions of a real variable.

The reader has no doubt an idea as to what is meant by a *continuous curve*. Thus he would call the curve C in Fig. 38 continuous, the curve C' discontinuous.

Either of these curves may be regarded as the graph of a function $\phi(x)$. It is natural to call a function *continuous* if its graph is a continuous curve, otherwise discontinuous. Let us take this as a provisional definition and try to distinguish more precisely some of the properties which are involved in it.

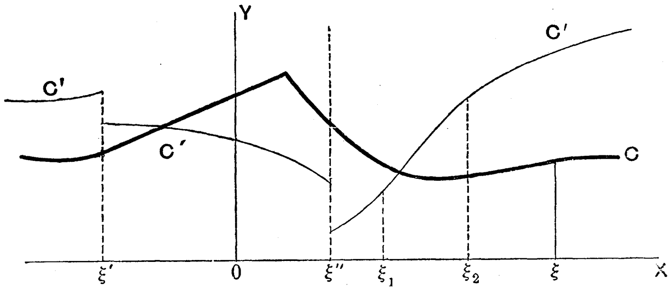

Fig. 38.

In the first place it is evident that the property of the function $y=\phi(x)$ of which C is the graph may be analysed into some property possessed by the curve at each of its points. To be able to define continuity *for all values of x* we must first define continuity *for any particular value of x*. Let us therefore fix on some particular value of x, say the value $x = \xi$ corresponding to the point P of the graph. What are the characteristic properties of $\phi(x)$ associated with this value of x?

In the first place *$\phi(x)$ is defined for $x = \xi$*. This is obviously essential. If $\phi(\xi)$ were not defined there would be a point missing from the curve.

Secondly *$\phi(x)$ is defined for all values of x near $x = \xi$*; i.e. we can find an interval, including $x = \xi$ in its interior, for all points of which $\phi(x)$ is defined.

Thirdly *if x approaches the value ξ from either side $\phi(x)$ approaches the limit $\phi(\xi)$*.

The properties thus defined are far from exhausting those which are possessed by the curve as pictured by the eye of common sense. This picture of a curve is a generalisation from particular curves such as straight lines and circles. But they are the simplest and most fundamental properties: and the graph of any function which has these properties would, so far as drawing it is practically possible, satisfy our geometrical intuition of what a continuous curve should be. We therefore select these properties as embodying the mathematical notion of continuity. We are thus led to the following

DEFINITION. *The function $\phi(x)$ is said to be continuous for $x = \xi$ if it tends to a limit as x tends to ξ from either side, and each of these limits is equal to $\phi(\xi)$.*

We can now define *continuity throughout an interval.* The function $\phi(x)$ is said to be continuous throughout a certain interval of values of x if it is continuous for all values of x in that interval. It is said to be *continuous everywhere* if it is continuous for every value of x. Thus the function whose graph is C' is continuous for every value of x except $x = \xi'$ and $x = \xi''$. It is continuous in the interval (ξ_1, ξ_2).

If we recur to the definitions of a limit we see that our definition is equivalent to '*$\phi(x)$ is continuous for $x = \xi$ if, given ϵ, we can choose η so that $|\phi(x) - \phi(\xi)| < \epsilon$ if $0 \leqq |x - \xi| \leqq \eta$.*'

85. This form of the definition may be illustrated geometrically as follows. Draw the two horizontal lines $y = \phi(\xi) \pm \epsilon$ (Fig.

39). Then $|\phi(x)-\phi(\xi)|<\epsilon$ expresses the fact that the point on the curve corresponding to x lies between these two lines.

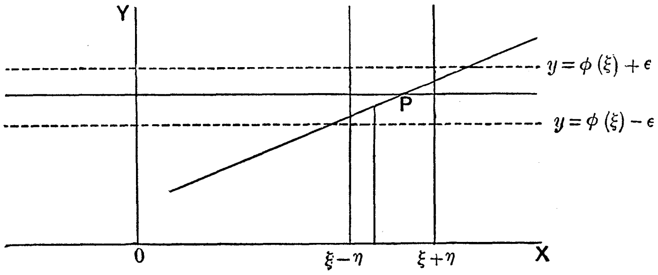

FIG. 39.

Similarly $|x-\xi|\leqq\eta$ expresses the fact that x lies in the interval $(\xi-\eta,\ \xi+\eta)$. Thus our definition asserts that if we draw two such horizontal lines, no matter how close together, we can always cut off a vertical strip of the plane by two vertical lines in such a way that all that part of the curve which is contained in the strip lies between the two horizontal lines.

This is evidently true of the curve C (Fig. 38), whatever value ξ may have.

We shall now discuss the continuity of some special types of functions. Some of the results which follow were (as was pointed out) tacitly assumed in Ch. II.

Examples XXXVIII. 1. The sum or product of two functions continuous at a point is continuous at that point. The quotient is also continuous unless the denominator vanishes at the point. [This follows at once from Ex. XXXVI. 1.]

2. Any polynomial is continuous for all values of x. Any rational fraction is continuous except for values of x for which the denominator vanishes. [This follows from Exs. XXXVI. 6, 7.]

3. $\sqrt{x}$ is continuous for all positive values of x (Ex. XXXVI. 10). It is not continuous for $x=0$, although it is equal to 0 when $x=0$ and also tends to the limit 0 as x tends to 0 by positive values. In fact $\sqrt{x}\to 0$ as $x\to +0$, but not as $x\to -0$, since $\sqrt{x}$ is not defined for negative values of x; and *both* conditions are wanted for continuity. Similarly for $x^{m/n}$ where m, n are any positive integers and n is even.

4. Discuss the continuity or discontinuity of $x^{m/n}$, where n is odd, for $x=0$.

5. $1/x$ is not continuous for $x=0$. It has no value for $x=0$, nor does it tend to a limit as $x\to 0$. In fact $1/x\to +\infty$ or $\to -\infty$ according as $x\to 0$ by positive or negative values.

6. Discuss the continuity of $x^{-m/n}$, where m and n are positive integers, for $x=0$.

7. The standard rational function $R(x)=P(x)/Q(x)$ is discontinuous for $x=a$, where a is any root of $Q(x)=0$.

Thus $(x^2+1)/(x^2-3x+2)$ is discontinuous for $x=1$. It will be noticed that in the case of these functions a discontinuity is always associated with (a) a failure of the definition for a particular value of x and (b) a tending of the function to $+\infty$ or $-\infty$ as x approaches this value from either side. Such a particular kind of point of discontinuity is usually described as **an infinity** of the function. 'Infinities' are far from being the only possible kind of points of discontinuity; but they are the kind of most common occurrence in ordinary work.

8. Discuss the continuity of

$$\sqrt{\{(x-a)(b-x)\}},\quad \sqrt[3]{\{(x-a)(b-x)\}},\quad \sqrt{\{(x-a)/(b-x)\}},\quad \sqrt[3]{\{(x-a)/(b-x)\}}.$$

9. $\sin x$, $\cos x$ are continuous for all values of x.

[For $$\sin(x+h)-\sin x=2\sin\tfrac{1}{2}h\cos(x+\tfrac{1}{2}h)$$
which is numerically less than the numerical value of h. Similarly for $\cos x$.]

10. For what values of x are $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$ continuous or discontinuous?

11. If $f(y)$ is continuous for $y=\eta$, and $\phi(x)$ is a continuous function of x which is equal to η when $x=\xi$, then $f\{\phi(x)\}$ is continuous for $x=\xi$.

12. $\sin(\sin x)$ is continuous for all values of x.

13. If $\phi(x)$ is continuous for any particular value of x, any polynomial in $\phi(x)$, such as $a\{\phi(x)\}^m+\dots$, is so too.

14. $a\cos^2 x+b\sin^2 x$ is continuous for all values of x.

15. Discuss the continuity of

$$1/(a\cos^2 x+b\sin^2 x),\quad \sqrt{(2+\cos x)},\quad \sqrt{(1+\sin x)},\quad 1/\sqrt{(1+\sin x)}.$$

16. $\sin(1/x)$, $x\sin(1/x)$, $x^2\sin(1/x)$ are continuous except for $x=0$. [Apply Exs. 9 and 11.]

17. The function which is equal to $x\sin(1/x)$ except when $x=0$, and to zero when $x=0$, is continuous for *all* values of x.

18. $[x]$, $x-[x]$ are discontinuous for all integral values of x.

19. For what (if any) values of x are the following functions discontinuous: $[x^2]$, $[\sqrt{x}]$, $(x-[x])^2$, $\sqrt{(x-[x])}$, $[x]+\sqrt{(x-[x])}$, $[2x]$, $[x]+[2x]$, $[x]+[-x]$?

20. **Classification of discontinuities.** Some of the preceding examples suggest a classification of different types of discontinuity.

(1) Suppose that $\phi(x)$ tends to a limit as $x\to a$ either by values greater than or by values less than a. Denote these limits by G and L respectively. Then, for continuity, it is necessary that $\phi(x)$ should be defined for $x=a$, and that $G=L=\phi(a)$. Discontinuity may arise in a variety of ways.

(α) G may be equal to L, but $\phi(a)$ may not be defined, or may differ from G and L. Thus if $\phi(x)=x\sin(1/x)$ and $a=0$, $G=L=0$, but $\phi(x)$ is not defined for $x=0$. Or if $\phi(x)=[1-x^2]$ and $a=0$, $G=L=0$, but $\phi(0)=1$.

(β) G and L may be unequal. In this case $\phi(a)$ may be equal to one or to neither, or be undefined. The first case is illustrated by $\phi(x)=[x]$, when (for $a=0$) $G=0=\phi(0)$, $L=-1$: the second by $\phi(x)=[x]-[-x]$, when $G=1$, $L=-1$, $\phi(0)=0$, and the third by $\phi(x)=[x]+x\sin(1/x)$, when $G=0$, $L=-1$, and $\phi(0)$ is undefined.

In any of these cases we say that $\phi(x)$ has a **simple discontinuity** at $x=a$. And to these cases we may add those in which one of G or L exists, but $\phi(x)$ is only defined on one side of $x=a$, so that there is no question of the existence of the other limit.

(2) It may be the case that only one (or neither) of G and L exists, but that (supposing for example G not to exist) $\phi(x)\to+\infty$ or $-\infty$ as $x\to a+0$: so that $\phi(x)$ tends to a limit or to $+\infty$ or $-\infty$ as x approaches a from either side. Such is the case, for instance, if $\phi(x)=1/x$ or $1/x^2$, and $a=0$. In such cases we say (cf. Ex. 7) that $x=a$ is an **infinity** of $\phi(x)$.

And again we may add to these cases those in which $\phi(x)\to+\infty$ or $-\infty$ as $x\to a$ from one side, but is not defined at all on the other side of $x=a$.

(3) Any point of discontinuity which is not a point of simple discontinuity nor an infinity is called a point of **oscillatory discontinuity**. Such is the point $x=0$ for the functions $\sin(1/x)$, $(1/x)\sin(1/x)$.

21. What is the nature of the discontinuities at $x=0$ of the functions: $(\sin x)/x$, $(1-\cos x)/x^2$, $\sqrt{x}$, $[x]+[-x]$, $\operatorname{cosec} x$, $\sqrt{(\operatorname{cosec} x)}$, $\operatorname{cosec}(1/x)$, $\sin(1/x)/\sin(1/x)$?

22. The function which is equal to 1 when x is rational and to 0 when x is irrational (Ch. II, Ex. XVII. 11) is discontinuous for *all* values of x. So too is any function which is only defined for rational or for irrational values of x.

23. The function which is equal to x when x is irrational and to $\sqrt{\{(1+p^2)/(1+q^2)\}}$ when x is a rational fraction p/q (Ch. II, Ex. XVII. 12) is discontinuous for all negative and for positive *rational* values of x, but continuous for positive *irrational* values. [This is not very obvious, and if the reader can see it he may be sure that he understands the nature of continuity and discontinuity.]

24. For what points are the functions considered in Ch. IV, Exs. XXXIII. discontinuous, and what is the nature of their discontinuities? [Consider, e.g., the function $y=\lim\limits_{n\to\infty} x^n$ (Ex. 5). Here y is only defined when $-1<x\leqq 1$: it is equal to 0 for $-1<x<1$ and to 1 for $x=1$. The points $x=\pm 1$ are points of simple discontinuity.]

86. The fundamental property of a continuous function. It may perhaps be thought that the way in which we stated (§ 84 above) the property which a function must have, in order that its graph may satisfy our common sense notion of what a continuous curve should be, is not the simplest or most natural way possible. Another method of analysing our idea of continuity is the following. Let A and B be two points, whose coordinates are x_0, $\phi(x_0)$ and x_1, $\phi(x_1)$ respectively, on the graph of $\phi(x)$. Draw any straight line L which passes between A and B. Then common sense certainly declares that if the graph of $\phi(x)$ is continuous it must cut L.

If we consider this property as an intrinsic geometrical property of continuous curves it is clear that there is no real loss of generality in supposing L to be parallel to the axis of x. In this case the ordinates of A and B cannot be equal: let us suppose, for definiteness, that $\phi(x_1) > \phi(x_0)$. And let L be the line $y = \eta$, where $\phi(x_0) < \eta < \phi(x_1)$. Then to say that the graph of $\phi(x)$ must cut L is the same thing as to say that there is a value of x between x_0 and x_1 for which $\phi(x) = \eta$.

We conclude then that a continuous function $\phi(x)$ must possess the following property: *if*

$$\phi(x_0) = y_0, \quad \phi(x_1) = y_1,$$

and $y_0 < \eta < y_1$, *there is a value of* x *between* x_0 *and* x_1 *for which* $\phi(x) = \eta$. In other words *as* x *varies from* x_0 *to* x_1, y *must assume at least once every value between* y_0 *and* y_1.

We shall now prove that if $\phi(x)$ is a continuous function of x in the sense defined in § 84 it does in fact possess this property. There is a certain range of values of x to the right of x_0, for which $\phi(x) < \eta$. For $\phi(x_0) < \eta$, and so $\phi(x)$ is certainly less than η if $\phi(x) - \phi(x_0)$ is numerically less than $\eta - \phi(x_0)$. But since $\phi(x)$ is continuous for $x = x_0$ this condition is certainly satisfied if x is near enough to x_0. Similarly there is a certain range of values to the left of x_1 for which $\phi(x) > \eta$.

We can now easily prove our theorem by a *reductio ad absurdum.* For suppose that there is no value of x between x_0 and x_1 for which $\phi(x) = \eta$. Then for every x in the interval (x_0, x_1) $\phi(x)$ is either greater or less than η.

Let us divide the values of x between x_0 and x_1 into two classes T, U as follows:

(1) in the class T we put all values ξ of x such that $\phi(x) < \eta$ for $x = \xi$ and for all values of x between x_0 and ξ;

(2) in the class U we put all the other values of x, i.e. all values ξ such that either $\phi(\xi) \geqq \eta$ or there is a value of x between x_0 and ξ for which $\phi(x) \geqq \eta$.

Then it is evident that these two classes are related like the classes T and U of Ch. I, § 5, i.e. that the points of the class U lie entirely to the right of those of the class T, and that there is a point P which divides the two classes.

Now, *ex hypothesi*, if ξ_0 is the abscissa of P, $\phi(\xi_0) \neq \eta$. First suppose $\phi(\xi_0) > \eta$, so that ξ_0 belongs to the upper class: and let $\phi(\xi_0) = \eta + k$ say. Then if $P'(x = \xi')$ is any point to the left of P, no matter how close, $\phi(\xi') < \eta$ and so

$$\phi(\xi_0) - \phi(\xi') > k,$$

which directly contradicts the condition of continuity at P.

Next suppose $\phi(\xi_0) = \eta - k < \eta$. Then if $P'(x = \xi')$ is any point to the right of P, no matter how close, either $\phi(\xi') \geqq \eta$ or we can find another point $P''(x = \xi'')$ between P and P' and such that $\phi(\xi'') \geqq \eta$. In any case we can find a point as near to P as we please and such that the corresponding values of $\phi(x)$ differ by more than k. And this again directly contradicts the hypothesis that $\phi(x)$ is continuous at P.

Hence the hypothesis that $\phi(x)$ is nowhere equal to η is untenable, and the theorem is established. The fact is, of course, that $\phi(\xi_0)$ must be equal to η.

87. It is easy to see that the converse of the theorem just proved is not true. Thus such a function as the function $\phi(x)$ whose graph is represented by Fig. 40 obviously assumes *at least* once every value between $\phi(x_0)$ and $\phi(x_1)$: yet $\phi(x)$ is obviously discontinuous. Indeed it is not even true that $\phi(x)$ must be continuous when it assumes each value *once and once only*. Thus let $\phi(x)$ be defined as follows from $x = 0$ to $x = 1$. If $x = 0$ let $\phi(x) = 0$; if $0 < x < \frac{1}{2}$ let $\phi(x) = \frac{1}{2} - x$; if $x = \frac{1}{2}$ let $\phi(x) = \frac{1}{2}$; if $\frac{1}{2} < x < 1$ let $\phi(x) = \frac{3}{2} - x$; and if $x = 1$ let $\phi(x) = 1$. The graph of the function is shown in Fig. 41; it includes the points O, C, F but *not* the points A, B, D, E. It is clear that, as x varies from

0 to 1, $\phi(x)$ assumes once and once only every value between $\phi(0) = 0$ and $\phi(1) = 1$; but $\phi(x)$ is discontinuous for $x = 0, \frac{1}{2}, 1$.

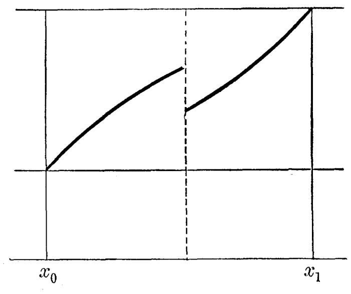

FIG. 40.

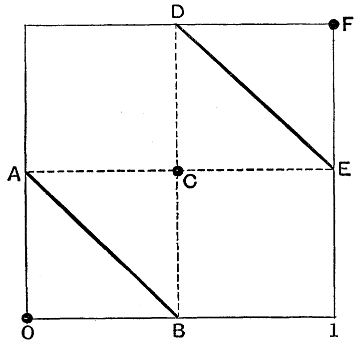

FIG. 41.

As a matter of fact, however, the curves which usually occur in elementary mathematics are all composed of *a finite number of pieces along which y always varies in the same direction.* It is easy to show that if $y = \phi(x)$ always varies in the same direction (i.e. steadily increases or decreases) as x varies from x_0 to x_1, the two notions of continuity are really equivalent, i.e. that if $\phi(x)$ takes every value between $\phi(x_0)$ and $\phi(x_1)$ it must be a continuous function in the sense of § 84. For let ξ be any value of x between x_0 and x_1. As $x \to \xi$ through values less than ξ, $\phi(x)$ tends to a limit l. Similarly as $x \to \xi$ through values greater than ξ it tends to a limit l'. Moreover $l \leqq \phi(\xi) \leqq l'$, and the function will be continuous for $x = \xi$ if and only if

$$l = \phi(\xi) = l'.$$

But if either of these equations is untrue, say the first, it is evident that $\phi(x)$ never assumes any value which lies between l and $\phi(\xi)$, which is contrary to our assumption. Thus $\phi(x)$ must be continuous. The net result of this and the last section is consequently to show that our commonsense notion of what we mean by continuity is substantially accurate, and capable of precise statement in mathematical terms.

88. Inverse functions. The reader is already familiar with some examples of inverse functions: thus $\arcsin x$ and $\arctan x$ are the functions inverse to $\sin x$ and $\tan x$. Generally, if $y = \phi(x)$

and x can be expressed in terms of y in the form $x = \psi(y)$, we call ψ the function inverse of ϕ. Thus if $y = x^2$, $x = \pm\sqrt{y}$. It will be observed in this case that the inverse function differs fundamentally from the original function in two respects: it is not defined for all values of y, and, when it is defined, it has two values. Similarly if $y = \sin x$, $x = \text{arc} \sin y$, x is only defined when $-1 \leqq y \leqq 1$, and then has an infinity of values.

Let us suppose now, however, that $\phi(x)$ is a function which steadily increases or decreases (suppose the former) as x varies from a to b, and let $\phi(a) = \alpha$, $\phi(b) = \beta$. If $\phi(x)$ is continuous, it assumes every value between α and β, and conversely: we shall suppose this to be the case. These suppositions, however, are not enough to ensure that $\phi(x)$ assumes each such value *once and only once*. In order to ensure this we must exclude the possibility of $\phi(x)$ *remaining stationary* for any part of the time during which x varies from a to b. We can do this by supposing that $a \leqq x' < x'' \leqq b$ involves $\phi(x'') > \phi(x')$, and not merely $\phi(x'') \geqq \phi(x')$, as we supposed in defining an increasing function in § 80. A function which satisfies this condition we may call an *increasing function in the stricter sense*, as opposed to the increasing functions in the wider sense with which we have hitherto been concerned.

Then, as x varies from a to b, $\phi(x)$ varies from α to β, *assuming each value between α and β once and once only*. Thus to a value of y between α and β corresponds one and only one value of x between a and b. And if we write $x = \psi(y)$, $\psi(y)$ is a function of y which has just one value for any value of y between α and β. Moreover it is evident that $\psi(y)$ increases steadily as y increases from α to β, assuming in turn, once and only once, each value between a and b. Finally, by § 87, $\psi(y)$ is continuous throughout the interval (α, β).

Thus *if $y = \phi(x)$ is a function of x which, throughout the interval (a, b), is one-valued, continuous, and increasing in the stricter sense, then $x = \psi(y)$ is a function of y which has the same properties throughout the corresponding interval of values of y.*

Examples XXXIX. 1. As x increases from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$, $y = \sin x$ is continuous and steadily increases, in the stricter sense, from -1 to $+1$. Hence $x = \text{arc} \sin y$ is a continuous and steadily increasing function of y from $y = -1$ to $y = +1$. Here $\text{arc} \sin y$ denotes the value of the inverse sine which lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

2. Apply similar reasoning to the function arc cos y.

3. Show that the numerically least value of arc tan y is a function of y continuous for all values of y, and increasing steadily from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$ as y varies through all real values positive and negative.

89. The range of values of a continuous function*. In this paragraph we shall state and prove some general theorems concerning continuous functions. These theorems are of a very abstract character, and the reader may well be tempted to regard them as obvious: but general theorems, as he probably realises by now, are apt to be a good deal less obvious than they seem.

Let us consider a function $\phi(x)$ about which we shall only assume at present that it is defined for every value of x in an interval (a, b).

It may be possible to assign a number G such that $\phi(x) \leqq G$ for all these values of x, i.e. such that the value of $\phi(x)$ cannot be greater than G, or, as we may say, $\phi(x)$ *cannot surpass* G. In this case we shall say that $\phi(x)$ is *limited above.*

If G can be assigned at all it can be assigned in an infinity of ways: for if $\phi(x)$ cannot surpass G it certainly cannot surpass any number greater than G. But there is a *least* number G which $\phi(x)$ cannot surpass. For if we divide the aggregate of real numbers into two classes T, U composed respectively of the numbers which $\phi(x)$ can and cannot surpass, it is clear that when T and U are represented along the line L, as in § 5, T lies entirely to the left of U. As in § 5, there is a point P which divides the two classes. But in this case, since the two classes between them include *all* real numbers, the coordinate M of P must belong to one class or the other. And it must belong to U. For if it belonged to T, $\phi(x)$ could assume a value greater than M, say $M + \delta$: and then all the numbers between M and $M + \delta$ could also be surpassed; so that there would be points of T to the right of P, which is not the case.

Thus M is *the least number which* $\phi(x)$ *cannot surpass*: we call M *the* **upper limit** *of* $\phi(x)$ *in the interval* (a, b).

Similarly it may be possible to find a number which $-\phi(x)$ cannot surpass; and if this is possible, it is possible to find a least number $-m$ which $-\phi(x)$ cannot surpass: or, what is the same thing, a *greatest* number m below which the value of $\phi(x)$ cannot sink. In this case we say that $\phi(x)$ is *limited below*, and call m the **lower limit** of $\phi(x)$.

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* In this section I have for the most part followed the exposition of de la Vallée Poussin (*Cours d'Analyse*, t. i. pp. 19—21).

If both M and m can be determined in this way we shall say that $\phi(x)$ is **limited** throughout (a, b). And then m and M are the greatest and least numbers respectively such that

$$m \leqq \phi(x) \leqq M$$

for all values of x in the interval (a, b).

If M cannot be determined $\phi(x)$ has no upper limit: in this case we can find values of $\phi(x)$ algebraically greater than any number we can assign. In the same way $\phi(x)$ may have no lower limit.

THEOREM 1. *If $\phi(x)$ is continuous throughout (a, b) it is limited throughout (a, b).*

We can certainly determine an interval (a, ξ), extending to the right from a, throughout which $\phi(x)$ is limited. For since $\phi(x)$ is continuous for $x = a$, we can, given any number δ, however small, determine an interval (a, ξ) throughout which $\phi(x)$ lies between $\phi(a) - \delta$ and $\phi(a) + \delta$; and obviously $\phi(x)$ is limited in this interval.

Now divide the points ξ of the interval (a, b) into two classes T, U, putting ξ in T if $\phi(\xi)$ is limited in (a, ξ), and in U if this is not the case. By what precedes T certainly exists: what we propose to prove is that U does not. Suppose that U does exist; and let β be the value of ξ which divides T from U. Since $\phi(x)$ is continuous at $x = \beta$ we can determine an interval $(\beta - \eta, \beta + \eta)$ throughout which $\phi(\beta) - \delta < \phi(x) < \phi(\beta) + \delta$, however small δ may be. Thus $\phi(x)$ is limited throughout $(\beta - \eta, \beta + \eta)$. But $\beta - \eta$ belongs to T. Therefore $\phi(x)$ is limited throughout $(a, \beta - \eta)$: and therefore it is limited throughout the whole interval $(a, \beta + \eta)$. But $\beta + \eta$ belongs to U and so $\phi(x)$ is *not* limited throughout $(a, \beta + \eta)$. This contradiction shows that U does not exist. And so $\phi(x)$ is limited throughout the whole interval (a, b).

THEOREM 2. *If $\phi(x)$ is continuous throughout (a, b) and M and m are its upper and lower limits, $\phi(x)$ assumes the values M and m at least once each in the interval.*

For since $M-\phi(x)$ assumes values less than any assignable number, $1/\{M-\phi(x)\}$ assumes values greater than any assignable number, and so, by Theorem 1, cannot be continuous. But $M-\phi(x)$ is a continuous function, and so $1/\{M-\phi(x)\}$ is continuous at any point at which its denominator does not vanish (Ex. XXXVIII. 1). There must therefore be one such point: at this point $\phi(x)=M$. Similarly for m.

Examples XL. 1. If $\phi(x)=1/x$ except for $x=0$, $\phi(x)=0$ when $x=0$, $\phi(x)$ has neither an upper nor a lower limit in any interval which includes $x=0$ in its interior, as e.g. the interval $(-1, +1)$.

2. If $\phi(x)=1/x^2$ except for $x=0$, $\phi(x)=0$ when $x=0$, $\phi(x)$ has the lower limit 0, but no upper limit, in the interval $(-1, +1)$.

3. Let $\phi(x)=\sin(1/x)$ except for $x=0$, $\phi(x)=0$ when $x=0$. Then $\phi(x)$ is discontinuous at $x=0$. In any interval $(-\delta, +\delta)$ the lower limit is -1, and the upper limit $+1$, and each of these values is assumed by $\phi(x)$ an infinity of times.

4. Let $\phi(x)=x-[x]$. This function is discontinuous for all integral values of x. In the interval $(0, 1)$ its lower limit is 0 and its upper limit 1. It is equal to 0 when $x=0$ or 1, but it is never equal to 1. Thus $\phi(x)$ never assumes a value equal to its upper limit.

5. Let $\phi(x)=0$ when x is irrational, $\phi(x)=q$ when x is a rational fraction p/q. In any interval (a, b), $\phi(x)$ has the lower limit 0, but no upper limit. But if $\phi(x)=(-1)^p q$ when $x=p/q$, $\phi(x)$ has neither an upper nor a lower limit in any interval.

90. Continuous functions of several variables. The notions of continuity and discontinuity may be extended to functions of several independent variables (Ch. II, §§ 21 *et seq.*). Their application to such functions, however, raises questions much more complicated and difficult than those which we have considered in this chapter. It would be impossible for us to discuss these questions in any detail here; but we shall, in the sequel, require to know what is meant by a continuous function of two variables, and we accordingly give the following definition. It is a straightforward generalisation of the last statement of § 84.

The function $\phi(x, y)$ of the two variables x and y is said to be **continuous** *for $x=\xi$, $y=\eta$ if, given any positive number ϵ however small, we can choose δ so that*

$$|\phi(x, y)-\phi(\xi, \eta)|<\epsilon$$

if $0\leqq|x-\xi|\leqq\delta$, $0\leqq|y-\eta|\leqq\delta$; that is to say if we can draw a square, whose sides are parallel to the axes of coordinates and of length 2δ, and whose centre is the point (ξ, η), and which is such that the value of $\phi(x, y)$ at any point inside it or on its boundary differs from $\phi(\xi, \eta)$ by less than ϵ[*].

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[*] The reader should draw a figure to illustrate the definition.

Another method of stating the definition is this: *$\phi(x, y)$ is continuous for $x=\xi$, $y=\eta$ if $\phi(\xi, \eta)$ is defined and $\phi(x, y) \to \phi(\xi, \eta)$ when $x \to \xi$, $y \to \eta$ in any manner*. This statement is apparently simpler; but it contains phrases the precise meaning of which has not yet been explained and can only be explained by the help of inequalities like those which occur in our original statement.

It is easy to prove that the sums, the products, and in general the quotients of continuous functions of two variables are themselves continuous. A polynomial in two variables is continuous for all values of the variables; and the ordinary functions of x and y which occur in every-day analysis are *generally* continuous—i.e. are continuous except for pairs of values of x and y connected by special relations.

MISCELLANEOUS EXAMPLES ON CHAPTER V.

1. Show that, if neither a nor b is zero, $ax^n+bx^{n-1}+\dots+k=ax^n(1+\epsilon_x)$, where ϵ_x is of the first order of smallness when x is large.

2. If $P(x)=ax^n+bx^{n-1}+\dots+k$, then as x increases $P(x)$ has ultimately the sign of a; and so has $P(x+\lambda)-P(x)$, if λ is any constant.

3. Show that in general

$$(ax^n+bx^{n-1}+\dots+k)/(Ax^n+Bx^{n-1}+\dots+K)=\alpha+(\beta/x)(1+\epsilon_x)$$

where $\alpha=a/A$, $\beta=(bA-aB)/A^2$ and ϵ_x is of the first order of smallness when x is large. Indicate any exceptional cases.

4. Express $$(ax^2+bx+c)/(Ax^2+Bx+C)$$
in the form $$\alpha+(\beta/x)+(\gamma/x^2)(1+\epsilon_x),$$
where ϵ_x is of the first order of smallness when x is large.

5. Show that $$\lim_{x\to\infty} \sqrt{x}\{\sqrt{(x+a)}-\sqrt{x}\}=\tfrac{1}{2}a.$$

[Use the formula $\sqrt{(x+a)}-\sqrt{x}=a/\{\sqrt{(x+a)}+\sqrt{x}\}$.]

6. Show that $\sqrt{(x+a)}=\sqrt{x}+\frac{1}{2}(a/\sqrt{x})(1+\epsilon_x)$, where ϵ_x is of the first order of smallness when x is large.

7. Find values of α and β such that $\sqrt{(ax^2+2bx+c)}-\alpha x-\beta$ has the limit 0 as $x\to\infty$; and prove that $\lim x\{\sqrt{(ax^2+2bx+c)}-\alpha x-\beta\}=(ac-b^2)/2a$.

8. Evaluate $$\lim_{x\to\infty} x\{\sqrt{[x^2+\sqrt{(x^4+1)}]}-x\sqrt{2}\}.$$

9. If $ax^2+2bxy+cy^2+2dx+2ey=0$ and $\Delta=2bde-ae^2-cd^2$, then one value of y is given by $y=\alpha x+\beta x^2+\gamma x^3(1+\epsilon_x)$, where

$$\alpha=-d/e,\quad \beta=\Delta/2e^3,\quad \gamma=(cd-be)\Delta/2e^5,$$

and ϵ_x is of the first order of smallness when x is small.

[If $y-\alpha x=\eta$ we have

$$-2e\eta=ax^2+2bx(\eta+\alpha x)+c(\eta+\alpha x)^2=Ax^2+2Bx\eta+C\eta^2,$$

say. It is evident that η is of the second order of smallness, $x\eta$ of the third, and η^2 of the fourth; and $-2e\eta = Ax^2 + 2ABx^3$, the error being of the fourth order.]

10. If $x = ay + by^2 + cy^3$, then one value of y is given by

$$y = \alpha x + \beta x^2 + \gamma x^3 (1 + \epsilon_x),$$

where $\alpha = 1/a$, $\beta = -b/a^3$, $\gamma = (2b^2 - ac)/a^5$, and ϵ_x is of the first order of smallness when x is small.

11. If $x = ay + by^n$, where n is an integer greater than unity, then one value of y is given by $y = \alpha x + \beta x^n + \gamma x^{2n-1}(1 + \epsilon_x)$, where $\alpha = 1/a$, $\beta = -b/a^{n+1}$, $\gamma = nb^2/a^{2n+1}$, and ϵ_x is of the $(n-1)$th order of smallness when x is small.

12. Prove that $(\sec x - \tan x) \to 0$ as $x \to \frac{1}{2}\pi$.

13. Prove that $\phi(x) = 1 - \cos(1 - \cos x)$ is of the fourth order of smallness when x is small; and find the limit of $\phi(x)/x^4$ as $x \to 0$.

14. Prove that $\phi(x) = x \sin(\sin x) - \sin^2 x$ is of the sixth order of smallness when x is small; and find the limit of $\phi(x)/x^6$ as $x \to 0$.

15. From a point T in a radius OA of a circle produced, a tangent TP is drawn to the circle, touching it in P, and PN is drawn perpendicular to OA. Show that $NA/AT \to 1$ as P moves up to A.

16. Tangents are drawn to a circular arc at its middle point and its extremities: Δ is the area of the triangle formed by the chord of the arc and the two tangents at the extremities, and Δ' the area of that formed by the three tangents. Show that, as the length of the arc tends to zero, $\Delta/\Delta' \to 4$.

17. For what values of a does $\{a + \sin(1/x)\}/x$ tend to (1) $+\infty$, (2) $-\infty$, as $x \to 0$? [To $+\infty$ if $a > 1$, to $-\infty$ if $a < -1$: if $-1 \leqq a \leqq 1$ the function oscillates.]

18. For what values of x is the function $\phi(x) = 1 - \lim_{n\to\infty} \sqrt[n]{(\cos^2 \pi x)}$ continuous or discontinuous?

19. Show that the least positive root of the equation $\sin x = ax$ is a continuous function of a throughout the interval $(0, 1)$, and increases steadily from 0 to π as a decreases from 1 to 0. [The function is the inverse of $(\sin x)/x$: apply § 88.]

20. The least positive root of $\tan x = ax$ is a continuous function of a throughout the interval $(1, \infty)$, and increases steadily from 0 to $\frac{1}{2}\pi$ as a increases from 1 towards ∞.

21. If $\phi(x) = 1/q$ when $x = p/q$, $\phi(x) = 0$ when x is irrational, $\phi(x)$ is continuous for all irrational and discontinuous for all rational values of x.

22. Let $\phi(x) = x$ when x is rational and $\phi(x) = 1 - x$ when x is irrational. Show that, as x increases from 0 to 1, $\phi(x)$ assumes every value, between and including 0 and 1, once and once only, although $\phi(x)$ is discontinuous for every value of x except $x = \frac{1}{2}$.

CHAPTER VI.

DERIVATIVES AND INTEGRALS

(A FIRST CHAPTER IN THE DIFFERENTIAL AND INTEGRAL CALCULUS).

91. Derivatives or Differential Coefficients. Let us return to the consideration of the properties which we naturally associate with the notion of a curve. The first and most obvious property is, as we saw in the last chapter, that which gives a curve its appearance of *connectedness*, and which we embodied in our definition of a *continuous function*, i.e. a function whose graph is a continuous curve, such as the curve C of Fig. 38.

The ordinary curves which occur in elementary geometry, such as straight lines (which we have agreed, for convenience, to class as curves), circles and conic sections, have of course many other properties of a general character. The simplest and most noteworthy of these is perhaps that they have a definite *direction* at every point, or what is the same thing, that at every point of the curve we can draw a *tangent* to it. The reader will probably remember that in elementary geometry the tangent to a curve at P is defined to be 'the limiting position of the chord PQ, when Q moves up towards coincidence with P.' Let us consider what is implied in the assumption of the existence of such a limiting position.

In the figure (Fig. 42) P is a fixed point on the curve, and Q a variable point; PM, QN are parallel to OY and PR to OX. We can denote the coordinates of P by x, y and those of Q by $x+h$, $y+k$: h will be positive or negative according as N lies to the right or left of M.

We have assumed that there is a tangent to the curve at P, or that there is a definite 'limiting position' of the chord PQ. Suppose that the tangent at P, PT, makes an angle ψ with OX.

Then to say that PT is the limiting position of PQ is equivalent to saying that the limit of the angle QPR is ψ, when Q approaches P along the curve from either side.

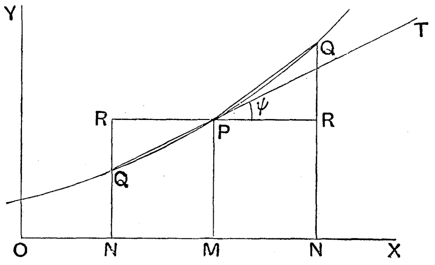

Fig. 42.

We have now to distinguish two cases, a general case and an exceptional one.

The general case is that in which $\psi \neq \frac{1}{2}\pi$: i.e. PT is not parallel to OY. In this case RPQ tends to the limit ψ, and

$$RQ/PR = \tan RPQ,$$

tends to the limit $\tan\psi$. Now

$$RQ/PR = (NQ - MP)/MN = \{\phi(x+h) - \phi(x)\}/h.$$

Hence $$\lim_{h\to 0}\{\phi(x+h) - \phi(x)\}/h = \tan\psi \quad \text{............(1)}.$$

The reader should be careful to note that in all these equations all lengths (such as RQ) are regarded as affected with the proper sign (e.g. RQ is negative in the figure when Q lies to the left of P), and that the convergence to the limit is unaffected by the sign of h.

Thus the assumption that the curve which is the graph of $\phi(x)$ has a tangent at P, which is not perpendicular to the axis of x, implies that $\phi(x)$ has, for the particular value of x corresponding to P, the property that $\{\phi(x+h) - \phi(x)\}/h$ *tends to a limit when* h *tends to zero.*

This of course implies that both of

$$\{\phi(x+h) - \phi(x)\}/h, \quad \{\phi(x-h) - \phi(x)\}/(-h)$$

tend to limits when $h\to 0$ *by positive values only*, and that the two limits are equal. The reader will easily convince himself that if these limits exist but are *not* equal, the curve $y=\phi(x)$ has an *angle* at the particular point considered. In such a case the curve might of course be said to have *two* directions at P, or two tangents. At any rate it has not *one definite tangent* or direction (see Fig. 43).

Now let us suppose that the curve has (like the circle or ellipse) a tangent at every point of its length, or at any rate every portion of its length which corresponds to a certain range of variation of x. Further let us suppose this tangent never perpendicular to the axis of x: in the case of a circle this would of course restrict us to considering an arc less than a semicircle. Then an equation such as (1) holds for all values of x which fall inside this range. To each such value of x corresponds a value of $\tan\psi$: $\tan\psi$ is a function of x, which is defined for all values of x in the range of values under consideration, and which may be calculated or *derived from* the original function $\phi(x)$. We shall call this function the **derivative** or *derived function* of $\phi(x)$, and we shall denote it by

$$\phi'(x).$$

Another name for the derived function of $\phi(x)$ is the **differential coefficient** of $\phi(x)$; and the operation of calculating $\phi'(x)$ from $\phi(x)$ is generally known as **differentiation**: these names are by no means happily chosen, but the reader should be familiar with them.

Before we proceed to consider the special case mentioned above, in which $\psi = \frac{1}{2}\pi$, we shall illustrate our definition by some general remarks and particular illustrations.

92. Some general remarks. (1) The existence of a derived function $\phi'(x)$ for all values of x in the interval $a \leqq x \leqq b$ implies that at every point of this interval $\phi(x)$ is continuous. For it is evident that

$$\lim \{\phi(x+h) - \phi(x)\}/h$$

cannot exist unless $\lim \phi(x+h) = \phi(x)$, and it is this which is the property denoted by continuity.

(2) It is natural to ask whether the converse is true, i.e. whether every continuous curve has a definite tangent at every point, and every function a differential coefficient for every value of x for which it is continuous[*]. The answer is obviously *No*: it

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[*] We leave out of account the exceptional case (which we have still to examine) in which the curve is supposed to have a tangent perpendicular to OX: apart from this possibility the two forms of the question stated above are equivalent.

is sufficient to consider the curve formed by two straight lines meeting to form an angle (Fig. 43). The reader will verify at once that in this case $\{\phi(x+h)-\phi(x)\}/h$ has the limit $\tan\alpha$ as $h \to 0$, if $x < x_0$, but the limit $\tan\beta$ if $x > x_0$. And

$$\{\phi(x_0+h)-\phi(x_0)\}/h$$

has no limit as $h \to 0$. In fact it has the limit $\tan\beta$ if $h \to 0$ by positive values and the limit $\tan\alpha$ if $h \to 0$ by negative values.

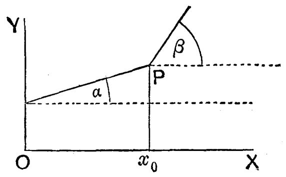

Fig. 43.

This is of course one of the cases in which a curve might reasonably be said to have *two* directions at a point. But the following example, although more difficult, shows conclusively that there are cases in which a continuous curve cannot be said to have either one definite direction or several directions at one of its points. Draw the graph of the function $x\sin(1/x)$ (Fig. 16, Ch. II). The function is not defined for $x=0$ and so is discontinuous for $x=0$. But (Exs. XXXVIII. 17) the function defined by the equations

$$\phi(x)=x\sin(1/x) \quad (x\neq 0), \quad \phi(x)=0 \quad (x=0)$$

is continuous for $x=0$: and the graph of this function, if it could be drawn completely (and we can draw it quite adequately enough to obtain a general idea of its appearance) would satisfy our common sense intuition of what a continuous curve should be like[*].

But $\phi(x)$ has no derivative for $x=0$. For $\phi'(0)$ would be, by definition, $\lim\{\phi(h)-\phi(0)\}/h$ or $\lim\sin(1/h)$ and this limit does not exist. The reader, on studying the figure carefully, will probably agree that the curve does not look as if it had a tangent or a definite direction at the origin.

It has even been shown that a function of x may be continuous and yet have no derivative for *any* value of x. But the proof of this is much more difficult; the reader who is interested in the question may be referred to Bromwich's *Infinite Series*, pp. 490–1, or Hobson's *Theory of Functions of a Real Variable*, pp. 620–5.

(3) **Rates of Variation.** There is another general point of view from which the notion of the derivative of a function $\phi(x)$ may be considered. Let us suppose that OM, measured along OX, represents the value of x, and ON, measured along OY, represents the value of $y=\phi(x)$: so that the corresponding point on the graph of $\phi(x)$ is obtained by completing the rectangle $OMPN$. Further, let us imagine the variation of x as taking

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[*] No doubt it is a somewhat peculiar curve, but there is obviously no breach of continuity. The apparently arbitrary assignment of the value 0 to $\phi(x)$ for $x=0$ is in reality natural enough—it merely amounts to the filling in, so to say, of a single point previously missing in the curve $y=x\sin(1/x)$.

place in time, so that M moves along the axis of x: it is most convenient to suppose that it moves steadily in the same direction with uniform velocity V, so that at time t, $OM = x = Vt$.

Let us suppose that $t + \tau$ is a time a little after or a little before t, according as τ is positive or negative: and let M', N', P' denote the corresponding positions of M, N, P. Finally let $OM' = x + h$, so that $h = V\tau$, and let NP cut $M'P'$ in R.

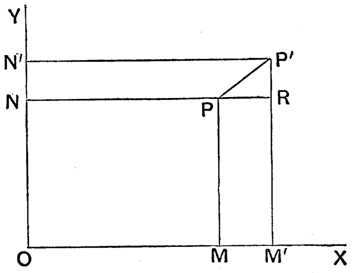

FIG. 44.

Then if $\phi(x)$ is a function such as those which we have been considering, the ratio RP'/PR converges to a limit, say l, when $\tau \rightarrow 0$: l is in fact the same as $\tan \psi$ in § 91.

Also $NN'/\tau = RP'/\tau = (RP'/PR)(PR/\tau)$, which has the limit Vl when $\tau \rightarrow 0$. But the limit of NN'/τ is precisely what is meant by the *velocity* of N. Thus the value of the derivative $\phi'(x)$ represents *the velocity of N compared with the velocity of M*. If, as is simplest, we suppose that M is moving with unit velocity, then $\phi'(x)$ is actually equal to the velocity of N. Another way of expressing the same thing is to say that $\phi'(x)$ represents *the rate of increase* of $\phi(x)$, taking the rate of increase of x as our standard; if $\phi(x)$ is *decreasing*, this 'rate of increase' is of course negative.

It is evident that the geometrical apparatus, by means of which we have arrived at this notion of the meaning of $\phi'(x)$, is not essential. There is no reason why we should imagine the values of y represented along a line perpendicular to OX: this representation was adopted merely to show that our present point of view is not essentially distinct from that of § 91. Another way of looking at the matter is as follows. Taking $V = 1$ as above we have $x = t$: and there is no need for any special geometrical representation of x at all: x *is* the time t, simply, and we may regard the time t itself as the independent variable. We may simply take a line L and suppose the values of $y = \phi(t)$ represented by the lengths OQ measured along this line from a fixed point O. If, at time t, $OQ = \phi(t)$, $\phi'(t)$ is the velocity of Q at time t.

Example. The reader is no doubt familiar with the formulae

$$v=u+ft, \quad s=ut+\tfrac{1}{2}ft^2$$

for a particle moving in a line with 'uniform acceleration f.' He will easily prove from the definition of a derivative that if $s=ut+\frac{1}{2}ft^2=\phi(t)$, then $\phi'(t)=u+ft$, i.e. $\phi'(t)=v$. The rate of increase of s is v; and similarly the rate of increase of v is f.

Examples XLI. 1. If $\phi(x)$ is a constant, $\phi'(x)=0$. Interpret this result geometrically.

2. If $\phi(x)=x$, $\phi'(x)=1$. Prove this (a) from the formal definition and (b) by geometrical considerations.

3. If $\phi(x)=ax+b$, $\phi'(x)=a$. Again interpret geometrically.

4. If $\phi(x)=x^m$, where m is a positive integer, $\phi'(x)=mx^{m-1}$.

[For
$$\begin{aligned}\phi'(x)&=\lim\{(x+h)^m-x^m\}/h\\ &=\lim\left\{mx^{m-1}+\frac{m(m-1)}{1.2}x^{m-2}h+\dots+h^{m-1}\right\}.\end{aligned}$$

The reader should observe that this method cannot be applied to $x^{p/q}$, where p/q is a fraction, as we have no means of expressing $(x+h)^{p/q}$ as a finite series of powers of h. We shall show later on (§ 98) that the result of this example holds for all rational values of m. Meanwhile the reader will find it instructive to determine $\phi'(x)$ when m has some special fractional value (e.g. $\frac{1}{2}$), by means of some special device.]

5. If $\phi(x)=\sin x$, $\phi'(x)=\cos x$; and if $\phi(x)=\cos x$, $\phi'(x)=-\sin x$.

[For example, if $\phi(x)=\sin x$,

$$\{\phi(x+h)-\phi(x)\}/h=\{2\sin\tfrac{1}{2}h\cos(x+\tfrac{1}{2}h)\}/h$$

the limit of which is $\cos x$ when $h\to 0$, since $\lim\cos(x+\frac{1}{2}h)=\cos x$ (the cosine being a continuous function) and $\lim\{(\sin\frac{1}{2}h)/\frac{1}{2}h\}=1$ (Ex. XXXVII. 14).]

6. **Equations of the tangent and normal to a curve** $y=\phi(x)$. The tangent to the curve at the point (x_0, y_0) is the line through (x_0, y_0) which makes an angle ψ with OX, where $\tan\psi=\phi'(x_0)$. Its equation is therefore

$$y-y_0=(x-x_0)\,\phi'(x_0);$$

and the equation of the normal (the perpendicular to the tangent at the point of contact) is

$$(y-y_0)\,\phi'(x_0)+x-x_0=0.$$

We have assumed that the tangent is not parallel to the axis of y. In this special case it is obvious that the tangent and normal are $x=x_0$ and $y=y_0$ respectively.

7. Write down the equations of the tangent and normal at any point of the parabola $x^2=4ay$. Show that if $x_0=2a/m$, $y_0=a/m^2$ is a point of the curve, the tangent there is $x=my+(a/m)$.

8. Find the equations of the tangent and normal at any point $(m, c^2/m)$ of the rectangular hyperbola $xy=c^2$.

9. Write down the equations of the tangents at any points of the curves $y=\sin x$, $y=\cos x$; and find where they make a given angle a with the axis of x.

93. We have seen that if $\phi(x)$ is not continuous for a value of x it cannot possibly have a derivative for that value of x. Thus $1/x^2$ cannot have a derivative for $x=0$: nor can any function, such as $\sin(1/x)$, which is not defined for $x=0$ and so is necessarily discontinuous for $x=0$. Or again the function $[x]$, which is discontinuous for every integral value of x, has no derivative for any such value of x.

Example. Since $[x]$ is constant between every two integral values of x, its derivative is zero for all values of x for which it is defined. Thus the derivative of $[x]$, which we may represent by $[x]'$, is a function equal to zero for all values of x save integral values and undefined for integral values. It is interesting to note that the functions $[x]'$ and $1-\dfrac{\sin \pi x}{\sin \pi x}$ are *the same function.*

We saw too in Ex. XXXVIII. 7 that the types of discontinuity which occur most commonly, when we are dealing with the very simplest and most obvious kinds of functions, such as polynomials or rational or trigonometrical functions, are associated with an equation of the type

$$\phi(x) \to +\infty \quad (\text{or } -\infty).$$

In all these cases, as in such cases as those considered above, there is no derivative for certain special values of x. In fact, as was pointed out above, *all discontinuities of* $\phi(x)$ *are also discontinuities of* $\phi'(x)$. But the converse is not true, as we may easily see by considering the special case referred to in § 91, in which the graph of $\phi(x)$ has a tangent parallel to OY. The most typical cases are shown in Fig. 45. In cases (c) and (d) the function is

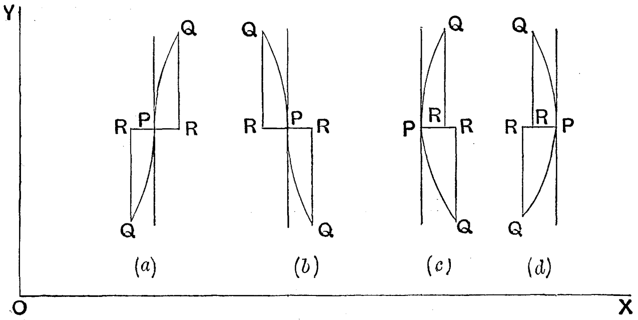

Fig. 45.

two valued on one side of P and not defined on the other. In such cases we may consider the two sets of values of $\phi(x)$, which occur on one side of P or the other, as defining distinct functions $\phi_1(x)$ and $\phi_2(x)$, the upper part of the curve corresponding to $\phi_1(x)$.

The reader will easily convince himself that in (*a*)

$$\{\phi(x+h)-\phi(x)\}/h \to +\infty,$$

as $h \to 0$, and in (*b*)

$$\{\phi(x+h)-\phi(x)\}/h \to -\infty;$$

while in (*c*)

$$\{\phi_1(x+h)-\phi_1(x)\}/h \to +\infty, \quad \{\phi_2(x+h)-\phi_2(x)\}/h \to -\infty,$$

and in (*d*)

$$\{\phi_1(x+h)-\phi_1(x)\}/h \to -\infty, \quad \{\phi_2(x+h)-\phi_2(x)\}/h \to +\infty,$$

though of course in (*c*) only positive and in (*d*) only negative values of h can be considered, a fact which by itself would preclude the existence of a derivative at P.

94. Some general rules for differentiation. Throughout the theorems which follow we assume that the functions $f(x)$ and $F(x)$ have derivatives $f'(x)$ and $F'(x)$, for the values of x considered.

(1) *If* $\phi(x)=f(x)+F(x)$, $\phi(x)$ *has a derivative*

$$\phi'(x)=f'(x)+F'(x).$$

(2) *If* $\phi(x)=cf(x)$, *where* c *is a constant,* $\phi(x)$ *has a derivative* $\phi'(x)=cf'(x)$.

We leave it as an exercise to the reader to deduce these results from the general theorems on limits stated in Ex. XXXVI. 1.

(3) *If* $\phi(x)=f(x)\,F(x)$, $\phi(x)$ *has a derivative*

$$\phi'(x)=f(x)\,F'(x)+f'(x)\,F(x).$$

For $\phi'(x)=\lim\{f(x+h)\,F(x+h)-f(x)\,F(x)\}/h$

$$=\lim\left\{f(x+h)\frac{F(x+h)-F(x)}{h}+F(x)\frac{f(x+h)-f(x)}{h}\right\}$$

$$=f(x)\,F'(x)+F(x)f'(x).$$

(4) $\phi(x)=1/f(x)$ *has a derivative* $\phi'(x)=-f'(x)/\{f(x)\}^2$.

In this theorem we of course suppose that $f(x)$ is not zero for the particular value of x under consideration. Then

$$\phi'(x) = \lim \frac{1}{h}\left\{\frac{f(x)-f(x+h)}{f(x+h)\,f(x)}\right\} = -\frac{f'(x)}{\{f(x)\}^2}.$$

(5) *$\phi(x) = f(x)/F(x)$ has a derivative*

$$\phi'(x) = \frac{f'(x)\,F(x) - f(x)\,F'(x)}{\{F(x)\}^2}.$$

This follows at once from (3) and (4).

(6) *$\phi(x) = f(x+a)$ has a derivative $\phi'(x) = f'(x+a)$.*

(7) *$\phi(x) = f(ax)$ has a derivative $\phi'(x) = af'(ax)$.*

The proof of the first of these two theorems we leave as an exercise to the reader. As regards the second, we have

$$\phi'(x) = a \lim \{f(ax+ah) - f(ax)\}/ah,$$

or, putting $ax = u$ and $ah = k$,

$$\phi'(x) = a \lim_{k \to 0} \{f(u+k) - f(u)\}/k = af'(u) = af'(ax).$$

(8) *$\phi(x) = f(ax+b)$ has a derivative $\phi'(x) = af'(ax+b)$.*

This follows at once from (6) and (7).

Thus if $f(x) = \sin x$, $\phi(x) = \sin(ax+b)$, $f'(x) = \cos x$, and $\phi'(x) = a\cos(ax+b)$. In particular, if $a = 1$ and $b = \frac{1}{2}\pi$, $\phi(x) = \cos x$ and $\phi'(x) = -\sin x$.

The theorems (6), (7), (8) are capable of simple geometrical interpretations. Thus the graph of $\phi(x) = f(x+a)$ is obtained by translating that of $f(x)$ through a distance a parallel to the axis OX; and the tangent to the curve $y = \phi(x)$, at the point whose abscissa is x, is parallel to the tangent to the curve $y = f(x)$ at the point whose abscissa is $x + a$.

The graph of $\phi(x) = f(ax)$ may be obtained from that of $f(x)$ by a geometrical construction which is probably sufficiently clear from Fig. 46 (where $a = 2$).

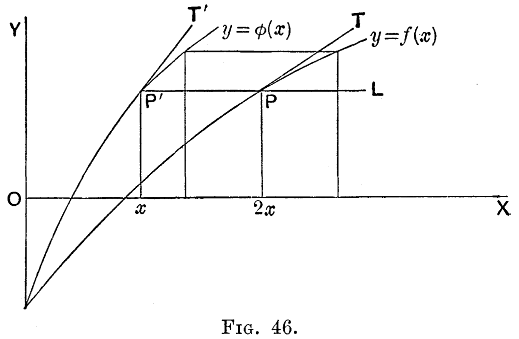

FIG. 46.

Here $\phi'(x) = \tan LP'T'$ and $f'(2x) = \tan LPT$, and the equation $\phi'(x) = 2f'(2x)$ expresses the fact that the 'gradient' at P' of the locus of P' is twice that at P of the locus of P.

Later (§ 108) we shall prove a much more general theorem of which (6), (7), (8) are particular cases.

(9) *If* $y = \phi(x)$, *where* ϕ *is the function inverse to* ψ, *so that* $x = \psi(y)$, *and* $\psi(y)$ *has a derivative* $\psi'(y)$ *which is not equal to zero, then* $\phi(x)$ *has a derivative* $\phi'(x) = 1/\psi'(y)$.

For if $\phi(x+h) = y + k$, $k \rightarrow 0$ as $h \rightarrow 0$, and

$$\phi'(x) = \lim_{h \rightarrow 0} \frac{\phi(x+h) - \phi(x)}{(x+h) - x} = \lim_{k \rightarrow 0} \frac{(y+k) - y}{\psi(y+k) - \psi(y)} = \frac{1}{\psi'(y)}.$$

The last function may now be expressed in terms of x by means of the relation $y = \phi(x)$, so that $\phi'(x) = 1/\psi'\{\phi(x)\}$. This theorem enables us to differentiate any function if we know the derivative of the inverse function.

95. Derivatives of complex functions. So far we have supposed that $y = \phi(x)$ is a purely *real* function of x. If y is a complex function $\phi(x) + i\psi(x)$, we simply define the derivative of y as being $\phi'(x) + i\psi'(x)$. The reader will have no difficulty in seeing that all the theorems proved above, except (9) (to which we cannot at present attach any meaning when y is complex), retain their validity in this case.

96. The notation of the differential calculus. We have already explained that what we call a *derivative* is often called a *differential coefficient*. Not only a different name but a different notation is often used; the derivative of the function $y = \phi(x)$ is often denoted by one or other of the expressions

$$D_x y, \quad \frac{dy}{dx}.$$

Of these the last is the most usual and convenient: the reader must however be careful to remember that dy/dx does *not* mean 'a certain number dy divided by another number dx': it means 'the result of a certain operation D_x or d/dx applied to $y = \phi(x)$,' the operation being that of forming the quotient $\{\phi(x+h) - \phi(x)\}/h$ and making $h \rightarrow 0$.

Of course a notation at first sight so peculiar would not have been adopted without some reason, and the reason was as follows. The denominator h of the fraction $\{\phi(x+h)-\phi(x)\}/h$ is the difference of the values $x+h$, x of the independent variable x; similarly the numerator is the difference of the corresponding values $\phi(x+h)$, $\phi(x)$ of the dependent variable y. These differences may be called the *increments* of x and y respectively, and denoted by δx and δy. Then the fraction is $\delta y/\delta x$, and it is for some purposes convenient to denote the limit of the fraction, which is the same thing as $\phi'(x)$, by dy/dx. But this notation is purely symbolical: the dy and dx which occur in it cannot be separated; standing by themselves they would mean nothing: in particular dy and dx do not mean $\lim \delta y$ and $\lim \delta x$, these limits being simply zero. The reader will have to become familiar with this notation, but so long as it puzzles him he will be wise to avoid it by writing the differential coefficient in the form $D_x y$, or using the notation $\phi(x)$, $\phi'(x)$, as we have done in the preceding sections of this chapter.

The first five theorems of § 94 may of course at once be translated into this notation. They may be stated as follows:

$$(1)\quad \text{if } y=y_1+y_2,\qquad \frac{dy}{dx}=\frac{dy_1}{dx}+\frac{dy_2}{dx};$$

$$(2)\quad \text{if } y=cy_1,\qquad \frac{dy}{dx}=c\frac{dy_1}{dx};$$

$$(3)\quad \text{if } y=y_1y_2,\qquad \frac{dy}{dx}=y_1\frac{dy_2}{dx}+y_2\frac{dy_1}{dx};$$

$$(4)\quad \text{if } y=\frac{1}{y_1},\qquad \frac{dy}{dx}=-\frac{1}{y_1^2}\frac{dy_1}{dx};$$

$$(5)\quad \text{if } y=\frac{y_1}{y_2},\qquad \frac{dy}{dx}=\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)\Big/ y_2^2.$$

The new notation is less convenient for the theorems (6), (7), (8). But theorem (9) may be expressed simply by the equation

$$(9)\qquad \frac{dy}{dx}=1\Big/\left(\frac{dx}{dy}\right).$$

Examples XLII. 1. If $y=\sin x\cos x=\frac{1}{2}\sin 2x$, find dy/dx by means of Theorem (3) and also by means of Theorems (2) and (7), and verify that the results are in agreement.

2. Since $(\cos^2 x+\sin^2 x)^n=1$, it follows that $D_x(\cos^2 x+\sin^2 x)^n=0$. Verify this by expanding and differentiating each term separately.

3. If $y=y_1y_2y_3$ then

$$\frac{dy}{dx}=y_2y_3\frac{dy_1}{dx}+y_3y_1\frac{dy_2}{dx}+y_1y_2\frac{dy_3}{dx},$$

and if $y=y_1y_2\dots y_n$,

$$\frac{dy}{dx}=\sum_{r=1}^{n} y_1y_2\dots y_{r-1}y_{r+1}\dots y_n\frac{dy_r}{dx}.$$

In particular, if $y=z^n$, $dy/dx=nz^{n-1}(dz/dx)$.

4. If $y=y_1 y_2 \dots y_n$, then

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{y_1}\frac{dy_1}{dx}+\frac{1}{y_2}\frac{dy_2}{dx}+\dots+\frac{1}{y_n}\frac{dy_n}{dx}.$$

In particular, if $y=z^n$, $\quad \dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{n}{z}\dfrac{dz}{dx}.$

5. Employ Theorem (9) and Ex. 3 to show that the differential coefficient of x^m is mx^{m-1} for all rational values of m.

[First suppose $m=1/q$, where q is an integer. If $y=x^{1/q}$, $x=y^q$ and $dy/dx=1/(dx/dy)=(1/q)y^{1-q}=(1/q)x^{(1/q)-1}$. Next, if $m=p/q$, $y=z^p$, where $z=x^{1/q}$, and the proof may be completed by means of the result of Ex. 3.]

6. If $y=\arcsin x$, $x=\sin y$ and

$$\frac{dy}{dx}=1\bigg/\left(\frac{dx}{dy}\right)=\frac{1}{\cos y}=\pm\frac{1}{\sqrt{(1-x^2)}}.$$

The positive sign must be chosen if $\cos y$ is positive, and in particular if $-\frac{1}{2}\pi<y<\frac{1}{2}\pi$.

97. Standard forms. We shall now investigate more systematically the forms of the derivatives of a few of the simplest types of functions.

A. **Polynomials.** If $\phi(x)=a_0x^n+a_1x^{n-1}+\dots+a_n$, then

$$\phi'(x)=na_0x^{n-1}+(n-1)a_1x^{n-2}+\dots+a_{n-1}.$$

It is sometimes more convenient to use for the standard form of a polynomial of degree n in x what is known as the *binomial form*, or the form with binomial coefficients, viz.

$$a_0x^n+\binom{n}{1}a_1x^{n-1}+\binom{n}{2}a_2x^{n-2}+\dots+a_n$$

which differs from the first form in that the coefficients are taken to be $\binom{n}{1}a_1$, $\binom{n}{2}a_2$, ... instead of a_1, a_2, ... In this case

$$\phi'(x)=n\left\{a_0x^{n-1}+\binom{n-1}{1}a_1x^{n-2}+\binom{n-1}{2}a_2x^{n-3}+\dots+a_{n-1}\right\}.$$

The binomial form of $\phi(x)$ is often written symbolically as

$$(a_0, a_1, \dots, a_n \between x, 1)^n;$$

and then $\quad \phi'(x)=n(a_0, a_1, \dots, a_{n-1} \between x, 1)^{n-1}.$

We shall see later that $\phi(x)$ can always be expressed as the product of n factors in the form

$$\phi(x)=a_0(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n),$$

where the α's are real or complex numbers. Then

$$\phi'(x)=a_0\Sigma(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n),$$

the notation implying that we form all possible products of $n-1$ factors, and add them all together. This form of the result holds even if several of the numbers α are equal; but of course then some of the terms on the right-hand side are repeated. The reader will easily verify that if

$$\phi(x) = a_0(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\dots(x-\alpha_\nu)^{m_\nu}$$

then $$\phi'(x) = a_0 \Sigma\, m_1(x-\alpha_1)^{m_1-1}(x-\alpha_2)^{m_2}\dots(x-\alpha_\nu)^{m_\nu}.$$

Examples XLIII. 1. Show that if $\phi(x)$ is a polynomial, $\phi'(x)$ is the coefficient of h in the expansion of $\phi(x+h)$ in powers of h.

2. If $\phi(x)$ is divisible by $(x-\alpha)^2$, $\phi'(x)$ is divisible by $x-\alpha$: and generally, if $\phi(x)$ is divisible by $(x-\alpha)^m$, $\phi'(x)$ is divisible by $(x-\alpha)^{m-1}$.

3. Conversely, if $\phi(x)$ and $\phi'(x)$ are *both* divisible by $x-\alpha$, $\phi(x)$ is divisible by $(x-\alpha)^2$; and if $\phi(x)$ is divisible by $x-\alpha$ and $\phi'(x)$ by $(x-\alpha)^{m-1}$, $\phi(x)$ is divisible by $(x-\alpha)^m$.

4. The multiple roots (if any) of $\phi(x)=0$ are the values of x given by equating to zero the H.C.F. of $\phi(x)$ and $\phi'(x)$.

5. Find all the roots, with their degrees of multiplicity, of

$$x^4+3x^3-3x^2-11x-6=0, \quad x^6+2x^5-8x^4-14x^3+11x^2+28x+12=0.$$

6. If $ax^2+2bx+c$ has a double root, i.e. is of the form $a(x-\alpha)^2$, then $2(ax+b)$ must be divisible by $x-\alpha$: i.e. $\alpha=-b/a$. This value of x must satisfy $ax^2+2bx+c=0$. Verify that the condition thus arrived at is $ac-b^2=0$.

7. The equation $1/(x-a)+1/(x-b)+1/(x-c)=0$ can have a pair of equal roots only if $a=b=c$. (*Math. Trip.* 1905.)

8. Show that $$ax^3+3bx^2+3cx+d=0,$$ has a double root if $G^2+4H^3=0$, where $H=ac-b^2$, $G=a^2d-3abc+2b^3$.

[Put $ax+b=y$, when the equation reduces to $y^3+3Hy+G=0$. This must have a root in common with $y^2+H=0$.]

9. The reader may verify that if α, β, γ, δ are the roots of

$$ax^4+4bx^3+6cx^2+4dx+e=0,$$

the equation whose roots are

$$\tfrac{1}{12}a\{(\alpha-\beta)(\gamma-\delta)-(\gamma-\alpha)(\beta-\delta)\},$$

and two similar expressions formed by permuting α, β, γ cyclically, is

$$4\theta^3-g_2\theta-g_3=0,$$

where $$g_2=ae-4bd+3c^2, \quad g_3=ace+2bcd-ad^2-eb^2-c^3.$$

It is clear that if two of α, β, γ, δ are equal two of the roots of this cubic will be equal. Using the result of Ex. 8 we deduce $g_2{}^3-27g_3{}^2=0$.

10. *If $\phi(x)$ is any polynomial, then between any pair of roots of $\phi(x)=0$ lies a root of $\phi'(x)=0$.*

A general proof of this theorem, applying not only to polynomials but to other classes of functions, will be given later. The following is an algebraical proof valid for polynomials only. We suppose that a, β are two successive roots, repeated respectively m and n times, so that

$$\phi(x)=(x-a)^m(y-\beta)^n\theta(x),$$

where $\theta(x)$ is a polynomial which has the same sign, say the positive sign, for $a \leqq x \leqq \beta$. Then

$$\begin{aligned}\phi'(x)&=(x-a)^m(y-\beta)^n\theta'(x)+\{m(x-a)^{m-1}(y-\beta)^n+n(x-a)^m(y-\beta)^{n-1}\}\theta(x)\\ &=(x-a)^{m-1}(y-\beta)^{n-1}[(x-a)(y-\beta)\theta'(x)+\{m(y-\beta)+n(x-a)\}\theta(x)]\\ &=(x-a)^{m-1}(y-\beta)^{n-1}F(x),\end{aligned}$$

say. Now $F(a)=m(a-\beta)\theta(a)$ and $F(\beta)=n(\beta-a)\theta(\beta)$, which have opposite signs. Hence $F(x)$, and so $\phi'(x)$, vanishes for some value of x between a and β.

98. B. **Rational Functions.** If

$$R(x)=P(x)/Q(x)$$

where P and Q are polynomials, it follows at once from § 94 that

$$R'(x)=\{P'(x)Q(x)-P(x)Q'(x)\}/\{Q(x)\}^2,$$

and this enables us to write down the derivative of any rational function. The form in which we obtain it, however, may or may not be the simplest possible. It will be the simplest possible if $Q(x)$ and $Q'(x)$ have no common factor, i.e. if $Q(x)$ has no repeated factor. But if $Q(x)$ has a repeated factor the expression which we obtain for $R'(x)$ will be capable of further reduction.

It is very often convenient, in differentiating a rational function, to employ the method of partial fractions. We shall suppose that $Q(x)$, as in § 97 above, is expressed in the form

$$a_0(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\dots(x-\alpha_\nu)^{m_\nu}.$$

Then it is proved in treatises on Algebra* that $R(x)$ can be expressed in the form

$$\begin{aligned}\Pi(x)&+\frac{A_{1,1}}{x-\alpha_1}+\frac{A_{1,2}}{(x-\alpha_1)^2}+\dots+\frac{A_{1,m_1}}{(x-\alpha_1)^{m_1}}\\ &+\frac{A_{2,1}}{x-\alpha_2}+\frac{A_{2,2}}{(x-\alpha_2)^2}+\dots+\frac{A_{2,m_2}}{(x-\alpha_2)^{m_2}}+\dots\dots\end{aligned}$$

where $\Pi(x)$ is a polynomial, i.e. as the sum of a polynomial and the sum of a number of terms of the type

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* See e.g. Chrystal's *Algebra*, vol. I, pp. 151 *et seq.*

$$A/(x-\alpha)^p$$

where α is a root of $Q(x)=0$. We know already how to find the derivative of the polynomial: all that remains, therefore, is that we should determine the derivative of the function written above, in which, it must be remembered, α may be real or complex.

We shall first establish the following proposition: **the derivative of x^m is mx^{m-1}, for all integral or rational values of m, positive or negative.**

We have already seen (Ex. XLI. 4) that this is so when m is a positive integer. The more general result (which was proved indirectly in Ex. XLII. 5) is a mere corollary from Ex. XXXVII. 4. For, if $\phi(x)=x^m$,

$$\begin{aligned}\phi'(x) &= \lim_{h\to 0}\{(x+h)^m-x^m\}/h \\ &= \lim_{\xi\to x}(\xi^m-x^m)/(\xi-x)=mx^{m-1}.\end{aligned}$$

From this result it follows at once, by Theorem (6) (§ 94) that if $\phi(x)=A/(x-\alpha)^p$ then $\phi'(x)=-pA/(x-\alpha)^{p+1}$, provided α is real.

We shall however require the result for complex as well as for real values of a. We shall therefore give a direct proof applicable to the case in which p is a positive integer and a has any value whatever. Then

$$\{(x+h-a)^{-p}-(x-a)^{-p}\}/h=-(x+h-a)^{-p}(x-a)^{-p}\{(x+h-a)^p-(x-a)^p\}/h$$

which, when we expand $(x+h-a)^p$ in powers of h by the Binomial Theorem, and make h tend to 0, obviously tends to

$$-(x-a)^{-2p}\{p(x-a)^{p-1}\}=-p(x-a)^{-p-1}.$$

We are now able to write down the derivative of the general rational function $R(x)$, in the form

$$\Pi'(x)-\frac{A_{1,1}}{(x-\alpha_1)^2}-\frac{2A_{1,2}}{(x-\alpha_1)^3}-\dots-\frac{A_{2,1}}{(x-\alpha_2)^2}-\frac{2A_{2,2}}{(x-\alpha_2)^3}-\dots$$

Examples XLIV. 1. Differentiate

$$x/(1-x^2),\quad (1+x^2)/(1-x^2),\quad \{(1+x)/(1-x)\}^2,\quad (1-x+x^2)/(1+x+x^2),$$
$$x(1-x)/(1+x^3),\quad x(1+x^2)/(1+x^4),\quad x(1-x^2)/(1+x^2+x^4),$$
$$1/(x-a)(x-b)(x-c),\quad (x-a)/(x-b)(x-c),\quad (x-a)(x-b)/(x-c)(x-d),$$
$$x/(x^2+a^2)(x^2+b^2),\quad (x^2+a^2)/(x^2+b^2),\quad (x-a)^2/(x+a)^2(x^2+b^2).$$

2. The derivative of $(ax^2+2bx+c)/(Ax^2+2Bx+C)$ is

$$\{(ax+b)(Bx+C)-(bx+c)(Ax+B)\}/(Ax^2+2Bx+C)^2.$$

3. Show that $1/(x^2+1)=\{\frac{1}{2}i/(x+i)\}-\{\frac{1}{2}i/(x-i)\}$
and $1/(1+x^3)=\{\frac{1}{3}/(x+1)\}+\{\frac{1}{3}\omega/(x+\omega)\}+\{\frac{1}{3}\omega^2/(x+\omega^2)\}$
where ω and ω^2 are the complex cube roots of unity. Write down the derivatives of the functions and verify that they agree with the results obtained by means of the formula $R'=(P'Q-PQ')/Q^2$.

4. If Q has a factor $(x-a)^m$, the denominator of R' (when reduced to its lowest terms) is divisible by $(x-a)^{m+1}$ but by no higher power of $(x-a)$.

5. In no case can the denominator of R' have a *simple* factor $x-a$. Hence no rational function (such as $1/x$) whose denominator contains any simple factor can be the derivative of another rational function.

99. C. Algebraical Functions. We add a few examples connected with the differentiation of irrational algebraical functions. At present we are not in a position to consider this question systematically, as the theorems which we have proved do not furnish us with any simple or convenient method of finding the derivative even of so elementary a function as $\sqrt{(1+x^2)}$. The further consideration of the subject must be postponed until §§ 108–9.

Examples XLV. 1. The derivative of x^m is mx^{m-1} for any rational value of m (§ 98). Deduce from Th. (8) of § 94 that the derivative of $(ax+b)^m$ is $ma(ax+b)^{m-1}$.

2. Find the derivatives of

$$\sqrt{(1+x)},\quad \sqrt{(1-x)},\quad \sqrt{\{(1+x)/(1-x)\}},\quad \sqrt{\{(ax+b)/(cx+d)\}},\quad \{(1+x)/(1-x)\}^m,$$
$$(1+\sqrt{x})/(1-\sqrt{x}),\ \{\sqrt{(x+1)}+\sqrt{(x-1)}\}/\{\sqrt{(x+1)}-\sqrt{(x-1)}\},\ (ax+b)^m(cx+d)^n.$$

3. Show that if $y=\sqrt{x}+\sqrt{(x+\sqrt{x})}$ then $x=y^4/(2y+1)^2$. Show that $dx/dy=4y^3(y+1)/(2y+1)^3$ and deduce that

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}\left\{1+\frac{1+2\sqrt{x}}{2\sqrt{(x+\sqrt{x})}}\right\}.$$

100. D. Transcendental Functions. We have already proved (Ex. XLI. 5) that

$$D_x \sin x=\cos x,\quad D_x \cos x=-\sin x.$$

By means of Theorems (4) and (5) of § 94 the reader will easily verify the following formulae:

$$D_x \tan x=\sec^2 x,\qquad D_x \cot x=-\operatorname{cosec}^2 x,$$
$$D_x \sec x=\tan x \sec x,\quad D_x \operatorname{cosec} x=-\cot x \operatorname{cosec} x.$$

By means of Theorem (8) the reader can write down the derivatives of $\tan(ax+b)$, $\cot(ax+b)$, etc. And by means of Theorem (9) we can determine the derivatives of the ordinary inverse trigonometrical functions, as was shown in Ex. XLII. 6 for the function $\operatorname{arc\,sin} x$. The reader should verify the following formulae:

$$D_x \operatorname{arc\,sin} x = \pm 1/\sqrt{(1-x^2)}, \qquad D_x \operatorname{arc\,cos} x = \mp 1/\sqrt{(1-x^2)},$$
$$D_x \operatorname{arc\,tan} x = 1/(1+x^2), \qquad D_x \operatorname{arc\,cot} x = -1/(1+x^2),$$
$$D_x \operatorname{arc\,sec} x = \pm 1/\{x\sqrt{(x^2-1)}\}, \quad D_x \operatorname{arc\,cosec} x = \mp 1/\{x\sqrt{(x^2-1)}\}.$$

In the case of the inverse sine and cosecant the ambiguous sign is the same as that of $\cos(\operatorname{arc\,sin} x)$; in the case of the inverse cosine and secant the same as that of $\sin(\operatorname{arc\,cos} x)$.

The more general formulae

$$D_x \operatorname{arc\,sin}(x/a) = \pm 1/\sqrt{(a^2-x^2)}, \quad D_x \operatorname{arc\,tan}(x/a) = a/(x^2+a^2),$$

—which are easily derived from Theorem (7) of § 94—are also of considerable importance. In the first of them the ambiguous sign is the same as that of $a\cos\{\operatorname{arc\,sin}(x/a)\}$, since

$$a\sqrt{\{1-(x^2/a^2)\}} = \pm\sqrt{(a^2-x^2)}$$

according as a is positive or negative.

The classes of functions which we can now differentiate are limited, and will have to be extended later on. But the reader will find the results which have been proved sufficient for many interesting applications of the Calculus.

Examples XLVI. 1. Differentiate

$$\sin^2 \tfrac{1}{2}x, \quad \cos^2(ax+b), \quad \sec^2 x, \quad \tan x + \sec x, \quad \cot x + \operatorname{cosec} x,$$
$$(1+\sin x)/(1-\sin x), \quad (a+b\cos x)/(a-b\cos x).$$

2. Prove that if $y=(\tan x+\sec x)^m$ then $dy/dx = my\sec x$; and establish a similar result for $(\cot x+\operatorname{cosec} x)^m$.

3. If $y=\cos ax+i\sin ax$, $dy/dx=aiy$.

4. Differentiate $x\cos x$, $(\sin x)/x$. Show that the values of x for which the tangents to the curves $y=x\cos x$, $y=(\sin x)/x$ are parallel to the axis of x are roots of $\cot x = x$, $\tan x = x$ respectively. These equations may be solved graphically in the way explained in § 20 and Exs. XVIII.

5. Verify by differentiation that $\operatorname{arc\,sin} x+\operatorname{arc\,cos} x$ is constant for all values of x between 0 and $\frac{1}{2}\pi$, and $\operatorname{arc\,tan} x+\operatorname{arc\,cot} x$ for all positive values of x.

6. It is easy to see (cf. Exs. XVIII.) that the equation $\sin x = ax$, where a is positive, has no real roots except $x=0$ if $a \geqq 1$, and if $a<1$ a finite number of roots which increases as a diminishes. Prove that the values of a for which the number of roots changes are the values of $\cos\xi$, where ξ is a positive root of the equation $\tan\xi=\xi$. [The values required are the values of a for which $y=ax$ touches $y=\sin x$.]

7. Show that

$$D_x \operatorname{arc\,cos}(x/a) = \mp 1/\surd(a^2-x^2), \quad D_x \operatorname{arc\,sec}(x/a) = \pm 1/\{x\surd(x^2-a^2)\},$$

determining in each case how the ambiguous sign depends on that of a and the value of the function chosen.

101. Repeated differentiation. We may form a new function $\phi''(x)$ from $\phi'(x)$ just as we formed $\phi'(x)$ from $\phi(x)$. This function is called the *second derivative* or *second differential coefficient* of $\phi(x)$. If $y = \phi(x)$, we may also write $\phi''(x)$ in any of the forms

$$D_x{}^2 y, \quad \left(\frac{d}{dx}\right)^2 y, \quad \frac{d^2y}{dx^2}.$$

In exactly the same way we may define the *nth derivative* or *nth differential coefficient* of $y = \phi(x)$, which may be written in any of the forms

$$\phi^{(n)}(x), \quad D_x{}^n y, \quad \left(\frac{d}{dx}\right)^n y, \quad \frac{d^ny}{dx^n}.$$

But it is only in a few cases that it is easy to write down a general formula for the nth differential coefficient of a given function. Some of these cases will be found in the examples which follow.

Examples XLVII. 1. If $\phi(x) = x^m$, $\phi^{(n)}(x) = m(m-1)\dots(m-n+1)x^{m-n}$.

2. If $\phi(x) = (ax+b)^m$, $\phi^{(n)}(x) = m(m-1)\dots(m-n+1)a^n(ax+b)^{m-n}$. In these two examples m may have any rational value. If m is a positive integer and $n > m$ it is clear that $\phi^{(n)}(x) = 0$.

3. The result of Ex. 1 enables us to write down the nth derivative of any polynomial. Thus if $\phi(x) = x^7 - 3x^6 + x^4 - 11$,

$$\phi^{(3)}(x) = 7.6.5x^4 - 3.6.5.4x^3 + 4.3.2x = 210x^4 - 360x^3 + 24x.$$

4. The formula

$$D_x{}^n\{A/(x-a)^p\} = (-1)^n p(p+1)\dots(p+n-1)A/(x-a)^{p+n},$$

enables us to write down the nth derivative of any rational function expressed in the standard form as a sum of partial fractions.

5. Prove that the nth derivative of $1/(1-x^2)$ is

$$\tfrac{1}{2}(n!)\{(1-x)^{-n-1} + (-1)^n(1+x)^{-n-1}\}.$$

6. **Leibniz' Theorem.** If y is a product y_1y_2, and we can form the first n derivatives of y_1 and y_2, we can form the nth derivative of y by means of *Leibniz' Theorem*, which gives the rule

$$D_x{}^n(y_1y_2) = D_x{}^n y_1 . y_2 + \binom{n}{1} D_x{}^{n-1}y_1 . D_x y_2 + \binom{n}{2} D_x{}^{n-2}y_1 . D_x{}^2 y_2 + \dots + y_1 . D_x{}^n y_2$$
$$= \sum_{r=0}^{n} \binom{n}{r} D_x{}^{n-r} y_1 . D_x{}^r y_2.$$

In this formula $D_x{}^0y_1$, $D_x{}^0y_2$ must be interpreted as meaning simply y_1 and y_2. To prove the theorem we observe that

$$D_x(y_1y_2)=D_xy_1.y_2+y_1.D_xy_2,$$
$$D_x{}^2(y_1y_2)=D_x{}^2y_1.y_2+2D_xy_1.D_xy_2+y_1D_x{}^2y_2;$$

and so on. It is obvious that by repeating this process we arrive at a formula of the type

$$D_x{}^n(y_1y_2)=D_x{}^ny_1.y_2+a_{n,1}D_x{}^{n-1}y_1.D_xy_2+a_{n,2}D_x{}^{n-2}y_1.D_x{}^2y_2+\dots+y_1.D_x{}^ny_2.$$

Let us assume that $a_{n,r}=\binom{n}{r}$ for $r=1, 2, \dots n-1$, and show that if this is so then $a_{n+1,r}=\binom{n+1}{r}$ for $r=1, 2, \dots n$. It will then follow by the principle of mathematical induction that $a_{n,r}=\binom{n}{r}$ for all values of n and r in question.

When we form $D_x{}^{n+1}(y_1y_2)$ by differentiating $D_x{}^n(y_1y_2)$ it is clear that the coefficient of $D_x{}^{n+1-r}y_1.D_x{}^ry_2$ is

$$a_{n,r}+a_{n,r-1}=\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}.$$

This establishes the theorem.

7. Find $\phi^{(n)}(x)$ when $\phi(x)$ is any one of the functions $x/(1+x)$, $x^2/(1-x)$, $x\sin x$, $x^2\cos x$. Verify, in the case of the first two functions, that the results agree with those obtained by the method of partial fractions.

8. The nth derivative of $x^mf(x)$ is

$$\frac{m!}{(m-n)!}x^{m-n}f(x)+n\frac{m!}{(m-n+1)!}x^{m-n+1}f'(x)$$
$$+\frac{n(n-1)}{1.2}\frac{m!}{(m-n+2)!}x^{m-n+2}f''(x)+\dots$$

the series being continued for $n+1$ terms or until zero terms occur.

9. Differentiate $1/(1-x^2)=\{1/(1+x)\}\{1/(1-x)\}$ n times by means of Leibniz' theorem. [The result is $n!\sum\limits_{r=0}^{n}\{(-1)^r/(1+x)^{r+1}(1-x)^{n-r+1}\}$. It is easy to verify that this result agrees with that of Ex. 5.]

10. Prove that $D_x{}^n\cos x=\cos(x+\frac{1}{2}n\pi)$, $D_x{}^n\sin x=\sin(x+\frac{1}{2}n\pi)$.

11. If $y=A\cos mx+B\sin mx$, then $D_x{}^2y+m^2y=0$. And if

$$y=A\cos mx+B\sin mx+P_n(x),$$

where $P_n(x)$ is a polynomial of degree n, then $D_x{}^{n+3}y+m^2D_x{}^{n+1}y=0$.

12. If $x^2D_x{}^2y+xDy+y=0$, then

$$x^2D_x{}^{n+2}y+(2n+1)xD^{n+1}y+(n^2+1)D_x{}^ny=0.$$

[Differentiate n times by Leibniz' Theorem.]

13. If U_n denotes the nth derivative of $(Lx+M)/(x^2-2Bx+C)$, then

$$\frac{x^2-2Bx+C}{(n+1)(n+2)}U_{n+2}+\frac{2(x-B)}{n+1}U_{n+1}+U_n=0. \quad (\textit{Math. Trip.}\ 1900.)$$

[First obtain the equation for $n=0$; then differentiate n times by Leibniz' Theorem.]

14. **The nth derivatives of $a/(a^2+x^2)$ and $x/(a^2+x^2)$.** Since

$$\frac{a}{a^2+x^2}=\frac{1}{2i}\left(\frac{1}{x-ai}-\frac{1}{x+ai}\right),\quad \frac{x}{a^2+x^2}=\frac{1}{2}\left(\frac{1}{x-ai}+\frac{1}{x+ai}\right),$$

we have

$$D_x{}^n\left(\frac{a}{a^2+x^2}\right)=\frac{(-1)^n n!}{2i}\left\{\frac{1}{(x-ai)^{n+1}}-\frac{1}{(x+ai)^{n+1}}\right\},$$

and a similar formula for $D_x{}^n\{x/(a^2+x^2)\}$. If $\rho=\sqrt{(x^2+a^2)}$, and θ is the numerically smallest angle whose cosine and sine are x/ρ and a/ρ, then $x+ai=\rho \operatorname{Cis}\theta$ and $x-ai=\rho\operatorname{Cis}(-\theta)$, and so

$$\begin{aligned} D_x{}^n\{a/(a^2+x^2)\}&=\{(-1)^n n!/2i\}\rho^{-n-1}[\operatorname{Cis}\{(n+1)\theta\}-\operatorname{Cis}\{-(n+1)\theta\}]\\ &=(-1)^n n!\,(x^2+a^2)^{-(n+1)/2}\sin\{(n+1)\operatorname{arc}\tan(a/x)\}.\end{aligned}$$

Similarly

$$D_x{}^n\{x/(a^2+x^2)\}=(-1)^n n!\,(x^2+a^2)^{-(n+1)/2}\cos\{(n+1)\operatorname{arc}\tan(a/x)\}.$$

These results enable us to write down the nth derivative of $(\lambda x+\mu)/(x^2+a^2)$; that of $(\lambda x+\mu)/(Ax^2+2Bx+C)$, where $B^2<AC$, so that the roots of the denominator are complex, may be found in the same way if we first make the substitution $Ax+B=At$. When the roots are real the function can be expressed in the form $\{H/(x-\alpha)\}+\{K/(x-\beta)\}$, where H, K, α, β are real, and the nth derivative can be written down immediately in a real form.

15. Obtain identities from the equations

$$D_x{}^r(x^m\,.\,x^n)=D_x{}^r x^{m+n},\quad D_x{}^r(\cos x\sin x)=\tfrac{1}{2}D_x{}^r\sin 2x,$$

calculating the left-hand sides by Leibniz' theorem and the right-hand sides directly.

16. Prove that

$$\begin{aligned} D_x{}^n\{(\cos x)/x\}&=\{P_n\cos(x+\tfrac{1}{2}n\pi)+Q_n\sin(x+\tfrac{1}{2}n\pi)\}/x^{n+1},\\ D_x{}^n\{(\sin x)/x\}&=\{P_n\sin(x+\tfrac{1}{2}n\pi)-Q_n\cos(x+\tfrac{1}{2}n\pi)\}/x^{n+1},\end{aligned}$$

where P_n and Q_n are polynomials in x of degree n and $n-1$ respectively.

17. Establish the formulae

$$\frac{dx}{dy}=1\bigg/\left(\frac{dy}{dx}\right),\quad \frac{d^2x}{dy^2}=-\frac{d^2y}{dx^2}\bigg/\left(\frac{dy}{dx}\right)^3,\quad \frac{d^3x}{dy^3}=-\left\{\frac{d^3y}{dx^3}\frac{dy}{dx}-3\left(\frac{d^2y}{dx^2}\right)^2\right\}\bigg/\left(\frac{dy}{dx}\right)^5.$$

18. If $yz=1$ and $y_r=(1/r!)\,D_x{}^r y$, $z_s=(1/s!)\,D_x{}^s z$, then

$$\frac{1}{y^3}\begin{vmatrix} z & z_1 & z_2\\ z_1 & z_2 & z_3\\ z_2 & z_3 & z_4\end{vmatrix}=\frac{1}{y^2}\begin{vmatrix} y_2 & y_3\\ y_3 & y_4\end{vmatrix}.$$

(*Math. Trip.* 1905.)

19. If $W(y,z,u)=\begin{vmatrix} y & z & u\\ y' & z' & u'\\ y'' & z'' & u''\end{vmatrix}$, dashes denoting differentiations with respect to x, then $$W(y,z,u)=y^3\,W\left(1,\frac{z}{y},\frac{u}{y}\right).$$

20. If $$ax^2+2hxy+by^2+2gx+2fy+c=0,$$
then $$dy/dx=-(ax+hy+g)/(hx+by+f)$$
and $$d^2y/dx^2=-(abc+2fgh-af^2-bg^2-ch^2)/(hx+by+f)^3.$$

102. Some general theorems concerning derived functions. In all that follows we suppose that $\phi(x)$ is a function of x which has a derivative $\phi'(x)$ for all values of x in question. This assumption of course involves the continuity of $\phi(x)$ (§ 92 (1)).

The meaning of the sign of $\phi'(x)$. THEOREM A. *If* $\phi'(x_0)>0$, $\phi(x)$ *increases with* x *in the neighbourhood of* $x=x_0$*:* it is, moreover, an increasing function of x in the stricter sense (§ 88); that is to say we can find a small interval of values of x, stretching away from x_0 on both sides, and such that if x_1 and x_2 are any values of x in this interval, and $x_1<x_2$, then $\phi(x_1)<\phi(x_2)$.

For, as $h\to 0$, $\{\phi(x_0+h)-\phi(x_0)\}/h$ converges to a positive limit $\phi'(x_0)$. This can only be the case if, for sufficiently small values of h, $\phi(x_0+h)-\phi(x_0)$ and h have the same sign; and this is precisely what the theorem states.

Of course from a geometrical point of view the result is intuitive, the inequality $\phi'(x)>0$ expressing the fact that the tangent to the curve $y=\phi(x)$ makes a positive acute angle with the axis of x.

COR. 1. *If* $\phi'(x)>0$ *for all values of* x *in a certain interval,* $\phi(x)$ *is an increasing function of* x *(in the stricter sense of* § 88*) throughout that interval.*

COR. 2. *If* $\phi'(x)>0$ *throughout the interval* (a, b), *and* $\phi(a)\geqq 0$, *then* $\phi(x)$ *is positive throughout the interval* (a, b).

The reader should formulate for himself the corresponding theorems for the case in which $\phi'(x)<0$.

An immediate deduction from Theorem A is the following important theorem, generally known as Rolle's Theorem:

THEOREM B. *If* $\phi(a)=0$ *and* $\phi(b)=0$ *there must be at least one value of* x *which lies between* a *and* b *and for which* $\phi'(x)=0$.

There are two possibilities: the first is that $\phi(x)$ is equal to zero throughout the whole interval (a, b). In this case $\phi'(x)$ is also equal to zero throughout the interval. If this is not so there must be at least one value of x for which $\phi(x)$ is positive or negative, say positive. By Theorem 2 of § 89, there is a value

ξ of x, not equal to a or b, and such that $\phi(\xi)$ is at least as great as the value of $\phi(x)$ at any other point in the interval. And $\phi'(\xi)$ must be equal to zero. For if it were positive, $\phi(\xi)$ would be an increasing function in the neighbourhood of $x = \xi$, and so there would be greater values of $\phi(x)$ than $\phi(\xi)$. Similarly we can show that $\phi'(\xi)$ cannot be negative.

COR. *If $\phi(a) = \phi(b) = k$ there must be a value of x between a and b, such that $\phi'(x) = 0$.*

We have only to put $\phi(x) - k = \psi(x)$ and apply Theorem B to $\psi(x)$. The reader should consider the obvious geometrical meaning of this theorem and its corollary.

103. Maxima and Minima. We shall say that the value $\phi(\xi)$ assumed by $\phi(x)$ for $x = \xi$ is a *maximum* if $\phi(\xi)$ is greater than any other value assumed by $\phi(x)$ in the immediate neighbourhood of $x = \xi$, i.e. if we can find an interval $(\xi - \delta, \xi + \delta)$ of values of x such that $\phi(\xi) > \phi(x)$ for $\xi - \delta < x < \xi$ and $\xi < x < \xi + \delta$: and similarly we define a *minimum*. Thus in the figure the points A correspond to maxima, the points B to minima of the function whose graph is there shown. It is to be observed that the fact that A_3 corresponds to a maximum and B_1 to a minimum is in no way inconsistent with the fact that the value of the function is greater at B_1 than at A_3.

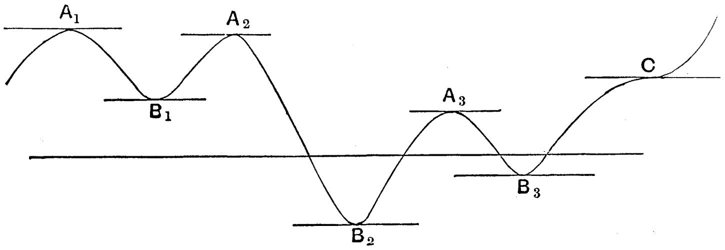

FIG. 47.

THEOREM C. *A* **necessary** *condition for a maximum or minimum value of $\phi(x)$ at $x = \xi$ is that $\phi'(\xi) = 0$**. This follows at once from the fact that $\phi(x)$ cannot be an increasing or a decreasing function at $x = \xi$. In geometrical language *the tangent at $x = \xi$ is parallel to the axis of x.*

---

* A function which, like those referred to in § 92 (2), is continuous but has no derivative, may have maxima and minima. We are of course assuming the existence of the derivative.

That this condition is not *sufficient* is evident from a glance at the point C in the figure. Thus if $y = x^3$, $\phi'(x) = 3x^2$, which vanishes when $x = 0$. But $x = 0$ does not give either a maximum or a minimum of x^3, as is obvious from the form of the graph of x^3 (§ 13, Fig. 11).

But there *will certainly be a maximum for* $x = \xi$ *if* $\phi'(\xi) = 0$, $\phi'(x) > 0$ *for all values of x less than but near to x, and* $\phi'(x) < 0$ *for all values of x greater than but near to x*: and if the signs of these two inequalities are reversed there will certainly be a minimum.

For then we can determine an interval $(\xi - \delta, \xi)$ throughout which $\phi(x)$ increases with x, and an interval $(\xi, \xi + \delta)$ throughout which it decreases as x increases: and obviously this ensures that $\phi(\xi)$ shall be a maximum.

This result may also be stated thus—if the sign of $\phi'(x)$ changes at $x = \xi$ from positive to negative, then $x = \xi$ gives a maximum of $\phi(x)$: and if the sign of $\phi'(x)$ changes in the opposite sense, then $x = \xi$ gives a minimum.

104. There is another way of stating the conditions for a maximum or minimum which is often useful. Let us assume that $\phi(x)$ has a second derivative $\phi''(x)$: this of course does not follow from the existence of $\phi'(x)$, any more than the existence of $\phi'(x)$ follows from that of $\phi(x)$. But in such cases as we are likely to meet with at present the condition is generally satisfied.

THEOREM D. *If* $\phi'(\xi) = 0$ *and* $\phi''(\xi) \neq 0$, $\phi(x)$ *has a maximum or minimum for* $x = \xi$: *a maximum if* $\phi''(\xi) < 0$, *a minimum if* $\phi''(\xi) > 0$.

Suppose, e.g., $\phi''(\xi) < 0$. Then $\phi'(x)$ is decreasing near $x = \xi$, and so its sign changes from positive to negative. Thus $x = \xi$ gives a maximum.

105. In what has preceded (apart from the last paragraph) we have assumed simply that $\phi(x)$ has a derivative for all values of x in the interval under consideration. If this condition is not fulfilled the theorems cease to be true. Thus Theorem B fails in the case of the function

$$y = 1 - \sqrt{(x^2)},$$

where the square root is to be taken positive. The graph of this function is shown in Fig. 48. Here $\phi(-1) = \phi(1) = 0$: but $\phi'(x)$ (as is evident from the figure) is equal to $+1$ if x is negative and to -1 if x is positive, and never vanishes. There is no derivative for $x=0$, and no definite tangent to the graph at P. And in this case $x=0$ obviously gives a maximum of $\phi(x)$, but $\phi'(0)$, as it does not exist, cannot be equal to zero, so that the test for a maximum fails.

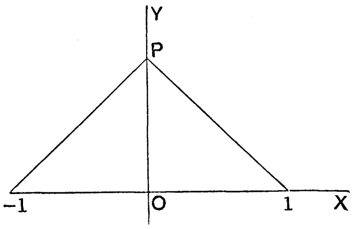

FIG. 48.

The bare existence of the derivative $\phi'(x)$, however, is *all* that we have assumed. And there is one assumption in particular that we have not made, and that is that *$\phi'(x)$ itself is a continuous function.* This raises a rather subtle but still a very interesting point. *Can* a function $\phi(x)$ have a derivative for all values of x which is not itself continuous? In other words can a curve have a tangent at every point, and yet the direction of the tangent not vary continuously? The reader if he considers what the question means and tries to answer it in the light of common sense will probably incline to the answer, *No.* That this answer is wrong is shown in the next set of examples (Ex. XLVIII. 44).

However let us leave this point for the moment and evade the difficulty by *assuming* that the derivative $\phi'(x)$ is continuous; an assumption which is needed in what follows, although it was not needed in what precedes. We might of course have assumed this all along, in proving the theorems of §§ 102–103; and there would, for practical purposes, have been no harm in making the assumption, as in all ordinary cases there is no doubt about the continuity of the derivative: but as the assumption would in no way have shortened or simplified the proofs of the theorems we should have gained nothing by making it. But in the next theorem, and some theorems which will be proved later on in this chapter, the fact that $\phi'(x)$ is continuous is essential to the argument.

THEOREM E. *The maxima and minima of $\phi(x)$ occur alternately.* For at a maximum $\phi'(x)$ changes its sign from positive to negative: and, being continuous, it can only become positive again by passing through zero and changing sign from negative to positive; and this gives a minimum of $\phi(x)$.

Examples XLVIII. 1. Verify Theorem B when $\phi(x)=(x-a)^m(x-b)^n$ or $(x-a)^m(x-b)^n(x-c)^p$, where m, n, p are positive integers and $a<b<c$.

[The first function vanishes for $x=a$ and $x=b$. And

$$\phi'(x)=(x-a)^{m-1}(x-b)^{n-1}\{(m+n)x-mb-na\}$$

vanishes for $x=(mb+na)/(m+n)$, which lies between a and b. In the second case we have to verify that the quadratic equation

$$(m+n+p)x^2-\{m(b+c)+n(c+a)+p(a+b)\}x+mbc+nca+pab=0$$

has roots between a and b, and between b and c.]

2. Show that the polynomials

$$2x^3+3x^2-12x+7, \qquad 3x^4+8x^3-6x^2-24x+19,$$

are positive for $x>1$.

3. Show that $x-\sin x$ is an increasing function for all values of x, and that $\tan x - x$ increases as x varies from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. For what values of a is $ax-\sin x$ a steadily increasing or decreasing function of x?

4. Show that $\tan x - x$ also increases from $x=\frac{1}{2}\pi$ to $x=\frac{3}{2}\pi$, from $x=\frac{3}{2}\pi$ to $x=\frac{5}{2}\pi$, and so on, and deduce that there is one and only one root of the equation $\tan x = x$ in each of these intervals (cf. Ex. XVIII. 5).

5. Deduce from Ex. 3 that $\sin x - x<0$ if $x>0$, from this that $\cos x - 1+\frac{1}{2}x^2>0$, and from this that $\sin x - x+\frac{1}{6}x^3>0$. And, generally, prove that if

$$C_{2m}=\cos x-1+\frac{x^2}{2\,!}-\dots-(-1)^m\frac{x^{2m}}{2m\,!},$$

$$S_{2m+1}=\sin x-x+\frac{x^3}{3\,!}-\dots-(-1)^m\frac{x^{2m+1}}{(2m+1)\,!},$$

and $x>0$, then $C_m \gtrless 0$ and $S_{2m+1}\gtrless 0$ according as m is odd or even.

6. Show that $x\sin x+\cos x+\frac{1}{2}\cos^2 x$ increases as x increases from 0 to $\frac{1}{2}\pi$.

7. If $f(x)$ and $f''(x)$ are continuous and have the same sign at every point of an interval (a, b), this interval can include at most one root of $f(x)=0$.

8. The functions u, v and their derivatives u', v' are continuous throughout a certain interval of values of x, and $uv'-u'v$ never vanishes at any point of the interval. Show that between any two roots of $u=0$ occurs one of $v=0$, and conversely.

[If v does not vanish between two roots of $u=0$, say α and β, the function u/v is continuous throughout the interval (α, β) and vanishes at its extremities. Hence $(u/v)'=(u'v-uv')/v^2$ must vanish between α and β, which contradicts our hypotheses.]

9. Verify the preceding theorem when $u=\cos x$, $v=\sin x$.

10. Show how to determine as completely as possible the multiple roots of $P(x)=0$, where $P(x)$ is a polynomial, with their degrees of multiplicity, by means of the elementary algebraical operations.

[If H_1 is the highest common factor of P and P', H_2 the highest common factor of H_1 and P'', H_3 that of H_2 and P''', and so on, then the roots of $H_1/H_2=0$ are the *double* roots of $P=0$, the roots of $H_2/H_3=0$ the *treble* roots, and so on. But it may not be possible to complete the solution of $H_1/H_2=0$, $H_2/H_3=0$, etc. Thus if $P(x)=(x-1)^3(x^5-x-7)^2$, $H_1/H_2=x^5-x-7$ and $H_2/H_3=x-1$; and we cannot solve the first equation.]

11. Show that $x^5-10x^2+15x-6=0$ has a treble root, and find it.

12. If $\phi(x)$ is a polynomial and $f(x)=(x-a)^r\phi(x)$, then

$$f'(a)=f''(a)=\dots=f^{(r-1)}(a)=0, \qquad f^{(r)}(a)=r\,!\,\phi(a).$$

13. Determine the maxima and minima (if any) of $(x-1)^2(x+2)$, x^3-3x, $2x^3-3x^2-36x+10$, $4x^3-18x^2+27x-7$, $3x^4-4x^3+1$, x^5-15x^3+3. In each case sketch the form of the graph of the function.

[Consider the last function, for example. Here $\phi'(x)=5x^2(x^2-9)$, which vanishes for $x=0$, $x=\pm 3$. It is easy to see that $x=-3$ gives a maximum, $x=+3$ a minimum, while $x=0$ gives neither, as $\phi'(x)$ is negative on both sides of $x=0$.]

14. Discuss the maxima and minima of $x(x-1)$, $x^2(x-1)^2$, $x^3(x-1)^3$, $x^4(x-1)^4$. Sketch the graphs of the functions.

15. Discuss similarly the function $(x-a)(x-b)^2(x-c)^3$, distinguishing the different forms of the graph which correspond to different hypotheses as to the relative magnitudes of a, b, c.

16. Discuss similarly the function $(x-a)^m(x-b)^n$, where m and n are any positive integers, considering the different cases which occur according as m and n are odd or even.

17. Show that $(ax+b)/(cx+d)$, whatever values a, b, c, d may have, has no maxima or minima. Draw a graph of the function.

18. Discuss the maxima and minima of $\{(1-x)/(1+x)\}^2$, $2x/(1+x^2)$, $2x/(1-x^2)$, $(1-x^2)/(1+x^2)$, $(1+x^2)/(1-x^2)$.

19. Discuss the maxima and minima of the function

$$y=(ax^2+2bx+c)/(Ax^2+2Bx+c),$$

when the denominator has complex roots.

[We may suppose a and A positive. The derivative vanishes if

$$(ax+b)(Bx+c)-(Ax+B)(bx+c)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This equation must have real roots. For if not the derivative would always have the same sign, and this is impossible, since y is continuous for all values of x, and $y \to a/A$ as $x \to +\infty$ or $x \to -\infty$.

It is easy to verify that the curve cuts the line $y=a/A$ in one and only one point, and that it lies above this line for large positive values of x, and below it for large negative values, or *vice versa*, according as $b/a \gtrless B/A$. Thus the algebraically greater root of (1) gives a maximum if $b/a > B/A$, a minimum in the contrary case.]

20. The maximum and minimum values themselves are the values of λ for which $ax^2+2bx+c-\lambda(Ax^2+2Bx+C)$ is a perfect square. [This is the condition that $y=\lambda$ should touch the curve.]

21. In general the maxima and minima of $R(x)=P(x)/Q(x)$ are among the values of λ obtained by expressing the condition that $P(x)-\lambda Q(x)=0$ should have a pair of equal roots.

22. If $Ax^2+2Bx+C=0$ has real roots it is convenient to proceed as follows. We have

$$y-(a/A)=(\lambda x+\mu)/\{A(Ax^2+2Bx+C)\}$$

where $\lambda = bA - aB$, $\mu = cA - aC$. Writing further ξ for $\lambda x + \mu$ and η for $(A/\lambda^2)(Ay - a)$ we obtain an equation of the form

$$\eta = \xi/\{(\xi - p)(\xi - q)\}.$$

This transformation from (x, y) to (ξ, η) amounts only to a shifting of the origin, keeping the axes parallel to themselves, a change of scale along each axis, and (if $\lambda < 0$) a reversal in direction of the axis of abscissae, and so a minimum of y, considered as a function of x, corresponds to a minimum of η considered as a function of ξ, and *vice versa*; and similarly for a maximum.

The derivative of η with respect to ξ vanishes if

$$(\xi - p)(\xi - q) - \xi(\xi - p) - \xi(\xi - q) = 0,$$

or if $\xi^2 = pq$. Thus there are two roots of the derivative if p and q have the same sign, none if they have opposite signs. In the latter case the form of the graph of η is as shown in Fig. 49 *a*.

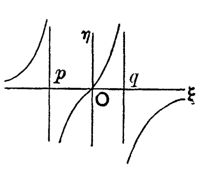

Fig. 49 *a*.

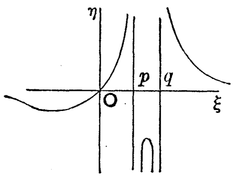

Fig. 49 *b*.

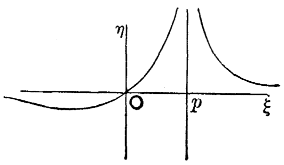

Fig. 49 *c*.

When p and q are positive the general form of the graph is as shown in Fig. 49 *b*, and it is easy to see that $\xi = \sqrt{pq}$ gives a maximum and $\xi = -\sqrt{pq}$ a minimum*.

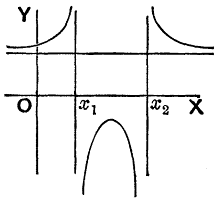

Fig. 50.

In the particular case in which $p = q$ the graph of

$$\eta = \xi/(\xi - p)^2$$

is of the form shown in Fig. 49 *c*.

The preceding discussion fails if $\lambda = 0$, i.e. if $a/A = b/B$. But in this case we have

$$\begin{aligned} y - (a/A) &= \mu/\{A\,(Ax^2 + 2Bx + C)\} \\ &= (\mu/A^2)/\{(x - x_1)(x - x_2)\}, \end{aligned}$$

say, and $dy/dx = 0$ gives the single value $x = \frac{1}{2}(x_1 + x_2)$. On drawing a graph it is clear that this is a maximum or minimum according as μ is positive or negative. The graph shown in Fig. 50 corresponds to the former case.

[A full discussion of the general function $y = (ax^2 + 2bx + c)/(Ax^2 + 2Bx + C)$, will be found in Chrystal's *Algebra*, vol. I. pp. 464–7: there however only purely algebraical methods are used.]

23. Show that $(x - \alpha)(x - \beta)/(x - \gamma)$ assumes all real values as x varies, if α lies between β and γ, and otherwise assumes all values except those included in an interval of length $4\sqrt{\{(\alpha \sim \beta)(\alpha \sim \gamma)\}}$.

---

* The maximum is $-1/(\sqrt{p} - \sqrt{q})^2$, the minimum $-1/(\sqrt{p} + \sqrt{q})^2$, of which the latter is the greater.

24. Discuss $(x^2-x+1)/(x^2+x+1)$, $(x^2-x+3)/(x^2-3x+2)$.

25. The maximum and minimum of $(x+a)(x+b)/(x-a)(x-b)$, where a and b are positive, are

$$-\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)^2, \quad -\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)^2.$$

26. The maximum value of $(x-1)^2/(x+1)^3$ is $\frac{2}{27}$.

27. Discuss the maxima and minima of

$$x(x-1)/(x^2+3x+3), \quad x^4/(x-1)(x-3)^3.$$

28. Discuss the maxima and minima of

$$(x-1)^2(3x^2-2x-37)/(x+5)^2(3x^2-14x-1).$$

(*Math. Trip.* 1898.)

[If the function be denoted by $P(x)/Q(x)$, it will be found that

$$P'Q-PQ'=72(x-7)(x-3)(x-1)(x+1)(x+2)(x+5).]$$

29. Find the maxima and minima of $a\cos x+b\sin x$. Verify the result by expressing the function in the form $A\cos(x-\alpha)$.

30. Find the maxima and minima of

$$a^2\cos^2 x+b^2\sin^2 x, \quad A\cos^2 x+2H\cos x\sin x+B\sin^2 x.$$

31. Show that $\sin(x+a)/\sin(x+b)$ has no maxima or minima. Draw a graph of the function.

32. The least value of $a^2\sec^2 x+b^2\operatorname{cosec}^2 x$ is $(a+b)^2$.

33. Show that $\tan 3x\cot 2x$ cannot lie between $\frac{1}{9}$ and $\frac{3}{2}$.

34. Find the maxima and minima of $(1+2x\arctan x)/(1+x^2)$.

35. The base of a triangle is equal to a, and the ratio of the other two sides is r. Show that the maximum value of its area is $\frac{1}{2}a^2 r/(r^2\sim 1)$.

36. A line is drawn through a fixed point (a, b) to meet the axes OX, OY in P and Q. Show that the minimum values of PQ, $OP+OQ$, and $OP \, . \, OQ$ are respectively $(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$, $(\sqrt{a}+\sqrt{b})^2$, $4ab$.

37. A tangent to an ellipse meets the axes in P and Q. Show that the least value of PQ is equal to the sum of the semiaxes of the ellipse.

38. Find the lengths and directions of the axes of the conic

$$ax^2+2hxy+by^2=1.$$

[The length r of the semidiameter which makes an angle θ with the axes of x is given by

$$1/r^2=a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta.$$

The condition for a maximum or minimum value of r is $\tan 2\theta=2h/(a-b)$: eliminating θ between these two equations we find

$$\{a-(1/r^2)\}\{b-(1/r^2)\}=h^2.]$$

39. The greatest value of the product of two positive numbers whose sum is constant is obtained by supposing them equal. Extend the result to any number of numbers.

40. The greatest value of $x^m y^n$, where $x+y=k$, is $m^m n^n k^{m+n}/(m+n)^{m+n}$.

41. The greatest value of $ax+by$, where $x^2+xy+y^2=3\kappa^2$, is

$$2\kappa\sqrt{(a^2-ab+b^2)}.$$

[If $ax+by$ is a maximum, $a+b\,(dy/dx)=0$. The relation between x and y gives $(2x+y)+(x+2y)\,(dy/dx)=0$. Equate the two values of dy/dx.]

42. If θ and ϕ are acute angles connected by the relation $a\sec\theta+b\sec\phi=c$, where a, b, c are positive, then $a\cos\theta+b\cos\phi$ is a minimum when $\theta=\phi$.

43. Two particles are moving with uniform velocities u, v along two straight lines which make an angle ω with one another. Originally they are at distances a, b from the point of intersection of the lines, and they are moving towards the point of intersection. Show that they are nearest after time $\{au+bv-(av+bu)\cos\omega\}/(u^2-2uv\cos\omega+v^2)$ has elapsed, and that then their distance is $(av\sim bu)\sin\omega/\sqrt{(u^2-2uv\cos\omega+v^2)}$.

44. **Example of a function whose derivative exists for every value of x, but is discontinuous.** In the case of a function whose graph has such a form as that shown in Fig. 48 the derivative is obviously discontinuous for $x=0$: but the derivative does not exist for $x=0$ (it is obvious from the figure that there is no definite tangent at P). We shall now give an example of a function (curve) which has a derivative (tangent) *everywhere*, although the derivative (tangent) is discontinuous (discontinuous in direction).

Let us consider how we might set about constructing such a curve geometrically. Draw a double curve (as shown in Fig. 51 *a*, where the upper and lower curves are $y=\pm x^2$ respectively) which touches the axis of x at the origin on both sides.

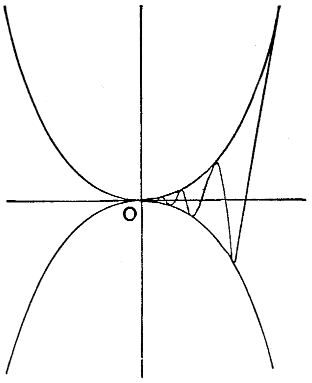

Fig. 51 *a*.

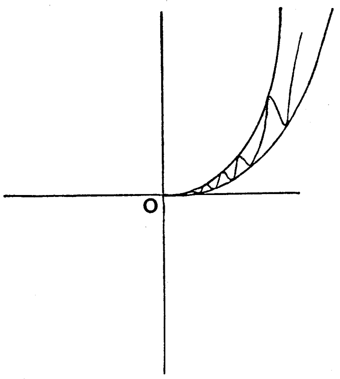

Fig. 51 *b*.

Now draw a wavy curve, as in the figure, oscillating continually between the two curves, and forming a single continuous branch passing through the origin.

The reader will easily convince himself that it is possible to draw the curve so that the tangent to the curve at P continually oscillates through an angle greater than some fixed angle as P approaches O. Thus $\tan\psi$

cannot tend to a limit as $x \to 0$. None the less the curve has a perfectly definite tangent at O. The common sense way of looking at it is that the curve always lies *between* the curves $y = +x^2$ and $y = -x^2$; and that as each of these has a definite direction at O, viz. along OX, the other curve must have the same direction too. Analytically we may regard the matter as follows. If $\phi(x)$ is the function represented by the curve, $\phi(0) = 0$, and $-x^2 \leqq \phi(x) \leqq x^2$, so that $\lim \{\phi(x) - \phi(0)\}/x = 0$, i.e. $\phi'(0) = 0$.

We shall see later (Ex. L. 17) that the function $\phi(x)$ which is equal to $x^2 \sin(1/x)$ when $x \neq 0$, and to 0 when $x = 0$, is such a function.

The reader may conceivably object that the tangent at O cuts the curve infinitely often near O, and that this is contrary to his notion of the tangent to a curve. The reply to this is that even if this is so it does not affect the purpose of the example, which is to show that $\phi'(x)$ may exist for every value of x and yet be discontinuous. And as a matter of fact it is easy to construct curves which possess the same peculiarity and for which the tangent at O does not meet the curve at all except at O. The nature of such examples will be sufficiently illustrated by Fig. 51 *b*.

45. It *is* however impossible that $\phi'(x)$ should have what was in Chap. V. called a *simple* discontinuity: e.g. that we should have $\phi'(x) \to a$ when $x \to 0$ by positive values and $\phi'(x) \to b$ when $x \to 0$ by negative values, and $\phi'(0) = c$, unless a, b, c are all equal and so $\phi'(x)$ continuous. A proof of this will be given shortly (Ex. XLIX. 3); from a geometrical point of view the result is obvious, since if a curve has an *angle* (as in Figs. 43, 48) it cannot have a definite tangent at the vertex of the angle.

106. The Mean Value Theorem. We can proceed now to the proof of another general and exceedingly important theorem, commonly known as '*The Mean Value Theorem*' or '*The Theorem of the Mean.*'

THEOREM. *If $\phi(x)$ has a derivative for all values of x in the interval (a, b), it must be possible to find a value ξ of x between a and b, such that*

$$\phi(b) - \phi(a) = (b - a)\,\phi'(\xi).$$

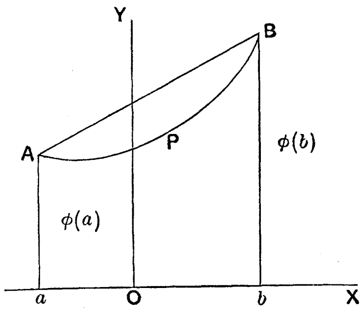

Fig. 52.

Before we give a strict proof of this theorem, which is perhaps the most important theorem in the Differential Calculus, it will be well to point out its obvious geometrical meaning. This is simply (see Fig. 52) that if the curve APB has a tangent at all points of its length, there must be a point, such as P, where the tangent is parallel to AB. For $\phi'(\xi)$ is the tangent of the angle which the tangent at P makes with OX, and $\{\phi(b)-\phi(a)\}/(b-a)$ the tangent of the angle which AB makes with OX.

It is easy to give a strict analytical proof. Consider the function

$$\phi(b)-\phi(x)-\frac{b-x}{b-a}\{\phi(b)-\phi(a)\},$$

which vanishes when $x=a$ and $x=b$. It follows from Theorem B above that there is a value ξ for which its derivative vanishes. But this derivative is

$$\frac{\phi(b)-\phi(a)}{b-a}-\phi'(x),$$

which proves the theorem. It should be observed that in this proof it has not been assumed that $\phi'(x)$ is continuous.

Examples XLIX. 1. Show that the expression

$$\phi(b)-\phi(x)-\frac{b-x}{b-a}\{\phi(b)-\phi(a)\}$$

is the difference between the ordinates of a point on the curve and the corresponding point on the chord.

2. Verify the theorem for $\phi(x)=x^2$ and for $\phi(x)=x^3$.

[In the latter case we have to prove that $(b^3-a^3)/(b-a)=3\xi^2$, where $a<\xi<b$: i.e. that if $\frac{1}{3}(b^2+ab+a^2)=\xi^2$, then ξ lies between a and b.]

3. Prove the result of Ex. XLVIII. 45 by means of the mean value theorem.

[Since $\phi'(0)=c$ we can find a small positive value of x such that $\{\phi(x)-\phi(0)\}/x$ is nearly equal to c; and therefore, by the theorem, a small positive value of ξ such that $\phi'(\xi)$ is nearly equal to c, which is inconsistent with $\lim_{x\to+0}\phi'(x)=a$, unless $a=c$. Similarly $b=c$.]

107. Another form of the Mean Value Theorem. It is often convenient to express the Mean Value Theorem in the form

$$\phi(b)=\phi(a)+(b-a)\phi'\{a+\theta(b-a)\}$$

where θ is a number lying between 0 and 1. Of course $a+\theta(b-a)$ is merely another way of writing 'some number ξ between a and b.' If we put $b=a+h$ we obtain

$$\phi(a+h)=\phi(a)+h\phi'(a+\theta h)$$

which is the form in which the theorem is most often quoted.

108. Differentiation of a function of a function. It is convenient at this stage to introduce a very important theorem which enables us to complete our results relating to the differentiation of the functions which commonly occur in analysis.

THEOREM. *If $f(x)$ and $F(x)$ are functions with derivatives $f'(x)$ and $F'(x)$, then $F\{f(x)\}$ has a derivative $F'\{f(x)\}f'(x)$.*

For let

$$f(x)=y,\quad f(x+h)=y+k,\quad F(y)=z,\quad F(y+k)=z+l.$$

Then k and l are functions of h which tend to zero with h, and $f'(x)$ and $F'(y)$ or $F'\{f(x)\}$ are by definition $\lim (k/h)$ and $\lim (l/h)$ respectively. And the derivative of $F\{f(x)\}$ is by definition

$$\lim (l/h) = \lim (l/k) \times \lim (k/h) = F'\{f(x)\}f'(x).$$

The following alternative method of proof, depending on the Mean Value Theorem, is interesting. It will be noticed that in using it we have to assume the *continuity* and not merely the *existence* of the derivatives of the functions f and F. This is of course a theoretical limitation, but not one of any practical importance.

The derivative of $F\{f(x)\}$ is by definition

$$\lim \frac{F\{f(x+h)-f(x)\}}{h}.$$

But, by the Mean Value Theorem, $f(x+h)=f(x)+hf'(\xi)$, where ξ is a number lying between x and $x+h$. And

$$F\{f(x)+hf'(\xi)\}=F\{f(x)\}+hf'(\xi)F'(\xi_1),$$

where ξ_1 is a number lying between $f(x)$ and $f(x)+hf'(\xi)$. Hence the derivative of $F\{f(x)\}$ is

$$\lim f'(\xi)F'(\xi_1)=f'(x)F'\{f(x)\},$$

since $\xi \rightarrow x$ and $\xi_1 \rightarrow f(x)$ as $h \rightarrow 0$.

The most important cases of this theorem are the following:

(i) If $f(x)=ax+b$, $f'(x)=a$, and so the derivative of $F(ax+b)$ is $aF'(ax+b)$, as result was proved independently in § 94, Theorem (8).

(ii) If $f(x)=x^m$, $f'(x)=mx^{m-1}$, and so the derivative of $F(x^m)$ is $mx^{m-1}F'(x^m)$.

(iii) If $f(x)=\sin x$, $f'(x)=\cos x$, and so the derivative of $F(\sin x)$ is $\cos x F'(\sin x)$. Similarly the derivative of $F(\cos x)$ is $-\sin x F'(\cos x)$.

109. By means of the theorem proved in the last section we are able practically to complete the rules for differentiation given in § 94 and to write down the derivative of any common function whatsoever. For the only ordinary class of functions with which the rules there given did not enable us to deal is precisely that to which this theorem applies, viz. what may be called *composite* functions, or functions of functions. The manner of applying the theorem will be seen from the ensuing examples.

Examples L*. 1. Consider the function $\phi(x)=\sqrt{(a^2+x^2)}$. This function may be expressed in the form $F\{f(x)\}$, where $f(x)=a^2+x^2$, $F(y)=\sqrt{y}$, and so $f'(x)=2x$, $F'(y)=\tfrac{1}{2}\sqrt{y}$. Thus $\phi'(x)=x/\sqrt{(a^2+x^2)}$.

2. Find the derivatives of

$$\sqrt{(a^2-x^2)},\quad 1/\sqrt{(a^2+x^2)},\quad 1/\sqrt{(a^2-x^2)},\quad x/\sqrt{(a^2+x^2)},\quad x/\sqrt{(a^2-x^2)},$$
$$(a^2+x^2)^m,\quad (a^2-x^2)^m,\quad (a^2+x^2)^m(a^2-x^2)^n,\quad x^m(a^2+x^2)^n,\quad \sqrt{(a^{2m}-x^{2m})},$$
$$\sqrt{\{(1+x+x^2)/(1-x+x^2)\}},\quad \sqrt{\{(a+2bx+cx^2)/(A+2Bx+Cx^2)\}},$$
$$\{\sqrt{(1+x)}+\sqrt{(1-x)}\}/\{\sqrt{(1+x)}-\sqrt{(1-x)}\},\quad \{x+\sqrt{(1+x^2)}\}^m,\quad x^m(a+bx^n)^p.$$

Here m, n, p are any rational numbers.

3. Find the derivatives of

$$(\sin x)^m,\quad (\cos x)^m,\quad \sin(x^m),\quad \cos(x^m),\quad \sin(\cos x),\quad \cos(\sin x),$$
$$\sqrt{(a+b\sin x)},\quad \cos x/\sqrt{(a+b\sin x)},\quad \sqrt{\{(a\cos x+b\sin x)/(\alpha\cos x+\beta\sin x)\}},$$
$$\sqrt{(a^2\cos^2 x+b^2\sin^2 x)},\quad \sin x\cos x/\sqrt{(a^2\cos^2 x+b^2\sin^2 x)}.$$

4. Find the derivatives of $x\arcsin x+\sqrt{(1-x^2)}$, $(1+x)\arctan\sqrt{x}-\sqrt{x}$, $\arctan\sqrt{\{(a+x)/(a-x)\}}$.

5. Find the derivatives of

$$\arcsin\sqrt{(1-x^2)},\quad \arctan(1/x),\quad \arcsin\{x/\sqrt{(1+x^2)}\},$$
$$\arcsin\{2x\sqrt{(1-x^2)}\},\quad \arctan\{2x/(1-x^2)\},\quad \arctan\{(a+x)/(1-ax)\}.$$

How do you explain the simplicity of the results?

6. Differentiate

$$\frac{1}{\sqrt{(ac-b^2)}}\arctan\frac{ax+b}{\sqrt{(ac-b^2)}},\qquad -\frac{1}{\sqrt{(-a)}}\arcsin\frac{ax+b}{\sqrt{(b^2-ac)}}.$$

7. The differential coefficient of $\arccos\sqrt{\{(\cos 3\theta)/\cos^3\theta\}}$ is

$$\sqrt{\{3/(\cos 3\theta\cos\theta)\}}.$$ (*Math. Trip.* 1904.)

8. Show that each of the functions

$$2\arccos\sqrt{\{(\alpha-x)/(\alpha-\beta)\}},\ 2\arcsin\sqrt{\{(x-\beta)/(\alpha-\beta)\}},$$
$$2\arctan\{\sqrt{(x-\beta)/(\alpha-x)}\},\ \arcsin[2\sqrt{\{(\alpha-x)(x-\beta)\}}/(\alpha-\beta)]$$

has the derivative $1/\sqrt{\{(\alpha-x)(x-\beta)\}}$.

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* In these examples it is assumed that the constants a, b, ... which occur have such values that the functions which involve them are real.

9. Show that

$$\frac{1}{\sqrt{\{C(Ac-aC)\}}}\frac{d}{dx}\left[\operatorname{arc\,cos}\sqrt{\left\{\frac{C(ax^2+c)}{c(Ax^2+C)}\right\}}\right]=\frac{1}{(Ax^2+C)\sqrt{(ax^2+c)}}.$$

10. Each of the functions

$$\frac{1}{\sqrt{(a^2-b^2)}}\operatorname{arc\,cos}\left(\frac{a\cos x+b}{a+b\cos x}\right),\quad \frac{2}{\sqrt{(a^2-b^2)}}\operatorname{arc\,tan}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\tfrac{1}{2}x\right\},$$

has the derivative $1/(a+b\cos x)$.

11. If $X=a+b\cos x+c\sin x$, and

$$y=\frac{1}{\sqrt{(a^2-b^2-c^2)}}\operatorname{arc\,cos}\frac{aX-a^2+b^2+c^2}{X\sqrt{(b^2+c^2)}},$$

then $dy/dx=1/X$.

12. Find the equations of the tangent and normal at the point (x_0, y_0) of the circle $x^2+y^2=a^2$.

[Here $y=\sqrt{(a^2-x^2)}$, $dy/dx=-x/\sqrt{(a^2-x^2)}$, and the tangent is

$$y-y_0=(x-x_0)\{-x_0/\sqrt{(a^2-x_0^2)}\},$$

which may be put in the form $xx_0+yy_0=a^2$. The normal is $xy_0-yx_0=0$, which of course passes through the origin.]

13. Find the equations of the tangent and normal at any point of the ellipse $(x/a)^2+(y/b)^2=1$ and the hyperbola $(x/a)^2-(y/b)^2=1$.

14. The equations of the tangent and normal to the curve $x=\phi(t)$, $y=\psi(t)$, at the point (t), are

$$\frac{x-\phi(t)}{\phi'(t)}=\frac{y-\psi(t)}{\psi'(t)},\quad \{x-\phi(t)\}\phi'(t)+\{y-\psi(t)\}\psi'(t)=0.$$

15. Prove that the derivative of $F[f\{\phi(x)\}]$ is $F'[f\{\phi(x)\}]f'\{\phi(x)\}\phi'(x)$, and extend the result to still more complicated cases.

16. If u and v are functions of x, then

$$D_x \operatorname{arc\,tan}(u/v)=(vD_x u-uD_x v)/(u^2+v^2).$$

17. If $\phi(x)=x^2\sin(1/x)$ when $x\neq 0$ and $\phi(0)=0$, then

$$\phi'(x)=2x\sin(1/x)-\cos(1/x)$$

when $x\neq 0$, and $\phi'(0)=0$. And $\phi'(x)$ is discontinuous for $x=0$ (cf. Ex. XLVIII. 44).

110. The Mean Value Theorem furnishes us with a proof of a result which is of essential importance for what follows:—*if $\phi'(x)=0$, throughout a certain interval of values of x, $\phi(x)$ is constant throughout that interval.*

For if a and b are any two values of x in the interval,

$$\phi(b)-\phi(a)=(b-a)\phi'\{a+\theta(b-a)\}=0.$$

An immediate corollary is that *if $\phi'(x)=\psi'(x)$, throughout a certain interval, the functions $\phi(x)$ and $\psi(x)$ differ, throughout that interval, by a constant.*

111. Integration. We have in this chapter seen how we can differentiate a given function $\phi(x)$—i.e. find its derivative—in a variety of cases, including all those of the commonest occurrence. It is natural to consider the converse question, that of determining *a function whose derivative is a given function.*

Suppose that $\psi(x)$ is the given function. Then we wish to determine a function such that $\phi'(x) = \psi(x)$. A little reflection shows us that this question may really be analysed into three parts.

(1) In the first place we want to know whether such a function as $\phi(x)$ *actually exists.* This is a purely theoretical question, and must be carefully distinguished from the practical question as to whether (supposing that there is such a function) we can find any simple formula to express it.

(2) We want to know whether it is possible that *more than one* such function should exist: i.e. we want to know whether our problem is one which admits of a *unique* solution or not; and if not, we want to know whether there is any simple relation between the different solutions which will enable us to express all of them in terms of any particular one.

(3) If there is a solution, we want to know *how to find an actual expression for it.*

It will throw light on the nature of these three distinct questions if we compare them with the three corresponding questions which arise with regard to the differentiation of functions.

(1) A function $\phi(x)$ may have a derivative for all values of x (like x^m, where m is a positive integer, or $\sin x$). It may generally, but not always have one (like $\sqrt[3]{x}$ or $\tan x$ or $\sec x$). Or again it may never have one: for example the function considered in Ex. XVII. 11, which is nowhere continuous, has obviously no derivative for any value of x. Of course, during this chapter, we have confined ourselves to functions which are continuous except for some special values of x. The example of the function $\sqrt[3]{x}$, however, shows that a continuous function may not have a derivative for some special value of x (in this case $x = 0$). Whether there are continuous functions which *never* have derivatives, or continuous curves which *never* have tangents, is a further question

which is at present beyond us. Common-sense says *No*: but, as we have already stated (§ 92 (2)), this is one of the cases in which higher mathematics has proved common-sense to be mistaken.

But at any rate it is clear enough that the theoretical question —has $\phi(x)$ a derivative $\phi'(x)$?—is one which has to be answered differently in different circumstances. And we may expect that the converse question—is there a function $\phi(x)$ of which $\psi(x)$ is the derivative?—will have different answers too. We have already seen that there are cases in which the answer is *No*: thus if $\psi(x)$ is the function which is equal to a, b, or c according as x is less than, equal to, or greater than 0, the answer is *No* (Exs. XLVIII. 45, XLIX. 3), unless $a=b=c$.

This is a case in which the given function is discontinuous. In what follows, however, we shall always suppose $\psi(x)$ continuous. And then the answer is, *Yes: if* $\psi(x)$ *is continuous there is always a function* $\phi(x)$ *such that* $\phi'(x)=\psi(x)$. To prove this would take us beyond our limits, however: in Ch. VII. we shall give a proof, not perfectly general, but general enough to deal with the simplest and most interesting cases that arise.

(2) The second question presents no difficulties. In the case of differentiation we have a direct definition of the derivative which makes it clear from the beginning that there cannot possibly be more than one. In the case of the converse problem the answer is almost equally simple. It is that if $\phi(x)$ is one solution of the problem $\phi(x)+C$ is another, for any value of the constant C: and that all possible solutions are comprised in the form $\phi(x)+C$. This follows at once from § 110.

(3) The practical problem of actually finding $\phi'(x)$ is as a rule a fairly simple one. We have already shown how it can be done in a number of cases, and the theorem of § 108, in conjunction with the rules of § 94, make the problem easy enough in the case of any function defined by some finite combination of the ordinary functional symbols. The converse problem is much more difficult. The nature of the difficulties will appear more clearly later on.

DEFINITIONS. *If* $\psi(x)$ *is the derivative of* $\phi(x)$, *we call* $\phi(x)$ *the* **integral** *or* **integral function** *of* $\psi(x)$. *The operation of forming* $\psi(x)$ *from* $\phi(x)$ *we call* **integration**.

We shall use the notation

$$\phi(x) = \int \psi(x)\, dx.$$

It is hardly necessary to point out that $\int \ldots dx$ like d/dx must, at present at any rate, be regarded purely as a *symbol of operation*: the $\int$ and the dx no more mean anything when taken by themselves than do the d and dx of the other operative symbol d/dx. The reason for this notation will be explained in Ch. VII.

112. The practical problem of integration. The results of the earlier part of this chapter enable us to write down at once the integrals of some of the commonest functions. Thus

$$\int x^m\, dx = \frac{x^{m+1}}{m+1}, \quad \int \cos x\, dx = \sin x, \quad \int \sin x\, dx = -\cos x \ \ldots(1).$$

These formulae must be understood as meaning that the function on the right-hand side is *one* integral of that under the sign of integration. The *most general* integral is of course obtained by adding to the former a constant C, known as the **arbitrary constant** of integration.

There is however one case of exception to the first formula, that in which $m = -1$. In this case the formula becomes nugatory, as is only to be expected, since we have already (Ex. XLIV. 5) seen that $1/x$ cannot be the derivative of any polynomial or rational fraction. And in fact it can be proved (though the proof is too detailed and tedious to be inserted here) that it is impossible to form, by means of a finite combination of the functional signs which correspond to any of the classes of functions which we have so far considered—signs such as $+$, $\times$, $\div$, $\sqrt{}$, sin, arc sin,—a function of x whose derivative is $1/x$. Some further discussion of this point will be found in Ch. IX. For the present we shall be content to assume that, if there is such a function, it is an *essentially new* function.

That there really is a function $F(x)$ such that $D_x F(x) = 1/x$ will be proved in the next chapter; and the properties of this function will be investigated in Ch. IX. For the present we shall simply assume the existence of such a function, and we shall call it **the logarithmic function** and denote it by **log** x; so that

$$\int \frac{dx}{x} = \log x \text{ ....(2).}$$

The four formulae (1) and (2) are the four most fundamental *standard forms* of the Integral Calculus. To them should perhaps be added two more, viz.

$$\int \frac{dx}{1+x^2} = \text{arc tan } x, \quad \int \frac{dx}{\sqrt{(1-x^2)}} = \pm \text{ arc sin } x^* \quad \text{...(3).}$$

113. Polynomials. All the general theorems of § 94 may of course be also stated as theorems in integration. Thus we have, to begin with, the formulae

$$\int \{f(x) + F(x)\}\, dx = \int f(x)\, dx + \int F(x)\, dx \text{ ....(1),}$$

$$\int \kappa f(x)\, dx = \kappa \int f(x)\, dx \text{ ....(2).}$$

Here it is assumed, of course, that the arbitrary constants are adjusted properly. Thus the formula (1) asserts that the sum of *any* integral of $f(x)$ and *any* integral of $F(x)$ is *an* integral of $f(x) + F(x)$.

These theorems enable us to write down at once the integral of any function of the form $\Sigma A_\nu f_\nu(x)$, the sum of a finite number of constant multiples of functions whose integrals are known. In particular we can write down the integral of any *polynomial*: in fact

$$\int (a_0 x^n + a_1 x^{n-1} + \dots + a_n)\, dx = \frac{a_0 x^{n+1}}{n+1} + \frac{a_1 x^n}{n} + \dots + a_n x.$$

114. Rational Functions. After integrating polynomials it is natural to turn our attention next to *rational functions.* Let us suppose $R(x)$ to be any rational function expressed in the standard form of § 98, viz. as the sum of a polynomial $\Pi(x)$ and a number of terms of the form $A/(x-\alpha)^p$.

We can at once write down the integrals of the polynomial and of all the other terms except those for which $p = 1$, since

$$\int \frac{A}{(x-\alpha)^p}\, dx = -\frac{A}{p-1} \frac{1}{(x-\alpha)^{p-1}},$$

whether α be real or complex (§ 98).

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* See § 100 for the rule for determining the ambiguous sign.

The terms for which $p=1$ present rather more difficulty. And it is convenient at this stage to introduce another general theorem in integration.

It follows immediately from Theorem (8) of § 94 that if $\int \psi(x)\,dx = \phi(x)$, and a and b are real, then

$$\int \psi(ax+b)\,dx = (1/a)\,\phi(ax+b) \quad \text{...........(3)}.$$

Thus, for example,

$$\int \frac{dx}{ax+b} = (1/a)\log(ax+b),$$

and, in particular, if α is real,

$$\int \frac{dx}{x-\alpha} = \log(x-\alpha).$$

We can therefore write down the integrals of all the terms in $R(x)$ for which $p=1$ and α is real. There remain the terms for which $p=1$ and α is complex.

In order to deal with these we shall introduce a restrictive hypothesis, viz. that $R(x)$ is a real function—i.e. that all its coefficients are real. Then if $\alpha=\gamma+i\delta$ is a root of $Q(x)=0$, m times repeated, so is $\alpha'=\gamma-i\delta$. Moreover, if the partial fractions corresponding to the factor $(x-\alpha)^m$ are $\Sigma A_p/(x-\alpha)^p$, those corresponding to the factor $(x-\alpha')^m$ are $\Sigma A_p'/(x-\alpha')^p$, where A_p' is conjugate (Ch. III, § 30) to A_p. This follows from the nature of the algebraical processes by means of which the partial fractions can be found, and which are explained at length in treatises on Algebra*.

Thus if a term $(\lambda+i\mu)/(x-\gamma-i\delta)$ occurs in the standard form of $R(x)$, so will a term $(\lambda-i\mu)/(x-\gamma+i\delta)$; and the sum of these two terms is

$$2\{\lambda(x-\gamma)-\mu\delta\}/\{(x-\gamma)^2+\delta^2\}.$$

This fraction is in reality the most general fraction of the form

$$(Ax+B)/(ax^2+2bx+c),$$

where $b^2<ac$. The reader will easily verify the equivalence of

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* See, for example, Chrystal's *Algebra*, vol. I, pp. 151–9.

the two forms, the formulae which express λ, μ, γ, δ in terms of A, B, a, b, c being

$$\lambda = A/2a, \quad \mu = -D/(2a\sqrt{\Delta}), \quad \gamma = -b/a, \quad \delta = \sqrt{\Delta}/a,$$

where $\Delta = ac - b^2$, and $D = aB - bA$.

We shall now introduce another general theorem in integration, which follows at once from the theorem of § 108: viz.

$$\int F'\{f(x)\}\, f'(x)\, dx = F\{f(x)\} \quad \ldots\ldots\ldots\ldots\ldots(4).$$

If in particular we suppose $F\{f(x)\}$ to be $\log f(x)$, so that $F'\{f(x)\} = 1/f(x)$, we obtain

$$\int \frac{f'(x)}{f(x)}\, dx = \log f(x);$$

and if we further suppose that $f(x) = (x - \lambda)^2 + \mu^2$ we obtain

$$\int \frac{2(x-\lambda)}{(x-\lambda)^2 + \mu^2}\, dx = \log\{(x-\lambda)^2 + \mu^2\}.$$

Again, in virtue of the equations (3) of § 112 and (3) above

$$\int \frac{-2\delta\mu}{(x-\lambda)^2 + \mu^2}\, dx = -2\delta \operatorname{arc\,tan}\left(\frac{x-\lambda}{\mu}\right).$$

These two formulae enable us to integrate the sum of the two terms which we have been considering in the expression of $R(x)$; and we are thus enabled to write down the integral of any rational function, if all the factors of its denominator can be determined. The integral of any such function is composed of *the sum of a polynomial, a number of rational fractions of the type*

$$-A/\{(p-1)(x-\alpha)^{p-1}\},$$

a number of logarithmic functions, and a number of inverse tangents. It only remains to add that if α is complex such a fraction as

$$-A/\{(p-1)(x-\alpha)^{p-1}\}$$

always occurs in conjunction with another in which A and α are replaced by the complex numbers conjugate to them, and that the sum of the two fractions is a real rational fraction.

Examples LI. 1. The integral of the function $(Ax+B)/(ax^2+2bx+c)$ may be expressed in the form

$$\frac{A}{2a}\log X + \frac{D}{2a\sqrt{(-\Delta)}}[\log\{ax+b-\sqrt{(-\Delta)}\} - \log\{ax+b+\sqrt{(-\Delta)}\}]$$

(where $X = ax^2+2bx+c$) if $\Delta < 0$, and in the form

$$\frac{A}{2a}\log X + \frac{D}{a\sqrt{\Delta}}\operatorname{arc\,tan}\left(\frac{ax+b}{\sqrt{\Delta}}\right)$$

if $\Delta > 0$.

2. In the particular case in which $ac=b^2$ the integral is

$$-D/\{a(ax+b)\}+(A/a)\log\{x+(b/a)\}.$$

3. Show that if the roots of $Q(x)=0$ are all real and distinct, and $P(x)$ is of lower degree than $Q(x)$, then

$$\int R(x)\,dx=\sum\frac{P(\alpha)}{Q'(\alpha)}\log(x-\alpha)$$

the summation applying to all the roots α of $Q(x)=0$.

[That the fraction corresponding to α is $\dfrac{P(\alpha)}{Q'(\alpha)}\dfrac{1}{x-\alpha}$ follows from the fact that $Q(x)/(x-\alpha)\to Q'(\alpha)$ and so $(x-\alpha)R(x)\to P(\alpha)/Q'(\alpha)$, as $x\to\alpha$.]

4. If all the roots of $Q(x)$ are real and one (α) is a double root, the rest being simple roots, and $P(x)$ is of lower degree than $Q(x)$, then the integral is $A/(x-\alpha)+A'\log(x-\alpha)+\sum B\log(x-\beta)$, where

$A=-2P(\alpha)/Q''(\alpha)$, $A'=\frac{2}{3}\{3P'(\alpha)Q''(\alpha)-P(\alpha)Q'''(\alpha)\}/\{Q''(\alpha)\}^2$, $B=P(\beta)/Q'(\beta)$,

and the summation applies to all roots β of $Q(x)=0$ other than α.

5. Find $$\int\frac{dx}{\{(x-1)(x^2+1)\}^2}.$$

[The expression in partial fractions is

$$\frac{1}{4(x-1)^2}-\frac{1}{2(x-1)}-\frac{i}{8(x-i)^2}+\frac{2-i}{8(x-i)}+\frac{i}{8(x+i)^2}+\frac{2+i}{8(x+i)},$$

and the integral is

$$-\frac{1}{4(x-1)}-\frac{1}{4(x^2+1)}-\tfrac{1}{2}\log(x-1)+\tfrac{1}{4}\log(x^2+1)+\tfrac{1}{4}\arctan x\text{*}.]$$

6. Integrate

$$x/\{(x-a)(x-b)(x-c)\},\quad x/\{(x-a)^2(x-b)\},\quad x/(x-a)^3,$$
$$x/\{(x^2+a^2)(x^2+b^2)\},\quad x^2/\{(x^2+a^2)(x^2+b^2)\},\quad x^3/\{(x^2+a^2)(x^2+b^2)\},$$
$$(x+1)/\{x^2(x-1)\},\quad (x+1)/\{x(x-1)^2\},\quad (x+1)/\{x(x-1)\}^2,$$
$$(x^2-1)/\{x^2(x^2+1)\},\quad (x^2-1)/\{x(x^2+1)^2\},\quad (x^2-1)/\{x(x^2+1)\}^2.$$

7. Prove the formulae:

$$\int\frac{dx}{1+x^4}=\frac{1}{4\sqrt{2}}\left\{\log(1+x\sqrt{2}+x^2)-\log(1-x\sqrt{2}+x^2)+2\arctan\left(\frac{x\sqrt{2}}{1-x^2}\right)\right\},$$

$$\int\frac{x^2\,dx}{1+x^4}=\frac{1}{4\sqrt{2}}\left\{\log(1-x\sqrt{2}+x^2)-\log(1+x\sqrt{2}+x^2)+2\arctan\left(\frac{x\sqrt{2}}{1-x^2}\right)\right\},$$

$$\int\frac{dx}{1+x^2+x^4}=\frac{1}{4\sqrt{3}}\left\{\sqrt{3}[\log(1+x+x^2)-\log(1-x+x^2)]+2\arctan\left(\frac{x\sqrt{3}}{1-x^2}\right)\right\}.$$

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* In this case the application of the general method of § 114 is fairly simple. In more complicated cases the labour involved is sometimes prohibitive, and other devices have to be used. We have, moreover, assumed that all the factors of the denominator can be determined. If this is not the case the method of partial fractions fails, and recourse must be had to other methods. For further information concerning the integration of rational functions the reader may be referred to Goursat's *Cours d'Analyse*, t. i, pp. 234 *et seq.*, and to the author's tract *The integration of functions of a single variable*, pp. 10 *et seq.*

115. Algebraical Functions. We naturally pass on next to the question of the integration of *algebraical* functions. We shall confine our attention to *explicit* algebraical functions (Ch. II, § 16).

We have to consider the problem of integrating y, where y is an explicit algebraical function of x. It is however convenient to consider an apparently more general integral, viz.

$$\int R(x, y)\,dx,$$

where $R(x, y)$ is any rational function of x and y. The greater generality of this form is only apparent, since (Ex. XV. 6) the function $R(x, y)$ is itself an algebraical function of x. The choice of this form is in fact dictated simply by motives of convenience: such a function as

$$\{x+\sqrt{(ax^2+2bx+c)}\}/\{x-\sqrt{(ax^2+2bx+c)}\}$$

is far more conveniently regarded as a rational function of x and the simple algebraical function $\sqrt{(ax^2+2bx+c)}$, than directly as itself an algebraical function of x.

116. Integration by substitution and rationalisation. It follows from equation (4) of § 114 that if $\int \psi(x)\,dx = \phi(x)$, then

$$\int \psi\{f(t)\}\,f'(t)\,dt = \phi\{f(t)\} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

This equation supplies us with a method for determining the integral of $\psi(x)$ in a large number of cases in which the form of the integral is not directly obvious. It may be stated as a rule as follows: *put* $x=f(t)$, *where* $f(t)$ *is any function of a new variable* t *which it may be convenient to choose; multiply by* $f'(t)$ *and determine (if possible) the integral of* $\psi\{f(t)\}\,f'(t)$; *express the result in terms of* x. It will often be found that the function of t to which we are led by the application of this rule is one whose integral can easily be calculated. This is always so, for example, if it is a rational function, and it is very often possible to choose the relation between x and t so that this shall be the case. Thus the integral of $R(\sqrt{x})$, where R denotes a rational function, is reduced by the substitution $x=t^2$ to the integral of $2tR(t^2)$, i.e. to the integral of a rational function of t. This method of integration is called **integration by rationalisation,** and is of extremely wide application.

Its application to the problem immediately under consideration is obvious. *If we can find a variable t such that x and y are both rational functions of t, say $x = R_1(t)$, $y = R_2(t)$, then*

$$\int R(x, y)\,dx = \int R\{R_1(t), R_2(t)\}\,R_1'(t)\,dt,$$

and the latter integral, being that of a rational function of t, can be calculated by the methods of § 114.

It would carry us beyond our present range to enter upon any general discussion as to when it is and when it is not possible to find an auxiliary variable t connected with x and y in the manner indicated above. We shall only consider a few simple and interesting special cases.

117. Integrals connected with conics. Let us suppose that x and y are connected by an equation of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0;$$

in other words that the graph of y, considered as a function of x, is a conic. Suppose that (ξ, η) is any point on the conic, and let $x - \xi = X$, $y - \eta = Y$. If the relation between x and y is expressed in terms of X and Y it assumes the form

$$aX^2 + 2hXY + bY^2 + 2GX + 2FY = 0,$$

where $F = h\xi + b\eta + f$, $G = a\xi + h\eta + g$. In this equation put $Y = tX$. It will then be found that X and Y can both be expressed as rational functions of t, and therefore x and y can be so expressed, the actual formulae being

$$x - \xi = -2(G + Ft)/(a + 2ht + bt^2), \quad y - \eta = -2t(G + Ft)/(a + 2ht + bt^2).$$

Hence the process of rationalisation described in the last section can be carried out.

The reader should verify that

$$hx + by + f = -\tfrac{1}{2}(a + 2ht + bt^2)\frac{dx}{dt},$$

so that
$$\int \frac{dx}{hx + by + f} = -2\int \frac{dt}{a + 2ht + bt^2},$$

a formula which will be useful later on.

118. The integral $\int R\{x, \sqrt{(ax^2+2bx+c)}\}\,dx$. The most important case is that in which the relation between x and y is

$$y^2 = ax^2 + 2bx + c.$$

Let ξ, $\sqrt{(a\xi^2+2b\xi+c)}$ be the coordinates of any point on the conic. The relation between X and Y is

$$aX^2 - Y^2 + 2(a\xi+b)X - 2\eta Y = 0,$$

and the formulae expressing x and y in terms of t are

$$x-\xi = -\frac{2(a\xi+b-t\eta)}{a-t^2}, \quad y-\eta = -\frac{2t(a\xi+b-t\eta)}{a-t^2} \quad \ldots(1).$$

Consider, for example, the integral $\int \frac{dx}{y}$. If $\Delta = ac-b^2$, there are three cases to consider, viz. those in which

(i) $a>0$, $\Delta>0$; (ii) $a>0$, $\Delta<0$; (iii) $a<0$, $\Delta<0$.

If $a<0$ and $\Delta>0$, $ax^2+2bx+c$ is always negative, so that this case is not of any interest.

If $a>0$, $\Delta>0$, the conic is a hyperbola with one branch entirely above the axis of x, which is described once as t varies from $-\sqrt{a}$ to $+\sqrt{a}$. If (ξ, η) lies on this branch, $\eta>0$ and $-\sqrt{a}<t<\sqrt{a}$. Since now $h=f=0$, $b=-1$, the last formula of § 117 gives

$$\int\frac{dx}{y} = 2\int\frac{dt}{a-t^2} = \frac{1}{\sqrt{a}}\{\log(\sqrt{a}+t) - \log(\sqrt{a}-t)\}.$$

But it follows from the equations (1) that

$$(x-\xi)\sqrt{a} \pm (y-\eta) = -2(a\xi+b-t\eta)/(\sqrt{a}\mp t)$$

and so*

$$\int\frac{dx}{y} = \frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{a}+t}{\sqrt{a}-t}\right) = \frac{1}{2\sqrt{a}}\log\left\{\frac{(x-\xi)\sqrt{a}+(y-\eta)}{(x-\xi)\sqrt{a}-(y-\eta)}\right\}^2 \quad \ldots\ldots\ldots(2),$$

the logarithm being written in this form in order to avoid any possible difficulty as to the sign of the function inside the large bracket. This equation is true for all pairs of values of ξ and η related as above. A particularly simple form of the integral may be found as follows. Since we may add any constant to the right-hand side, we have

$$\int\frac{dx}{y} = \frac{1}{2\sqrt{a}}\log\{(x-\xi)\sqrt{a}+(y-\eta)\}^2 + \frac{1}{2\sqrt{a}}\log\left\{\frac{\eta-\xi\sqrt{a}}{(x-\xi)\sqrt{a}-(y-\eta)}\right\}^2.$$

Now suppose that $\xi \to -\infty$. Then it is easy to see that $\eta \to +\infty$, $\xi\sqrt{a}+\eta \to -b/\sqrt{a}$, and that the contents of the last large bracket tend to unity. We thus obtain the formula

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* In the succeeding discussion we anticipate the fundamental properties of the logarithm, which will be proved later on, viz. that $\log u$ is continuous for all positive values of u, $\log 1 = 0$, $\log(1/u) = -\log u$, $\log uv = \log u + \log v$.

$$\int \frac{dx}{y} = \frac{1}{2\sqrt{a}} \log \left(x\sqrt{a} + y + \frac{b}{\sqrt{a}} \right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If in particular $a=1$, $b=0$, $c=a^2$ we obtain

$$\int \frac{dt}{\sqrt{(x^2+a^2)}} = \log\{x+\sqrt{(x^2+a^2)}\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The truth of this equation may be at once verified by differentiation of the right-hand side. If we transform the integral by the substitution $x^2+a^2=u^2$, $u=\sqrt{(x^2-a^2)}$ we obtain, on writing x again for u,

$$\int \frac{dx}{\sqrt{(x^2-a^2)}} = \log\{x+\sqrt{(x^2-a^2)}\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

This integral may also be calculated directly by an argument similar to that used for the integral (4). It is the simplest example of case (ii) above. The reader should associate with these two formulae the third formula

$$\int \frac{dx}{\sqrt{(a^2-x^2)}} = \arcsin(x/a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

This integral corresponds to the case (iii) above. The formula appears very different from (4) and (5): the reader will hardly be in a position to appreciate the connection between them until he has read Chs. IX and X. In the last formula it is supposed that a is positive: if a is negative the integral function is $\arcsin(x/|a|) = -\arcsin(x/a)$ (cf. § 100).

119. The integral $\int \frac{\lambda x + \mu}{\sqrt{(ax^2+2bx+c)}}\,dx$. This integral can be integrated in all cases by means of the results of the preceding sections. It is most convenient to proceed as follows. Since

$$\lambda x + \mu \equiv (\lambda/a)(ax+b) + \mu - (\lambda b/a),$$

$$\int \frac{ax+b}{\sqrt{(ax^2+2bx+c)}}\,dx = \sqrt{(ax^2+2bx+c)},$$

we have

$$\int \frac{(\lambda x+\mu)\,dx}{\sqrt{(ax^2+2bx+c)}} = (\lambda/a)\sqrt{(ax^2+2bx+c)} + \gamma \int \frac{dx}{\sqrt{(ax^2+2bx+c)}},$$

where $\gamma = \mu - (\lambda b/a)$. In the last integral a may be positive or negative. If it is positive we put $x\sqrt{a} + (b/\sqrt{a}) = t$, when we obtain

$$\frac{1}{\sqrt{a}} \int \frac{dt}{\sqrt{(t^2+\kappa)}}$$

where $\kappa = (ac-b^2)/a$. If a is negative we write A for $-a$ and put $x\sqrt{A} - (b/\sqrt{A}) = t$, when we obtain

$$\frac{1}{\sqrt{(-a)}} \int \frac{dt}{\sqrt{(-\kappa-t^2)}}.$$

It thus appears that in any case the calculation of the integral may be made to depend on that considered in § 118, viz. one or other of the three integrals

$$\int \frac{dt}{\sqrt{(t^2+\alpha^2)}}, \quad \int \frac{dt}{\sqrt{(t^2-\alpha^2)}}, \quad \int \frac{dt}{\sqrt{(\alpha^2-t^2)}}.$$

120. The integral $\int (\lambda x+\mu)\sqrt{(ax^2+2bx+c)}\,dx$. In exactly the same way we find

$$\int (\lambda x+\mu)\sqrt{(ax^2+2bx+c)}\,dx = (\lambda/3a)(ax^2+2bx+c)^{3/2}+\gamma\int\sqrt{(ax^2+2bx+c)}\,dx;$$

and the last integral may be reduced to one or other of the three forms

$$\int\sqrt{(t^2+a^2)}\,dt, \quad \int\sqrt{(t^2-a^2)}\,dt, \quad \int\sqrt{(a^2-t^2)}\,dt.$$

In order to obtain these integrals it is convenient to introduce at this point another general theorem in integration.

121. Integration by parts. The theorem of *integration by parts* is merely another way of stating the rule for the differentiation of a product (§ 94). It follows at once from Theorem (3) of § 94 that

$$\int f'(x)\,F(x)\,dx = f(x)\,F(x) - \int f(x)\,F'(x)\,dx.$$

It may happen that the function which we wish to integrate is expressible in the form $f'(x)\,F(x)$, and that $f(x)\,F'(x)$ can be integrated. Thus suppose that $\phi(x) = x\psi(x)$, where $\psi(x)$ is the second derivative of a known function $\chi(x)$. Then

$$\int \phi(x)dx = \int x\chi''(x)dx = x\chi'(x) - \int \chi'(x)dx = x\chi'(x) - \chi(x).$$

We can illustrate the working of this method of integration by applying it to the integrals of the last section. Taking

$$f(x) = ax+b, \quad F(x) = \sqrt{(ax^2+2bx+c)} = y,$$

we obtain

$$a\int y\,dx = (ax+b)y - \int\frac{(ax+b)^2}{y}\,dx = (ax+b)y - a\int y\,dx + (ac-b^2)\int\frac{dx}{y},$$

so that
$$\int y\,dx = \frac{ax+b}{2a}y + \frac{ac-b^2}{2a}\int\frac{dx}{y};$$

and we have already seen (§ 118) how to determine the last integral.

Examples LII. 1. Prove that, if $a>0$,

$$\int \sqrt{(x^2+a^2)}\,dx=\tfrac{1}{2}x\sqrt{(x^2+a^2)}+\tfrac{1}{2}a^2\log\{x+\sqrt{(x^2+a^2)}\},$$

$$\int \sqrt{(x^2-a^2)}\,dx=\tfrac{1}{2}x\sqrt{(x^2-a^2)}-\tfrac{1}{2}a^2\log\{x+\sqrt{(x^2-a^2)}\},$$

$$\int \sqrt{(a^2-x^2)}\,dx=\tfrac{1}{2}x\sqrt{(a^2-x^2)}+\tfrac{1}{2}a^2\arcsin(x/a).$$

2. Calculate the integrals $\int\frac{dx}{\sqrt{(a^2-x^2)}}$, $\int\sqrt{(a^2-x^2)}\,dx$ by means of the substitution $x=a\sin\theta$, and verify that the results agree with those obtained in § 118 and Ex. 1.

3. Calculate $\int x(x+a)^m\,dx$, m being any rational number, in three ways, (i) by integration by parts, (ii) by the substitution $(x+a)^m=t$, (iii) by writing $(x+a)-a$ for x; and verify that the results agree.

4. Prove, by means of the substitutions $ax+b=1/t$ and $x=1/u$, that (in the notation of § 118)

$$\int\frac{dx}{y^3}=\frac{ax+b}{\Delta y},\quad \int\frac{x\,dx}{y^3}=-\frac{bx+c}{\Delta y}.$$

5. Integrate

$$1/\{(1+x)\sqrt{x}\},\quad \sqrt{x}/(1+x),\quad 1/\{x\sqrt{(1+x)}\},\quad x/\sqrt{(1+x)},\quad 1/\sqrt{\{x(1+x)\}},$$
$$1/\sqrt{\{x(x-1)\}},\quad 1/\sqrt{\{x(1-x)\}},\quad \sqrt{\{(1+x)/(1-x)\}},\quad x\sqrt{(a+bx)},\quad x^2/\sqrt{(a+bx)},$$
$$1/\{x\sqrt{(x^2+a^2)}\},\quad 1/\{x^2\sqrt{(x^2-a^2)}\},\quad 1/\{x^3\sqrt{(a^2-x^2)}\},\quad x^3/\sqrt{(a^2-x^2)}.$$

6. Integrate $1/\sqrt{\{(x-a)(b-x)\}}$ in three ways, (i) by the methods of the preceding sections, (ii) by the substitution $(b-x)/(x-a)=t^2$, (iii) by the substitution $x=a\cos^2\theta+b\sin^2\theta$; and verify that the results agree.

7. Integrate $\sqrt{\{(x-a)(b-x)\}}$ and $\sqrt{\{(b-x)/(x-a)\}}$.

8. Show, by means of the substitution $2x+a+b=\frac{1}{2}(a-b)\{t^2+(1/t^2)\}$, or by multiplying numerator and denominator by $\sqrt{(x+a)}-\sqrt{(x+b)}$, that, if $a>b$,

$$\int\frac{dx}{\sqrt{(x+a)}+\sqrt{(x+b)}}=\tfrac{1}{2}\sqrt{(a-b)}\{t+(1/3t^3)\}.$$

9. Find a substitution which will reduce $\int\frac{dx}{(x+a)^{3/2}+(x-a)^{3/2}}$ to the integral of a rational function. (*Math. Trip.* 1899.)

10. Show that $\int R\{x,\sqrt[n]{(ax+b)}\}\,dx$ is reduced, by the substitution $ax+b=y^n$, to the integral of a rational function.

11. Prove that

$$\int f''(x)F(x)\,dx=f'(x)F(x)-f(x)F'(x)+\int f(x)F''(x)\,dx,$$

and generally

$$\int f^{(n)}(x)F(x)\,dx=f^{(n-1)}(x)F(x)-f^{(n-2)}(x)F'(x)+\dots+(-1)^n\int f(x)F^{(n)}(x)\,dx.$$

12. Hence integrate $x^n \phi^{(n+1)}(x)$, i.e. $x^n \psi(x)$, where $\psi(x)$ is a function which can be integrated $n+1$ times. In particular integrate $x^n(ax+b)^p$, n being a positive integer, and p any rational number other than -1.

13. The integral $\int (1+x)^p x^q\,dx$, where p and q are rational, can be found in three cases, viz. (i) if p is an integer, (ii) if q is an integer, (iii) if $p+q$ is an integer.

[In case (i) put $x=u^s$, where s is the denominator of q: in case (ii) put $1+x=t^s$, where s is the denominator of p: in case (iii) put $1+x=xt^s$, where s is the denominator of p.]

14. The integral $\int x^m(ax^n+b)^q\,dx$ can be reduced to the preceding integral by the substitution $ax^n=bt$.

[In practice it is often most convenient to calculate a particular integral of this kind by a *formula of reduction* (v. Misc. Ex. 40).]

15. The integral $\int R\{x, \sqrt{(ax+b)}, \sqrt{(cx+d)}\}\,dx$ can be reduced to that of a rational function by the substitution

$$4x = -(b/a)\{t+(1/t)\}^2 - (d/c)\{t-(1/t)\}^2.$$

16. Show how to calculate the integral $\int \frac{dx}{(x-p)\sqrt{(ax^2+2bx+c)}}$ by means of the substitution $x-p=1/t$.

17. Show by means of the substitution $y=\sqrt{(ax^2+2bx+c)}/(x-p)$ that

$$\int \frac{dx}{(x-p)\sqrt{(ax^2+2bx+c)}} = \int \frac{dy}{\sqrt{\{\lambda y^2-\mu\}}},$$

where $\lambda=ap^2+2bp+c$, $\mu=ac-b^2$; and hence evaluate the integral.

18. Calculate the integrals of

$$1/\{(x-1)\sqrt{(x^2+1)}\}, \quad 1/\{(x+1)\sqrt{(1+2x-x^2)}\}$$

by means of each of the preceding methods, and verify the agreement of the results*.

19. Reduce $\int R(x, y)\,dx$, where $y^2(x-y)=x^2$, to the integral of a rational function. [Putting $y=tx$ we obtain $x=1/\{t^2(1-t)\}$, $y=1/\{t(1-t)\}$.]

20. Reduce the integral in the same way when (*a*) $y(x-y)^2=x$, (*b*) $(x^2+y^2)^2=a^2(x^2-y^2)$. [In case (*a*) put $x-y=t$: in case (*b*) put $x^2+y^2=t(x-y)$, when we obtain $x=a^2t(t^2+a^2)/(t^4+a^4)$, $y=a^2t(t^2-a^2)/(t^4+a^4)$.]

21. If $y(x-y)^2=x$ then $\int \frac{dx}{x-3y} = \frac{1}{2}\log\{(x-y)^2-1\}$.

22. If $(x^2+y^2)^2=2c^2(x^2-y^2)$ then $\int \frac{dx}{y(x^2+y^2+c^2)} = -\frac{1}{c^2}\log\left(\frac{x^2+y^2}{x-y}\right)$.

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* See also Misc. Exs. 33 *et seq.*

122. Transcendental Functions. Owing to the immense variety of the different classes of transcendental functions, the theory of their integration is a good deal less systematic than that of the integration of rational or algebraical functions. We shall consider in order a few classes of transcendental functions whose integrals can always be found.

123. Polynomials in cosines and sines of multiples of x. We can always integrate any function which is the sum of a finite number of terms such as

$$A\,(\cos ax)^m(\sin ax)^{m'}(\cos bx)^n(\sin bx)^{n'}\dots$$

where $m, m', n, n', \dots$ are positive integers and $a, b, \dots$ any real numbers whatever. For such a term can be expressed as the sum of a finite number of terms of the types

$$\alpha\cos\{(pa+qb+\dots)x\},\quad \beta\sin\{(p'a+q'b+\dots)x\}$$

and the integrals of these terms may be at once written down.

Examples LIII. 1. Integrate $\sin^3 x\cos^2 2x$. In this case we use the formulae

$$\sin^3 x=\tfrac{1}{4}(3\sin x-\sin 3x),\quad \cos^2 2x=\tfrac{1}{2}(1+\cos 4x).$$

Multiplying these two expressions and replacing $\sin x\cos 4x$, for example, by $\frac{1}{2}(\sin 5x-\sin 3x)$, we obtain

$$\tfrac{1}{16}\int(7\sin x-5\sin 3x+3\sin 5x-\sin 7x)\,dx$$
$$=-\tfrac{7}{16}\cos x+\tfrac{5}{48}\cos 3x-\tfrac{3}{80}\cos 5x+\tfrac{1}{112}\cos 7x.$$

The integral may of course be obtained in a different form by different methods. For example

$$\int\sin^3 x\cos^2 2x\,dx=\int(4\cos^4 x-4\cos^2 x+1)(1-\cos^2 x)\sin x\,dx,$$

which reduces, on making the substitution $\cos x=t$, to

$$\int(4t^6-8t^4+5t^2-1)\,dt=\tfrac{4}{7}\cos^7 x-\tfrac{8}{5}\cos^5 x+\tfrac{5}{3}\cos^3 x-\cos x.$$

It may of course be verified that this expression and the integral already obtained differ only by a constant.

2. Integrate by any method $\cos ax\cos bx$, $\sin ax\sin bx$, $\cos ax\sin bx$, $\cos^2 x$, $\sin^3 x$, $\cos^4 x$, $\cos x\cos 2x\cos 3x$, $\cos\frac{1}{2}x\cos 2x$, $\cos^3 2x\sin^2 3x$, $\cos^5 x\sin^7 x$. [In cases of this kind it is also sometimes convenient to use a formula of reduction (Misc. Ex. 40).]

124. The integrals $\int x^n\cos x\,dx$, $\int x^n\sin x\,dx$ and associated integrals. The method of integration by parts enables us to generalise the preceding results. For

$$\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx,$$
$$\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx,$$

and clearly the integrals can be calculated completely by a repetition of this process. It follows that we can always calculate $\int x^n \cos ax\,dx$, $\int x^n \sin ax\,dx$; and so by a process similar to that of the preceding paragraph, we can calculate

$$\int P(x,\ \cos ax,\ \sin ax,\ \cos bx,\ \sin bx,\ \dots)\,dx,$$

where P is any polynomial.

Examples LIV. 1. Integrate $x\sin x$, $x^2\cos x$, $(x\cos x)^2$, $(x\sin x\sin 2x)^2$, $x\sin^2 x\cos^4 x$, $(x\sin\frac{1}{3}x)^3$.

2. Find polynomials P and Q such that

$$\int[(3x-1)\cos x+(1-2x)\sin x]\,dx = P\cos x + Q\sin x.$$

3. Prove that $\int x^n\cos x\,dx = P_n\cos x + Q_n\sin x$, where

$$P_n = nx^{n-1} - n(n-1)(n-2)x^{n-3} + \dots,\quad Q_n = x^n - n(n-1)x^{n-2} + \dots.$$

125. Rational Functions of $\cos x$ **and** $\sin x$. The integral of any rational function of $\cos x$ and $\sin x$ may be calculated by the substitution $\tan\frac{1}{2}x = t$. For

$$\cos x = \frac{1-t^2}{1+t^2},\quad \sin x = \frac{2t}{1+t^2},\quad \frac{dx}{dt} = \frac{2}{1+t^2},$$

so that the substitution reduces the integral to that of a rational function of t.

Examples LV. 1. Prove that

$$\int \sec x\,dx = \log(\sec x + \tan x),\quad \int \operatorname{cosec} x\,dx = \log\tan\tfrac{1}{2}x.$$

[Another form of the first integral is $\log\tan(\frac{1}{4}\pi + \frac{1}{2}x)$; a third form is $\frac{1}{2}\log\{(1+\sin x)/(1-\sin x)\}$.]

2. $\int \tan x\,dx = -\log\cos x$, $\int \cot x\,dx = \log\sin x$, $\int \sec^2 x\,dx = \tan x$, $\int \operatorname{cosec}^2 x\,dx = -\cot x$, $\int \tan x\sec x\,dx = \sec x$, $\int \cot x\operatorname{cosec} x\,dx = -\operatorname{cosec} x$.

[These integrals are included in the general form, but there is no need to use a substitution, as the results follow at once from § 100.]

3. Show that the integral of $1/(a+b\cos x)$, where $a+b$ is positive, may be expressed in one or other of the forms

$$\frac{2}{\sqrt{(a^2-b^2)}}\operatorname{arc\,tan}\left\{t\sqrt{\left(\frac{a-b}{a+b}\right)}\right\},\qquad \frac{1}{\sqrt{(b^2-a^2)}}\log\left\{\frac{\sqrt{(b+a)}+t\sqrt{(b-a)}}{\sqrt{(b+a)}-t\sqrt{(b-a)}}\right\},$$

where $t=\tan\frac{1}{2}x$, according as $a^2 \gtrless b^2$. If $a^2=b^2$ the integral reduces to a constant multiple of that of $\sec^2\frac{1}{2}x$ or $\operatorname{cosec}^2\frac{1}{2}x$, and its value may at once be written down. Deduce the forms of the integral when $a+b$ is negative.

4. Show that if y is defined in terms of x by means of the equation

$$(a+b\cos x)(a-b\cos y)=a^2-b^2,$$

where $a^2>b^2$, then as x varies from 0 to π one value of y varies from 0 to π. Show also that

$$\sin x=\frac{\sqrt{(a^2-b^2)}\sin y}{a-b\cos y},\qquad \frac{\sin x}{a+b\cos x}\frac{dx}{dy}=\frac{\sin y}{a-b\cos y};$$

and deduce that, if $0<x<\pi$,

$$\int\frac{dx}{a+b\cos x}=\frac{1}{\sqrt{(a^2-b^2)}}\operatorname{arc\,cos}\left(\frac{a\cos x+b}{a+b\cos x}\right).$$

Show that this result agrees with that of Ex. 3.

5. Show how to integrate $1/(a+b\cos x+c\sin x)$. [Express $b\cos x+c\sin x$ in the form $\sqrt{(b^2+c^2)}\cos(x-\alpha)$.]

6. Integrate $(a+b\cos x+c\sin x)/(\alpha+\beta\cos x+\gamma\sin x)$.

[Determine λ, μ, ν so that

$$a+b\cos x+c\sin x\equiv\lambda+\mu(\alpha+\beta\cos x+\gamma\sin x)+\nu(-\beta\sin x+\gamma\cos x).$$

Then the integral is

$$\mu x+\nu\log(\alpha+\beta\cos x+\gamma\sin x)+\lambda\int\frac{dx}{\alpha+\beta\cos x+\gamma\sin x}.]$$

7. Integrate $1/(5+3\cos x)$, $1/(3-5\cos x)$, $1/(2-\sin x)$, $1/(1-\cos x+2\sin x)$, $(5+3\cos x-7\sin x)/(11-\cos x+\sin x)$.

8. Integrate $1/(a\cos^2 x+2b\cos x\sin x+c\sin^2 x)$. [The subject of integration may be expressed in the form $1/(A+B\cos 2x+C\sin 2x)$, where $A=\frac{1}{2}(a+c)$, $B=\frac{1}{2}(a-c)$, $C=b$: but the integral may be calculated more simply by putting $\tan x=t$, when we obtain

$$\int\frac{\sec^2 x\,dx}{a+2b\tan x+c\tan^2 x}=\int\frac{dt}{a+2bt+ct^2}.]$$

126. Integrals involving $\operatorname{arc\,sin} x$, $\operatorname{arc\,tan} x$, **and** $\log x$. The integrals of the inverse sine and tangent and of the logarithm can easily be calculated by integration by parts. Thus

$$\int\operatorname{arc\,sin} x\,dx=x\operatorname{arc\,sin} x-\int\frac{x\,dx}{\sqrt{(1-x^2)}}=x\operatorname{arc\,sin} x+\sqrt{(1-x^2)},$$

$$\int\operatorname{arc\,tan} x\,dx=x\operatorname{arc\,tan} x-\int\frac{x\,dx}{1+x^2}=x\operatorname{arc\,tan} x-\tfrac{1}{2}\log(1+x^2),$$

$$\int\log x\,dx=x\log x-\int dx=x(\log x-1).$$

It is easy to see that if we can find the integral of $y = f(x)$ we can always find that of $x = \phi(y)$, ϕ being the function inverse to f. For on making the substitution $y = f(x)$ we obtain

$$\int \phi(y)\,dy = \int x f'(x)\,dx = x f(x) - \int f(x)\,dx.$$

The reader should evaluate the integrals of $\operatorname{arc}\sin y$ and $\operatorname{arc}\tan y$ in this way.

Any integrals of the form

$$\int P(x, \operatorname{arc}\sin x)\,dx, \quad \int P(x, \log x)\,dx,$$

where P is a polynomial, can be found. Take the first form, for example. We have to calculate a number of integrals of the type $\int x^m (\operatorname{arc}\sin x)^n\,dx$. Making the substitution $x = \sin y$ we obtain $\int y^n \sin^m y \cos y\,dy$, which can be found by the methods of § 124.

In the case of the second form we have to calculate a number of integrals of the form $\int x^m (\log x)^n\,dx$. Integrating by parts we obtain

$$\int x^m (\log x)^n\,dx = \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1}\int x^m (\log x)^{n-1}\,dx,$$

and it is evident that by repeating this process often enough we shall always arrive finally at the complete value of the integral*.

127. Areas of plane curves. One of the most important applications of the processes of integration which have been explained in the preceding sections is to the calculation of **areas** of plane curves. Suppose that P_0PP' (Fig. 53) is the graph of a continuous curve $y = \phi(x)$, P being the point (x, y) and P' the point $(x+h, y+k)$ and h being either positive or negative (positive in the figure).

The reader is of course familiar with the idea of an 'area,' and in particular with that of an area such as $ONPP_0$. This idea we shall at present take for granted. It is indeed one which needs and has received the most careful mathematical analysis: later on we shall return to it and explain precisely what is meant by ascribing an 'area' to such a region of space as $ONPP_0$. For the present we shall simply assume that this and similar regions have associated with them definite numbers which we call their areas, and that these areas can be compounded in the manner indicated by common sense—e.g. that

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* A more general account of the problem of integration (§§ 111–126) will be found in Goursat's *Cours d'Analyse* or the author's tract quoted on p. 225. The reader may also be referred to the text-books of Profs. Lamb and Gibson, to Prof. Greenhill's *A Chapter in the Integral Calculus*, and a paper by Mr Bromwich in vol. xxxv of the *Messenger of Mathematics*.

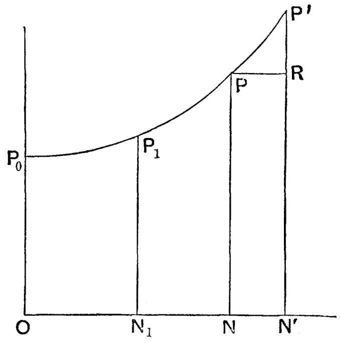

Fig. 53.

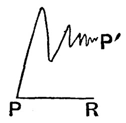

Fig. 53 *a*.

$$(\text{area } PRP') + (\text{area } NN'RP) = (\text{area } NN'P'P)$$

and so on.

Taking all this for granted it is obvious that the area $ONPP_0$ is *a function of* x; we denote it by $\Phi(x)$. Also $\Phi(x)$ is a *continuous* function. For

$$\Phi(x+h) - \Phi(x) = (\text{area } NN'P'P)$$
$$= (\text{area } NN'RP) + (\text{area } PRP') = h\phi(x) + (\text{area } PRP').$$

As the figure is drawn, $(\text{area } PRP') < h \,.\, RP' = hk$. This is not however necessarily true in general, because it is not necessarily the case (see for example Fig. 53 *a*) that the arc PP' should rise (or fall) steadily from P to P'. But the area PRP' is always less than $|h|\lambda_h$, where λ_h is the greatest distance of any point of the arc PP' from PR. Moreover, since $\phi(x)$ is a continuous function, $\lambda_h \to 0$ with h.

Thus we have

$$\Phi(x+h) - \Phi(x) = h\{\phi(x) + \mu_h\},$$

where $|\mu_h| < \lambda_h$ and $\lambda_h \to 0$ as $h \to 0$. From this it follows at once that $\Phi(x)$ is continuous. And not only so but

$$\Phi'(x) = \lim_{h \to 0} \frac{\Phi(x+h) - \Phi(x)}{h} = \lim_{h \to 0} \{\phi(x) + \mu_h\} = \phi(x).$$

Thus *the ordinate of the curve is the derivative of the area, and the area is the integral of the ordinate.*

We are thus able to formulate a rule for determining the area $ONPP_0$. *Calculate $\Phi(x)$, the integral of $\phi(x)$. This involves an arbitrary constant, which we suppose so chosen that $\Phi(0) = 0$. Then the area required is $\Phi(x)$.*

If it were the area N_1NPP_1 which was wanted, we should of course determine the constant so that $\Phi(x_1) = 0$, where x_1 is the abscissa of P_1.

128. Lengths of curves. The notion of the length of a curve (other than a straight line) is in reality a more difficult one even than that of an area. In fact the assumption that OP (Fig. 53) has a definite length (which we may denote by $S(x)$) does not suffice for our purposes, as did the corresponding assumption about areas.

In fact we cannot even prove that $S(x)$ is continuous—i.e. that $\lim\{S(P') - S(P)\} = 0$. This looks obvious enough in the large figure, but less so in such a case as is shown in the smaller figure.

Indeed it is not possible to proceed further (with any degree of rigour) without a careful analysis of precisely what is meant by the length of a curve.

It is however easy to see what the *formula* must be. The assumption that the curve has a length leads to the equation

$$\begin{aligned}\{S(x+h) - S(x)\}/h &= (\text{arc } PP')/h \\ &= \{(\text{chord } PP')/h\} \times \{(\text{arc } PP')/(\text{chord } PP')\}.\end{aligned}$$

Now $$PP' = \sqrt{(PR^2 + RP'^2)} = h\sqrt{\left(1 + \frac{k^2}{h^2}\right)},$$

and, if we assume that the curve has a tangent whose direction varies continuously,

$$k = \phi(x+h) - \phi(x) = h\phi'(\xi)$$

where ξ lies between x and $x+h$. Hence

$$\lim(PP'/h) = \lim \sqrt{\left\{1 + \left[\phi'(\xi)\right]^2\right\}} = \sqrt{\left\{1 + \left[\phi'(x)\right]^2\right\}}.$$

If also we assume that

$$\lim \{(\text{arc } PP')/(\text{chord } PP')\} = 1$$

we obtain the result

$$S'(x) = \lim \{S(x+h) - S(x)\}/h = \sqrt{\{1 + [\phi'(x)]^2\}}$$

and so

$$S(x) = \int \sqrt{\{1 + [\phi'(x)]^2\}}\, dx^*.$$

Examples LVI. 1. Calculate the area of the segment cut off from the parabola $y^2 = 4ax$ by the ordinate $x = \xi$, and the length of the arc which bounds it.

[The area is $2\int_0^\xi \sqrt{(4ax)}\, dx = \frac{8}{3}\xi\sqrt{(a\xi)}$ and the arc is $2\int_0^\xi \sqrt{\left\{1+\frac{a}{x}\right\}}\, dx$. Also

$$\int \sqrt{\left(1+\frac{a}{x}\right)} dx = \int \frac{x+a}{\sqrt{\{x(x+a)\}}} dx = \sqrt{\{x(x+a)\}} + \tfrac{1}{2}a \int \frac{dx}{\sqrt{\{x(x+a)\}}}$$
$$= \sqrt{\{x(x+a)\}} + a \log\{\sqrt{x} + \sqrt{(x+a)}\},$$

so that the length of the arc is

$$2\sqrt{\{\xi(a+\xi)\}} + 2a \log[\sqrt{(\xi/a)} + \sqrt{\{(a+\xi)/a\}}].$$

In particular, the length of the arc from one end of the latus rectum to another is $2a\{\sqrt{2} + \log(1+\sqrt{2})\}$.]

2. Answer the same questions for the curves $y = x^2/4a$, $ay^2 = x^3$, showing that the length of the arc of the latter curve, from the origin to the point for which $x = \xi$, is $(\frac{8}{27}a)[\{1+(9\xi/4a)\}^{3/2} - 1]$.

3. Calculate the areas and lengths of the circles $x^2+y^2 = a^2$, $x^2+y^2 = 2ax$ by means of the formulae of §§ 127–8.

4. Show that the area of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ is πab.

5. Find the area bounded by the curve $y = \sin x$ and the segment of the axis of x from $x = 0$ to $x = 2\pi$.

[Here $\Phi(x) = -\cos x$, and the difference between the values of $-\cos x$ for $x = 0$ and $x = 2\pi$ is *zero*. The explanation of this is of course that between $x = \pi$ and $x = 2\pi$ the curve lies below the axis of x, and so the corresponding part of the area is counted negative in applying the method. The area from $x = 0$ to $x = \pi$ is $-\cos\pi + \cos 0 = 2$; and the whole area required, when every part is counted positive, is twice this, i.e. is 4.]

6. Show that the curves $c^2y^2 = x^2(x-a)(b-x)$, $(a^2+x^2)y^2 = (a^2-x^2)x^2$, $(a-x)y^2 = (a+x)x^2$, and $a^2(y-x)^2 = (a+x)^3(a-x)$ each consist of a single loop, and find the area of each curve.

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[*] For some discussion of the question of the existence of areas and lengths see Ch. VII, § 137 *et seq.*

[Consider the last curve, for example. Here $y=x\pm(1/a)(a+x)\sqrt{(a^2-x^2)}$. The curve lies entirely between the lines $x=-a$ and $x=+a$ (Fig. 54). The area of the curve is $\Phi(a)-\Phi(-a)$, where

$$\Phi(x)=\int(y_1-y_2)\,dx=(2/a)\int(a+x)\sqrt{(a^2-x^2)}\,dx.$$

Putting $x=a\cos\theta$, we find without difficulty that the area of the curve is $a^2\pi$.]

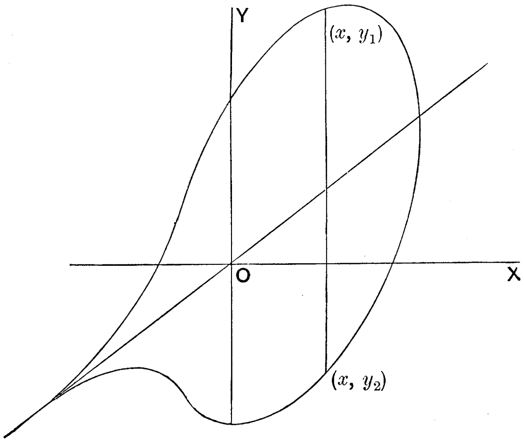

Fig. 54.

7. Suppose that the coordinates of any point on a curve are expressed as functions of a parameter t by equations of the type $x=\phi(t)$, $y=\psi(t)$, ϕ and ψ being one-valued and differentiable functions of t. Prove that if, as t varies from t_0 to t_1, x steadily increases, the area of the region bounded by the corresponding portion of the curve, the axis of x, and two ordinates, is $\int_{t_0}^{t_1}\psi(t)\,\phi'(t)\,dt$ or $\int_{t_0}^{t_1} y\frac{dx}{dt}\,dt$.

8. Suppose that C is a closed curve formed of a single loop and not met by any parallel to either axis in more than two points. And suppose that the coordinates of any point P on the curve can be expressed as in Ex. 7 in terms of t, and that, as t varies from t_0 to t_1, P moves in the same direction round the curve and returns after a single circuit to its original position. Show that the area of the loop is equal to the difference of the initial and final values of any one of the integrals

$$-\int y\frac{dx}{dt}\,dt,\quad \int x\frac{dy}{dt}\,dt,\quad \tfrac{1}{2}\int\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt,$$

this difference being of course taken positively.

9. Apply the result of Ex. 8 to determine the areas of the curves given by (i) $x/a=(1-t^2)/(1+t^2)$, $y/a=2t/(1+t^2)$, (ii) $x=a\cos t$, $y=b\sin t$, (iii) $x=a\cos^3 t$, $y=b\sin^3 t$. Determine also the perimeters of the first and third curves (for the second see Ex. 15).

10. Find the area of the loop of the curve $x^3+y^3=3axy$. [Putting $y=tx$ we obtain $x=3at/(1+t^3)$, $y=3at^2/(1+t^3)$. As t varies from 0 towards $+\infty$ the loop is described once. Also

$$\tfrac{1}{2}\int\left(y\frac{dx}{dt}-x\frac{dy}{dt}\right)dt=-\tfrac{1}{2}\int x^2\frac{d}{dt}\left(\frac{y}{x}\right)dt=-\tfrac{1}{2}\int\frac{9a^2t^2}{(1+t^3)^2}\,dt=\frac{3a^2}{2(1+t^3)},$$

which tends to 0 as $t\rightarrow\infty$. Thus the area of the loop is $\frac{3}{2}a^2$.]

11. Find the area of the loop of $x^5+y^5=5ax^2y^2$.

12. Find the length of one arch of the curve $x=a(t-\sin t)$, $y=a(1-\cos t)$, and the area included between the arch and the axis of x.

13. Find the area common to the two parabolas $y^2=4ax$, $x^2=4ay$.

14. The whole area between the curve $y(a^2+x^2)^4=a^5x^4$ and the positive half of the axis of x (i.e. the limit as $\xi\rightarrow\infty$ of the area between the curve, the axis and the ordinate $x=\xi$) is $\pi a^2/32$.

15. The arc of the ellipse given by $x=a\cos t$, $y=b\sin t$, between the points $t=t_1$ and $t=t_2$, is $F(t_2)-F(t_1)$, where

$$F(t)=a\int\sqrt{(1-e^2\sin^2 t)}\,dt$$

(e being the eccentricity). [This integral cannot however be evaluated in terms of such functions as are at present at our disposal.]

16. **Polar coordinates.** Show that the area bounded by the curve $r=f(\theta)$, where $f(\theta)$ is a one-valued function of θ, and the radii $\theta=\theta_1$, $\theta=\theta_2$, is $F(\theta_2)-F(\theta_1)$, where $F(\theta)=\frac{1}{2}\int r^2\,d\theta$. And the length of the corresponding arc of the curve is $\Phi(\theta_2)-\Phi(\theta_1)$, where

$$\Phi(\theta)=\int\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta.$$

Hence determine (i) the area and perimeter of the circle $r=2a\sin\theta$; (ii) the area between the parabola $r=\frac{1}{2}l\sec^2\frac{1}{2}\theta$ and its latus rectum, and the length of the corresponding arc of the parabola; (iii) the area of the limaçon $r=a+b\cos\theta$, distinguishing the cases in which $a>b$, $a=b$, and $a<b$; and (iv) the areas of the ellipses $1/r^2=a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta$ and $l/r=1+e\cos\theta$. [In the last case we are led to the integral $\int\frac{d\theta}{(1+e\cos\theta)^2}$, which may be calculated (cf. Ex. LV. 4) by the help of the substitution

$$(1+e\cos\theta)(1-e\cos\phi)=1-e^2.]$$

17. Trace the curve $2\theta=(a/r)+(r/a)$, and show that the area bounded by the radius vector $\theta=\beta$ and the two branches which touch at the point $r=a$, $\theta=1$ is $\frac{2}{3}a^2(\beta^2-1)^{3/2}$. (*Math. Trip.* 1900.)

18. A curve is given by an equation $p=f(r)$, r being the radius vector and p the perpendicular from the origin on to the tangent. Show that the calculation of the area of the region bounded by an arc of the curve and two radii vectores depends upon that of the integral $\frac{1}{2}\int\frac{pr\,dr}{\sqrt{(r^2-p^2)}}$.

MISCELLANEOUS EXAMPLES ON CHAPTER VI.

1. Denoting a, $ax+b$, $ax^2+2bx+c$, ... by u_0, u_1, u_2, ..., show that $u_0^2u_3-3u_0u_1u_2+2u_1^3$ and $u_0u_4-4u_1u_3+3u_2^2$ are independent of x.

2. If a_0, a_1, ..., a_{2n} are constants and $U_r=(a_0, a_1, \dots, a_r)(x, 1)^r$, then

$$U_0U_{2n}-2nU_1U_{2n-1}+\frac{2n(2n-1)}{1.2}U_2U_{2n-2}-\dots+U_{2n}U_0$$

is independent of x. (*Math. Trip.* 1896.)

[Differentiate and use the relation $U_r'=rU_{r-1}$.]

3. The first three derivatives of the function $\arcsin(\mu\sin x)-x$, where $\mu>1$, are positive for $0\leqq x\leqq\frac{1}{2}\pi$.

4. The constituents of a determinant are functions of x. Show that its differential coefficient is the sum of the determinants formed by differentiating the constituents of one row only, leaving the rest unaltered.

5. If f_1, f_2, f_3, f_4 are polynomials of degree not greater than 4, then

$$\begin{vmatrix} f_1 & f_2 & f_3 & f_4 \\ f_1' & f_2' & f_3' & f_4' \\ f_1'' & f_2'' & f_3'' & f_4'' \\ f_1''' & f_2''' & f_3''' & f_4''' \end{vmatrix}$$

is also a polynomial of degree not greater than 4. [Differentiate five times, using the result of Ex. 4, and rejecting vanishing determinants.]

6. If $y^3+3yx+2x^3=0$ then $x^2(1+x^3)y''-\frac{3}{2}xy'+y=0$. (*Math. Trip.* 1903.)

7. Verify that the differential equation $y=\phi\{\psi(y_1)\}+\phi\{x-\psi(y_1)\}$, where y_1 is the derivative of y, and ψ is the function inverse to ϕ', is satisfied by $y=\phi(c)+\phi(x-c)$ or by $y=2\phi(\frac{1}{2}x)$.

8. Verify that the differential equation $y=\{x/\psi(y_1)\}\phi\{\psi(y_1)\}$, where the notation is the same as that of Ex. 7, is satisfied by $y=c\Phi(x/c)$ or by $y=\beta x$, where $\beta=\phi(\alpha)/\alpha$ and α is any root of the equation $\phi(\alpha)-\alpha\phi'(\alpha)=0$.

9. If $ax+by+c=0$ it is clear that $y_2=0$ (suffixes denoting differentiations with respect to x). We may express this by saying that *the general differential equation of all straight lines* is $y_2=0$. Find the general differential equations of (i) all circles with their centres on the axis of x, (ii) all parabolas with their axes along the axis of x, (iii) all parabolas with their axes parallel to the axis of y, (iv) all circles, (v) all parabolas, (vi) all conics.

[The equations are (i) $1+y_1^2+yy_2=0$, (ii) $y_1^2+yy_2=0$, (iii) $y_3=0$, (iv) $(1+y_1^2)y_3=3y_1y_2^2$, (v) $5y_3^2=3y_2y_4$, (vi) $9y_2^2y_5-45y_2y_3y_4+40y_3^3=0$. In each case we have only to write down the general equation of the curves in question, and differentiate until we have enough equations to eliminate all the arbitrary constants.]

10. Show that the general differential equations of all parabolas and of all conics are respectively

$$D_x^2(y_2^{-2/3})=0,\quad D_x^3(y_2^{-2/3})=0.$$

[The equation of a conic may be put in the form

$$y=ax+b\pm\sqrt{(px^2+2qx+r)}.$$

From this we deduce

$$y_2=\pm(pr-q^2)/(px^2+2qx+r)^{3/2}.$$

If the conic is a parabola, $p=0$.]

11. Denoting $\dfrac{dy}{dx}$, $\dfrac{1}{2!}\dfrac{d^2y}{dx^2}$, $\dfrac{1}{3!}\dfrac{d^3y}{dx^3}$, $\dfrac{1}{4!}\dfrac{d^4y}{dx^4}$, ... by t, a, b, c, ... and $\dfrac{dx}{dy}$, $\dfrac{1}{2!}\dfrac{d^2x}{dy^2}$, $\dfrac{1}{3!}\dfrac{d^3x}{dy^3}$, $\dfrac{1}{4!}\dfrac{d^4x}{dy^4}$, ... by τ, α, β, γ, ..., show that

$$4ac-5b^2=(4\alpha\gamma-5\beta^2)/\tau^8,\quad bt-a^2=-(\beta\tau-\alpha^2)/\tau^6.$$

Establish similar formulae for the functions $a^2d-3abc-2b^3$, $(1+t^2)b-2a^2t$, $2ct-5ab$.

12. If y_k is the kth derivative of $y=\sin(n\arcsin x)$, prove that

$$(1-x^2)y_{k+2}-(2k+1)xy_{k+1}+(n^2-k^2)y_k=0.$$

[Prove first for $k=0$, and differentiate k times by Leibniz's Theorem.]

13. Prove the formula

$$vD_x^n u=D_x^n(uv)-nD_x^{n-1}(uD_xv)+\frac{n(n-1)}{1.2}D_x^{n-2}(uD_x^2v)-\ldots\ldots$$

where n is any positive integer. [Use the method of induction.]

14. A curve is given by

$$x=a(2\cos t+\cos 2t),\quad y=a(2\sin t-\sin 2t).$$

Prove (i) that the equations of the tangent and normal, at the point P whose parameter is t, are

$$x\sin\tfrac{1}{2}t+y\cos\tfrac{1}{2}t=a\sin\tfrac{3}{2}t,\quad x\cos\tfrac{1}{2}t-y\sin\tfrac{1}{2}t=3a\cos\tfrac{3}{2}t;$$

(ii) that the tangent at P meets the curve in the points Q, R whose parameters are $-\frac{1}{2}t$ and $\pi-\frac{1}{2}t$; (iii) that $QR=4a$; (iv) that the tangents at Q and R are at right angles and intersect on the circle $x^2+y^2=a^2$; (v) that the normals at P, Q, and R are concurrent and intersect on the circle $x^2+y^2=9a^2$; (vi) that the equation of the curve is

$$(x^2+y^2+12ax+9a^2)^2=4a(2x+3a)^3.$$

Sketch the form of the curve.

15. Show that the equations which define the curve of Ex. 14 may be replaced by $\xi/a=2u+(1/u^2)$, $\eta/a=(2/u)+u^2$, where $\xi=x+yi$, $\eta=x-yi$, $u=\mathrm{Cis}\,t$. Show that the tangent and normal, at the point defined by u, are

$$u^2\xi-u\eta=a(u^3-1),\quad u^2\xi+u\eta=3a(u^3+1),$$

and deduce the properties (ii)-(v) of Ex. 14.

16. Show that the condition that $x^4+4px^3-4qx-1=0$ should have equal roots may be expressed in the form $(p+q)^{2/3}-(p-q)^{2/3}=1$.

(*Math. Trip.* 1898.)

17. Show that the equation $\sin x = ax$ will have a repeated root if $a = \cos\xi$, where ξ is any root of the equation $\tan x = x$. Determine roughly the values of a which satisfy this condition.

18. Investigate the maxima and minima of $f(x)$, and the real roots of $f(x) = 0$, $f(x)$ being either of the functions

$$x - \sin x - \tan\alpha\,(1 - \cos x), \quad x - \sin x - (\alpha - \sin\alpha) - \tan\tfrac{1}{2}\alpha\,(\cos\alpha - \cos x),$$

and α an angle between 0 and π. Show that in the first case the condition for a double root is that $\tan\alpha - \alpha$ should be a multiple of π.

19. Show that by choice of the ratio $\lambda : \mu$ we can make the roots of $\lambda(ax^2 + bx + c) + \mu(a'x^2 + b'x + c') = 0$ real and having a difference of any magnitude, unless the roots of the two quadratics are all real and interlace; and that in the excepted case the roots are always real, but there is a lower limit for the magnitude of their difference. (*Math. Trip.* 1895.)

[Consider the form of the graph of the function $(ax^2 + bx + c)/(a'x^2 + b'x + c')$: cf. Exs. XLVIII. 19–22.]

20. A sheet of paper is folded over so that one corner just reaches the opposite side. Show how to fold the paper in this way so that the length of the crease shall be a maximum.

21. The greatest acute angle at which the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ can be cut by a concentric circle is $\operatorname{arc}\tan\{(a^2 - b^2)/2ab\}$. (*Math. Trip.* 1900.)

22. In a triangle the area Δ and the semi-perimeter s are fixed. Show that any maximum or minimum of one of the sides is a root of the equation $s(x - s)x^2 + 4\Delta^2 = 0$. Discuss the reality of the roots of this equation, and whether they correspond to maxima or minima.

[The equations $a + b + c = 2s$, $s(s - a)(s - b)(s - c) = \Delta^2$ determine a and b as functions of c. Differentiate with respect to c, and suppose $da/dc = 0$. It will be found that $b = c$, $s - b = s - c = \frac{1}{2}a$, from which we deduce that $s(a - s)a^2 + 4\Delta^2 = 0$.

This equation has three real roots if $s^4 > 27\Delta^2$, and one in the contrary case. In an equilateral triangle (the triangle of minimum perimeter for a given area) $s^4 = 27\Delta^2$; thus the case of $s^4 < 27\Delta^2$ cannot occur. Hence the equation in a has three real roots, and since their sum is positive and their product negative, two roots are positive and the third negative. Of the two positive roots one corresponds to a maximum and one to a minimum.]

23. The area of the greatest equilateral triangle which can be drawn with its sides passing through three given points A, B, C is

$$2\Delta + (1/2\surd 3)(a^2 + b^2 + c^2),$$

a, b, c being the sides and Δ the area of ABC. (*Math. Trip.* 1899.)

24. If Δ, Δ' are the areas of the two maximum isosceles triangles which can be described with their vertices at the origin and their base angles on the cardioid $r = a(1 + \cos\theta)$, prove that $256\Delta\Delta' = 25a^4\surd 5$. (*Math. Trip.* 1907.)

[One of Δ, Δ' will be found to be negative, and algebraically a *minimum*.]

25. Find the limiting values which $(x^2-4y+8)/(y^2-6x+3)$ approaches as the point (x, y) on the curve $x^2y-4x^2-4xy+y^2+16x-2y-7=0$ approaches the position $(2, 3)$. (*Math. Trip.* 1903.)

[If we take $(2, 3)$ as a new origin, the equation of the curve becomes $\xi^2\eta-\xi^2+\eta^2=0$, and the function given $(\xi^2+4\xi-4\eta)/(\eta^2+6\eta-6\xi)$. If we put $\eta=t\xi$ we obtain $\xi=(1-t^2)/t$, $\eta=1-t^2$. The curve has a loop branching at the origin, which corresponds to the two values $t=\pm 1$. Expressing the given function in terms of t and making t tend to ± 1 we obtain the limiting values $-\frac{3}{2}$, $-\frac{2}{3}$.]

26. If
$$f(x)=\{1/(\sin x-\sin a)\}-\{1/(x-a)\}\sec a,$$
then
$$\frac{d}{da}\{\lim_{x\to a} f(a)\}-\lim_{x\to a} f'(x)=\tfrac{3}{4}\sec^3 a-\tfrac{5}{12}\sec a.$$
(*Math. Trip.* 1896.)

27. Show that if $\phi(x)=1/(1+x^2)$, $\phi^{(n)}(x)=Q_n(x)/(1+x^2)^{n+1}$, where $Q_n(x)$ is a polynomial of degree n. Show also that

(i) $Q_{n+1}=(1+x^2)\,Q_n'-2(n+1)\,x\,Q_n$,

(ii) $Q_{n+2}+2(n+2)\,x\,Q_{n+1}+(n+2)(n+1)(1+x^2)\,Q_n=0$,

(iii) $(1+x^2)\,Q_n''-2nx\,Q_n'+n(n+1)\,Q_n=0$,

(iv) $Q_n=(-1)^n n!\left\{(n+1)\,x^n-\dfrac{(n+1)n(n-1)}{3!}x^{n-2}+\dots\right\}$,

(v) all the roots of $Q_n=0$ are real and separated by those of Q_{n-1}.

[To prove (ii) differentiate the equation $(1+x^2)\phi(x)=1$ $n+2$ times by Leibniz's theorem: (iii) follows from (i) and (ii); and (iv) can be deduced from (iii) or by writing $\phi(x)$ in the form $(1/2i)[\{1/(x-i)\}-\{1/(x+i)\}]$ before differentiating. Finally, to prove (v), we observe that, when $Q_n=0$, Q_{n+1} has the sign of Q_n', by (i), and that the sign of Q_n when x is numerically large is that of $(-1)^n/x^{n+2}$.]

28. If $f(x)$, $\phi(x)$, $\psi(x)$ have derivatives for $a\leqq x\leqq b$, there is a value of ξ, lying between a and b, and such that
$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(\xi) & \phi'(\xi) & \psi'(\xi) \end{vmatrix}=0.$$

[Consider the function formed by replacing the constituents of the third row by $f(x)$, $\phi(x)$, $\psi(x)$.]

29. Deduce from Ex. 28 the formula
$$\{f(b)-f(a)\}/\{\phi(b)-\phi(a)\}=f'(\xi)/\phi'(\xi).$$

30. If $\phi'(x)\to a$ as $x\to\infty$, then $\phi(x)/x\to a$. If $\phi'(x)\to\infty$ (or $-\infty$), then $\phi(x)\to\infty$ (or $-\infty$). [Use the formula $\phi(x)-\phi(x_0)=(x-x_0)\,\phi'(\xi)$, where $x_0<\xi<x$.]

31. If $\phi(x)\to a$ as $x\to\infty$, $\phi'(x)$ cannot tend to any limit other than zero.

32. If $\phi(x)+\phi'(x)\rightarrow a$ as $x\rightarrow+\infty$, then $\phi(x)\rightarrow a$ and $\phi'(x)\rightarrow 0$.

[Let $\phi(x)=a+\psi(x)$, so that $\psi(x)+\psi'(x)\rightarrow 0$. If $\psi'(x)$ is of constant sign, say positive, for all sufficiently large values of x, then $\psi(x)$ steadily increases; thus $\psi(x)$ tends to a limit l or to $+\infty$, and so $\psi'(x)$ to $-l$ or $-\infty$, which is impossible (Exs. 30, 31) unless $l=0$. But if $\psi'(x)$ changes sign for infinitely many values of x, these are the maxima and minima of $\psi(x)$; at these $\psi(x)+\psi'(x)$ is very small (when x is large), and $\psi'(x)=0$, so that $\psi(x)$ is very small. *A fortiori* are the other values of $\psi(x)$ very small when x is large.]

33. Prove that

$$\int\frac{dx}{(x-p)\sqrt{\{(x-p)(x-q)\}}}=\frac{2}{q-p}\sqrt{\left(\frac{x-q}{x-p}\right)}.$$

34. Show that $\displaystyle\int\frac{dx}{(x-x_0)y}$, where $y^2=ax^2+2bx+c$, may be expressed in one or other of the forms

$$-\frac{1}{y_0}\log\left\{\pm\frac{axx_0+b(x+x_0)+c+yy_0}{x-x_0}\right\},\quad \frac{1}{z_0}\arctan\left\{\frac{axx_0+b(x+x_0)+c}{yz_0}\right\},$$

according as $ax_0^2+2bx_0+c$ is positive and equal to y_0^2 or negative and equal to $-z_0^2$. In the first case the ambiguous sign is to be chosen so as to make the contents of the bracket positive. [Put $x-x_0=1/t$, when we obtain

$$\int\frac{dx}{(x-x_0)y}=\int\frac{dt}{\sqrt{\{a+2(ax_0+b)t+(ax_0^2+2bx_0+c)t^2\}}}.]$$

35. If $ag^2+ch^2=-\nu<0$, show that

$$\int\frac{dx}{(hx+g)\sqrt{(ax^2+c)}}=-\frac{1}{\sqrt{\nu}}\arctan\left[\frac{\sqrt{\{\nu(ax^2+c)\}}}{ch-agx}\right].$$

36. Show that the integral $\displaystyle\int\frac{\phi(x^2)\,dx}{\sqrt{(ax^2+c)}}$ is reduced to that of a rational function by the substitution $x/\sqrt{(ax^2+c)}=u$.

In particular show that

$$\int\frac{dx}{(Ax^2+C)\sqrt{(ax^2+c)}}=\int\frac{du}{C+(cA-aC)u^2},$$

and hence evaluate the integral.

[The integral $\displaystyle\int\frac{x\,dx}{(Ax^2+C)\sqrt{(ax^2+c)}}$ is reduced to this form by putting $x=1/y$ (or it may be calculated by putting $x^2=t$). Hence we can calculate any integral of the type $\displaystyle\int\frac{Lx+M}{Ax^2+C}\frac{dx}{\sqrt{(ax^2+c)}}$ in terms of $x/\sqrt{(ax^2+c)}$ and $1/\sqrt{(ax^2+c)}$.]

37. Show how to integrate $\displaystyle\int\frac{Lx+M}{Ax^2+2Bx+C}\,\frac{dx}{\sqrt{(ax^2+2bx+c)}}$, when $Ax^2+2Bx+C=0$ has complex roots.

[Put $x=(\mu t+\nu)/(t+1)$, where μ, ν are so chosen that

$$a\mu\nu+b(\mu+\nu)+c=0,\quad A\mu\nu+B(\mu+\nu)+C=0,$$

i.e. are the roots of $(aB-bA)\xi^2-(cA-aC)\xi+(bC-cB)=0$. It is easy to verify that μ and ν are real and distinct: they are in fact the maxima and minima of the function $(ax^2+2bx+c)/(Ax^2+2Bx+C)$ considered in Exs. XLVIII. 19 *et seq.* It will be found that the integral now reduces to one of the form $\int \frac{at+\beta}{\rho t^2+\sigma}\frac{dt}{\sqrt{(rt^2+s)}}$, and so may be calculated in terms of $(x-\mu)/y$ and $(x-\nu)/y$, where $y=\sqrt{(ax^2+2bx+c)}$*.]

38. Show how to reduce $\int R\left\{x, \sqrt{\left(\frac{ax+b}{mx+n}\right)}, \sqrt{\left(\frac{cx+d}{mx+n}\right)}\right\} dx$ to the integral of a rational function. [Put $mx+n=1/t$ and use Ex. LII. 15.]

39. Calculate the integrals:

$$\int\frac{dx}{(1+x^2)^3},\quad \int\sqrt{\left(\frac{x-1}{x+1}\right)}\frac{dx}{x},\quad \int\frac{dx}{x\sqrt{(x^2+2x+3)}},\quad \int\frac{(x+1)\,dx}{(x^2+4)\sqrt{(x^2+9)}},$$

$$\int\frac{dx}{(5x^2+12x+8)\sqrt{(5x^2+2x-7)}},\qquad \int\frac{x\,dx}{\sqrt{(1+x)}-\sqrt[3]{(1+x)}},$$

$$\int\sqrt{\left\{a^2+\sqrt{\left(b^2+\frac{c}{x}\right)}\right\}}\,dx,\qquad \int\operatorname{cosec}^3 x\,dx,\qquad \int\frac{5\cos x+6}{2\cos x+\sin x+3}\,dx,$$

$$\int\frac{dx}{(2-\sin^2 x)(2+\sin x-\sin^2 x)},\qquad \int\frac{\cos x\sin x\,dx}{\cos^4 x+\sin^4 x},\qquad \int\operatorname{cosec} x\sqrt{(\sec 2x)}\,dx,$$

$$\int\frac{dx}{\sqrt{\{(1+\sin x)(2+\sin x)\}}},\quad \int\frac{x+\sin x}{1+\cos x}\,dx,\quad \int\operatorname{arc}\sec x\,dx,\quad \int(\operatorname{arc}\sin x)^2\,dx,$$

$$\int x\operatorname{arc}\sin x\,dx,\quad \int\frac{x\operatorname{arc}\sin x}{\sqrt{(1-x^2)}}\,dx,\quad \int\frac{\operatorname{arc}\sin x}{x^2}\,dx,\quad \int\frac{\operatorname{arc}\sin x}{(1+x)^2}\,dx,$$

$$\int\frac{\operatorname{arc}\tan x}{x^2}\,dx,\quad \int\frac{\operatorname{arc}\tan x}{(1+x^2)^{3/2}}\,dx,\quad \int\frac{\log(a^2+\beta^2x^2)}{x^2}\,dx,\quad \int\frac{\log(a+\beta x)}{(a+bx)^2}\,dx.$$

40. **Formulae of Reduction.** (i) Show that

$$\int\frac{dx}{(x^2+px+q)^n}=\frac{1}{2(n-1)(q-\frac{1}{4}p^2)}\frac{x+\frac{1}{2}p}{(x^2+px+q)^{n-1}}$$
$$+\frac{2n-3}{2(n-1)(q-\frac{1}{4}p^2)}\int\frac{dx}{(x^2+px+q)^{n-1}}.$$

[Put $x+\frac{1}{2}p=t$, $q-\frac{1}{4}p^2=\lambda$: then we obtain

$$\int\frac{dt}{(t^2+\lambda)^n}=\frac{1}{\lambda}\int\frac{dt}{(t^2+\lambda)^{n-1}}-\frac{1}{\lambda}\int\frac{t^2\,dt}{(t^2+\lambda)^n}$$
$$=\frac{1}{\lambda}\int\frac{dt}{(t^2+\lambda)^{n-1}}+\frac{1}{2\lambda(n-1)}\int t\,\frac{d}{dt}\left\{\frac{1}{(t^2+\lambda)^{n-1}}\right\}dt,$$

and the result follows on integrating by parts.

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* The method fails if $a/A=b/B$; but then the integral may be reduced by the substitution $ax+b=t$. See Stolz, *Grundzüge der Diff. und Int. Calc.*, bd. I, pp. 331 *et seq.*; Goodwin, *Messenger of Mathematics*, vol. XXXVII, pp. 104–6. Another method of reduction has been given by Prof. Greenhill; see his *A Chapter in the Integral Calculus*, pp. 12 *et seq.*, and pp. 24 *et seq.* of the author's tract quoted on p. 225. Reference may also be made to the paper by Mr Bromwich quoted on p. 236.

A formula such as this is called a *formula of reduction*; it enables us to calculate $\int\frac{dx}{(x^2+px+q)^n}$ in terms of $\int\frac{dx}{(x^2+px+q)^{n+1}}$, and so to evaluate the integral whenever n is a positive integer.]

(ii) If $I_{p,q}=\int x^p(1+x)^q\,dx$, show that

$$I_{p,q}=\{1/(p+1)\}\,x^{p+1}(1+x)^q-\{q/(p+1)\}\,I_{p+1,q-1},$$

and hence that $I_{p,q}$ can be calculated if q is a positive or p a negative integer. Obtain a similar formula connecting $I_{p,q}$ with $I_{p-1,q+1}$, and show that $I_{p,q}$ can be calculated whenever p or q is an integer positive or negative. Finally, by means of the substitution $x=-y/(1+y)$, show that

$$I_{p,q}=(-1)^{p+1}\int y^p(1+y)^{-p-q-2}\,dy,$$

and that $I_{p,q}$ can also be calculated whenever $p+q$ is an integer.

(iii) Show that if $X=a+bx$,

$$\int xX^{-1/3}\,dx=-3(3a-2bx)X^{2/3}/10b^2,$$

$$\int x^2X^{-1/3}\,dx=3(9a^2-6abx+5b^2x^2)X^{2/3}/40b^3.$$

$$\int xX^{-1/4}\,dx=-4(4a-3bx)X^{3/4}/21b^2,$$

$$\int x^2X^{-1/4}\,dx=4(32a^2-24abx+21b^2x^2)X^{3/4}/231b^3.$$

(iv) If $I_{m,n}=\int\frac{x^m\,dx}{(1+x^2)^n}$ then

$$2(n-1)\,I_{m,n}=-x^{m-1}(1+x^2)^{-(n-1)}+(m-1)\,I_{m-2,n-1}.$$

(v) If $I_n=\int x^n\cos\beta x\,dx$, $J_n=\int x^n\sin\beta x\,dx$, then

$$\beta I_n=x^n\sin\beta x-n\,J_{n-1},\quad \beta J_n=-x^n\cos\beta x+n\,I_{n-1}.$$

(vi) If $I_n=\int\cos^n x\,dx$ and $J_n=\int\sin^n x\,dx$, then

$$nI_n=\sin x\cos^{n-1}x+(n-1)\,I_{n-2},\quad nJ_n=-\cos x\sin^{n-1}x+(n-1)\,J_{n-2}.$$

(vii) If $I_n=\int\tan^n x\,dx$, then $I_n+I_{n-2}=(\tan^{n-1}\theta)/(n-1)$.

(viii) If $I_{m,n}=\int\cos^m x\sin^n x\,dx$, then

$$\begin{aligned}(m+n)\,I_{m,n}&=-\cos^{m+1}x\sin^{n-1}x+(n-1)\,I_{m,n-2}\\&=\cos^{m-1}x\sin^{n+1}x+(m-1)\,I_{m-2,n}.\end{aligned}$$

[For
$$\begin{aligned}(m+1)\,I_{m,n}&=-\int\sin^{n-1}x\,\frac{d}{dx}(\cos^{m+1}x)\,dx\\&=-\cos^{m+1}x\sin^{n-1}x+(n-1)\int\cos^{m+2}x\sin^{n-2}x\,dx\\&=-\cos^{m+1}x\sin^{n-1}x+(n-1)\,(I_{m,n-2}-I_{m,n}),\end{aligned}$$

which leads to the first reduction formula: similarly for the second.]

(ix) Connect $I_{m,n}=\int \sin^m x \sin nx\, dx$ with $I_{m-2,n}$. (*Math. Trip.* 1897.)

(x) If $I_{m,n}=\int x^m (\operatorname{cosec} x)^n\, dx$, then

$$(n-1)(n-2)\, I_{m,n}=(n-2)^2\, I_{m,n-2}+m(m-1)\, I_{m-2,n-2} - x^{m-1}\{m \sin x+(n-2)\, x \cos x\}(\operatorname{cosec} x)^{n-1}.$$

(*Math. Trip.* 1896.)

(xi) If $I_n=\int (a+b\cos x)^{-n}\, dx$ then

$$(n-1)(a^2-b^2)\, I_n=-b\sin x\,(a+b\cos x)^{-(n-1)}+(2n-3)\, aI_{n-1}-(n-2)\, I_{n-2}.$$

(xii) If $I_n=\int (a\cos^2 x+2h\cos x\sin x+b\sin^2 x)^{-n}\, dx$ then

$$2(n+1)(ab-h^2)\, I_{n+2}-(2n+1)(a+b)\, I_{n+1}+2n\, I_n=-\frac{1}{2n}\frac{d^2 I_n}{dx^2}.$$

(*Math. Trip.* 1898.)

(xiii) If $I_{m,n}=\int x^m(\log x)^n\, dx$, $\quad (m+1)\, I_{m,n}=x^{m+1}(\log x)^n-n\, I_{m,n-1}$.

41. If n is a positive integer the value of $\int x^m(\log x)^n\, dx$ is

$$x^{m+1}\left\{\frac{(\log x)^n}{m+1}-\frac{n(\log x)^{n-1}}{(m+1)^2}+\frac{n(n-1)(\log x)^{n-2}}{(m+1)^3}-\dots+\frac{(-1)^n n!}{(m+1)^{n+1}}\right\}.$$

42. Show that if u and v are functions of x such that $du/dx=-v$, $dv/dx=u$, then u^2+v^2 is a constant; and that if $u=1$ and $v=0$ when $x=0$, then $u=\cos x$ and $v=\sin x$.

43. Show that the most general function $\phi(x)$, such that $\phi''+a^2\phi=0$ for all values of x, may be expressed in either of the forms $A\cos ax+B\sin ax$, $\rho\cos(ax+\epsilon)$, where A, B, ρ, ϵ are constants. [Multiplying by $2\phi'$ and integrating we obtain $\phi'^2+a^2\phi^2=\text{const.}=a^2b^2$, say, from which we deduce $ax=\int \frac{d\phi}{\sqrt{(b^2-\phi^2)}}$.]

44. Determine the most general functions y and z such that $y'+\omega z=0$, $z'-\omega y=0$, where ω is a constant and dashes denote differentiation with respect to x.

45. Trace the curve for which $x=2a(1-t^2)/(1+t^2)^2$, $y=4at/(1+t^2)^2$. Show that the point $(ty, -tx)$ lies on the normals at the two points t and $-1/t$; and deduce that the locus of their intersection is $x^2+y^2=ax$. Also find the area of the curve.

46. The area of the curve given by

$$x=\cos\phi+\{\sin\alpha\sin\phi/(1-\cos^2\alpha\sin^2\phi)\},\ y=\sin\phi-\{\sin\alpha\cos\phi/(1-\cos^2\alpha\sin^2\phi)\},$$

where α is a positive acute angle, is $\frac{1}{2}\pi(1+\sin\alpha)^2/\sin\alpha$. (*Math. Trip.* 1904.)

47. The projection of a chord of a circle of radius a on a diameter is of constant length $2a\cos\beta$; show that the locus of the middle point of the chord consists of two loops, and that the area of either is $a^2(\beta-\cos\beta\sin\beta)$.

(*Math. Trip.* 1903.)

48. If $f(x, y) \equiv (a, b, c, f, g, h \between x, y, 1)^2 = 0$ is the equation of a conic, prove that

$$\int \frac{dx}{(lx+my+n)(hx+by+f)} = \alpha \log \frac{PT}{PT'} + \beta,$$

where PT, PT' are the perpendiculars from a point P of the conic on the tangents at the ends of the chord $lx+my+n=0$, and α, β are constants.

(*Math. Trip.* 1902.)

[We may, in the formulae of § 117, suppose that (ξ, η) is one of the points in which $lx+my+n=0$ cuts the conic (say that corresponding to T). Then

$$lx+my+n = l(x-\xi)+m(y-\eta) = -\frac{2(l+mt)(G+Ft)}{a+2ht+bt^2},$$

$$\frac{1}{hx+by+f}\frac{dx}{dt} = -\frac{2}{a+2ht+bt^2},$$

and so

$$\int \frac{dx}{(lx+my+n)(hx+by+f)} = \int \frac{dt}{(l+mt)(G+Ft)} = \frac{1}{mG-lF} \log \left(\frac{l+mt}{G+Ft}\right).$$

But the equations of the chord and the tangent at (ξ, η) are respectively

$$lx+my+n=0, \quad G(x-\xi)+F(y-\eta)=0;$$

and, if PN is the perpendicular from P on to the chord,

$$PN = \frac{lx+my+n}{\sqrt{(l^2+m^2)}} = \frac{l(x-\xi)+m(y-\eta)}{\sqrt{(l^2+m^2)}}.$$

Similarly

$$PT = \frac{G(x-\xi)+F(y-\eta)}{\sqrt{(F^2+G^2)}}.$$

Also $t=(y-\eta)/(x-\xi)$, and $PT \,.\, PT' = k \,.\, PN^2$, where k is a constant. From these relations the result follows.

It is not difficult to prove that

$$(mG-lF)^2 = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix}.]$$

CHAPTER VII.

ADDITIONAL THEOREMS IN THE DIFFERENTIAL AND INTEGRAL CALCULUS.

129. Higher Mean Value Theorems. In the preceding chapter we proved (§ 106) that if $f(x)$ has a derivative $f'(x)$ throughout the interval (a, b), then

$$f(b)-f(a)=(b-a)f'\{a+\theta(b-a)\}$$

where θ lies between 0 and 1; or, if $b=a+h$,

$$f(a+h)-f(a)=hf'(a+\theta h) \quad \text{..............(1).}$$

This we proved by considering the function

$$f(b)-f(x)-\frac{b-x}{b-a}\{f(b)-f(a)\}$$

which vanishes for $x=a$ and $x=b$.

Let us now suppose that $f(x)$ has also a second derivative $f''(x)$ throughout (a, b) (an assumption which of course involves the continuity of the first derivative $f'(x)$), and consider the function

$$f(b)-f(x)-(b-x)f'(x)-\left(\frac{b-x}{b-a}\right)^2\{f(b)-f(a)-(b-a)f'(a)\}.$$

This function also vanishes for $x=a$ and $x=b$; and its derivative is

$$\frac{2(b-x)}{(b-a)^2}\{f(b)-f(a)-(b-a)f'(a)-\tfrac{1}{2}(b-a)^2f''(x)\},$$

and this must vanish (§ 102) for some value of x between a and b (exclusive of a and b). Hence there is a value ξ of x, between a and b, and therefore capable of representation in the form $a+\theta(b-a)$, where $0<\theta<1$, which is such that

$$f(b)=f(a)+(b-a)f'(a)+\tfrac{1}{2}(b-a)^2f''(\xi).$$

If we put $b = a + h$ we obtain the equation

$$f(a+h) = f(a) + hf'(a) + \tfrac{1}{2}h^2 f''(a+\theta h) \quad \ldots\ldots\ldots(2),$$

which is the standard form of what may be called **the Mean Value Theorem of the second order.**

The analogy suggested by (1) and (2) at once leads us to formulate the following theorem:

Taylor's or the General Mean Value Theorem. *If $f(x)$ is a function of x which has derivatives of the first n orders throughout the whole interval (a, b), then*

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2\,!}f''(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)\,!}f^{(n-1)}(a) + \frac{(b-a)^n}{n\,!}f^{(n)}(\xi)$$

where $a < \xi < b$: or, if $b = a + h$,

$$f(a+h) = f(a) + hf'(a) + \tfrac{1}{2}h^2 f''(a) + \ldots + \frac{h^{n-1}}{(n-1)\,!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h)$$

where $0 < \theta < 1$.

The proof proceeds on precisely the same lines as were adopted before in the special cases in which $n = 1, 2$. We consider the function

$$F_n(x) - \left(\frac{b-x}{b-a}\right)^n F_n(a)$$

where $$F_n(x) = f(b) - f(x) - (b-x)f'(x) - \frac{1}{2!}(b-x)^2 f''(x) - \ldots - \frac{1}{(n-1)!}(b-x)^{n-1}f^{(n-1)}(x).$$

This function vanishes for $x = a$ and $x = b$; its derivative is

$$\frac{n(b-x)^{n-1}}{(b-a)^n}\left\{F_n(a) - \frac{1}{n!}(b-a)^n f^{(n)}(x)\right\},$$

and there must be some value of x between a and b for which the derivative vanishes. This leads at once to the desired result.

In view of the great importance of this theorem we shall give at the end of this chapter another proof, not essentially distinct from that given above, but different in form and depending on the method of integration by parts.

Examples LVII. 1. Suppose that $f(x)$ is a polynomial of degree r. Then $f^{(n)}(x)$ is identically zero for $n > r$, and the theorem leads to the algebraical identity

$$f(a+h)=f(a)+hf'(a)+\frac{1}{2!}h^2f''(a)+\dots+\frac{1}{r!}h^rf^{(r)}(a).$$

Verify this when $f(x)=x,\ x^2,\ x^m,\ x^m+px^2+qx+r$.

2. By applying the theorem to $f(x)=1/x$, supposing x and $x+h$ positive, obtain the result

$$\frac{1}{x+h}=\frac{1}{x}-\frac{h}{x^2}+\frac{h^2}{x^3}\dots+\frac{(-1)^{n-1}h^{n-1}}{x^n}+\frac{(-1)^nh^n}{(x+\theta h)^{n+1}}.$$

[Since $$\frac{1}{x+h}\equiv\frac{1}{x}-\frac{h}{x^2}+\frac{h^2}{x^3}\dots+\frac{(-1)^{n-1}h^{n-1}}{x^n}+\frac{(-1)^nh^n}{x^n(x+h)},$$
we can verify the result by showing that $x^n(x+h)$ can be put in the form $(x+\theta h)^{n+1}$, or that $x^{n+1}<x^n(x+h)<(x+h)^{n+1}$, as is evidently the case.]

3. Obtain the formula

$$\sin(x+h)=\sin x+h\cos x-\frac{h^2}{2!}\sin x-\frac{h^3}{3!}\sin x+\dots$$
$$+(-1)^{n-1}\frac{h^{2n-1}}{(2n-1)!}\cos x+(-1)^n\frac{h^{2n}}{2n!}\sin(x+\theta h),$$

the corresponding formula for $\cos(x+h)$, and similar formulae involving powers of h extending up to h^{2n+1}.

4. Show that if m is a positive integer, and n a positive integer not greater than m,

$$(x+h)^m=x^m+\binom{m}{1}x^{m-1}h+\dots+\binom{m}{n-1}x^{m-n+1}h^{n-1}+\binom{m}{n}(x+\theta h)^{m-n}h^n.$$

Show also that if the interval $(x,\ x+h)$ does not include $x=0$, the formula holds for all real values of m and all positive integral values of n; and that, even if $x<0<x+h$ or $x+h<0<x$, the formula still holds if $m-n$ is positive.

5. The formula $f(x+h)=f(x)+hf'(x+\theta h)$ is not true if $f(x)=1/x$ and $x<0<x+h$. [For $f(x+h)-f(x)>0$ and $hf'(x+\theta h)=-h/(x+\theta h)^2<0$: it is evident that the conditions for the truth of the Mean Value Theorem are not satisfied.]

6. If $x=-a$, $h=2a$, $f(x)=x^{1/3}$ the equation $f(x+h)=f(x)+hf'(x+\theta h)$ is satisfied by $\theta=\frac{1}{2}\pm\frac{1}{18}\sqrt{3}$. [This example shows that the result of the theorem may hold even if the conditions under which it was proved are not satisfied.]

7. **Approximation to the roots of equations.** Let ξ be an approximation to a root of an algebraical equation $f(x)=0$, the actual root being $\xi+h$. Then

$$0=f(\xi+h)=f(\xi)+hf'(\xi)+\tfrac{1}{2}h^2f''(\xi+\theta h),$$

so that $$h=-\frac{f(\xi)}{f'(\xi)}-\tfrac{1}{2}h^2\frac{f''(\xi+\theta h)}{f'(\xi)}.$$

It follows that in general a better approximation than $x=\xi$ is

$$x=\xi-\{f(\xi)/f'(\xi)\}.$$

If the root is a simple root, so that $f'(\xi+h)\neq 0$, we can, when h is small enough, find a positive constant K such that $|f'(x)| > K$ for all the values of x which we are considering, and then, if h is regarded as of the first order of smallness, $f(\xi)$ is of the first order of smallness, and the error in taking $\xi - \{f(\xi)/f'(\xi)\}$ as the root is of the second order*.

8. Apply this process to the equation $x^2=2$, taking $\xi=3/2$ as the first approximation. [We find $h=-1/12$, $\xi+h=17/12=1{\cdot}417\ldots$, which is quite a good approximation, in spite of the roughness of the first. If now we repeat the process, taking $\xi=17/12$, we obtain $\xi+h=577/408=1{\cdot}414215\ldots$, which is correct to 5 places of decimals. The reader should also test the result of taking $\xi=7/5$.]

9. By considering in this way the equation $x^2-1-y=0$, where y is small, show that $\surd(1+y)=1+\frac{1}{2}y-\{\frac{1}{4}y^2/(2+y)\}$ approximately, the error being of the fourth order.

10. Show that the error in taking the root to be $\xi-(f/f')-\frac{1}{2}(f^2f''/f'^3)$, where ξ is the argument of every function, is in general of the fourth order.

11. The equation $\sin x=ax$, where a is small, has a root nearly equal to π. Show that $(1-a)\pi$ is a better approximation, and $(1-a+a^2)\pi$ a better still. [The method of Exs. 7—10 does not depend on $f(x)=0$ being an algebraical equation, so long as f' and f'' are continuous.]

12. Show that as $h\rightarrow 0$ the limit of the θ which occurs in the general Mean Value Theorem is $1/(n+1)$, provided $f^{(n+1)}(x)$ is continuous.

[For $f(x+h)$ is equal to each of

$$f(x)+\ldots+\frac{h^n}{n!}f^{(n)}(x+\theta h),\quad f(x)+\ldots+\frac{h^n}{n!}f^{(n)}(x)+\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(x+\theta_1 h),$$

where θ_1 as well as θ lies between 0 and 1. Hence

$$f^{(n)}(x+\theta h)=f^{(n)}(x)+\{hf^{(n+1)}(x+\theta_1 h)\}/(n+1).$$

But if we apply the original Mean Value Theorem to the function $f^{(n)}(x)$, taking θh in place of h, we find

$$f^{(n)}(x+\theta h)=f^{(n)}(x)+\theta hf^{(n+1)}(x+\theta\theta_2 h)$$

where θ_2 also lies between 0 and 1. Hence

$$\theta f^{(n+1)}(x+\theta\theta_2 h)=\{f^{(n+1)}(x+\theta_1 h)\}/(n+1),$$

from which the result follows, since $f^{(n+1)}(x+\theta\theta_2 h)$ and $f^{(n+1)}(x+\theta_1 h)$ tend to the same limit $f^{(n+1)}(x)$ as $h\rightarrow 0$.]

13. Prove that $\{f(x+2h)-2f(x+h)+f(x)\}/h^2\rightarrow f''(x)$ as $h\rightarrow 0$, provided $f''(x)$ is continuous. [Use equation (2), p. 252.]

14. Show that, if the first n derivatives of $f(x)$ are continuous for $x=0$, and $f^{(n)}(0)\neq 0$, then

$$f(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n(1+\epsilon_x),$$

where $a_r=f^{(r)}(0)/r!$ and $\epsilon_x\rightarrow 0$ with x.

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* This method of approximation is due to Newton. For further details see Gibson's *Calculus*, pp. 244 *et seq.*, and Tannery's *Leçons d'Algèbre et d'Analyse*, pp. 302 *et seq.*

15. Show that if

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n(1+\epsilon_x)=b_0+b_1x+b_2x^2+\ldots+b_nx^n(1+\eta_x),$$

where ϵ_x and η_x tend to zero with x, then $a_0=b_0$, $a_1=b_1$, ..., $a_n=b_n$. [Making $x\to 0$ we see that $a_0=b_0$. Now divide by x and afterwards make $x\to 0$. We thus obtain $a_1=b_1$; and this process may be repeated as often as is necessary. It follows that if $f(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n(1+\epsilon_x)$ then $a_r=f^{(r)}(0)/r\,!$.]

130. Taylor's Series. Suppose that $f(x)$ is a function all of whose differential coefficients are continuous in an interval $(a-\alpha,\ a+\alpha)$ surrounding the point $x=a$. Then, if h is numerically less than α,

$$f(a+h)=f(a)+hf'(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^{(n)}(a+\theta_nh),$$

where $0<\theta_n<1$, for all values of n. Or, if

$$S_n=\sum_0^{n-1}\frac{h^\nu}{\nu!}f^{(\nu)}(a),\quad R_n=\frac{h^n}{n!}f^{(n)}(a+\theta_nh),$$

we have

$$f(a+h)-R_n=S_n.$$

Now let us suppose, in addition, that we can prove that $R_n\to 0$ as $n\to\infty$. Then evidently

$$f(a+h)=\lim_{n\to\infty}S_n=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots.$$

This expansion of $f(a+h)$ is known as **Taylor's Series.** The particular case in which $a=0$, when the formula reduces to

$$f(h)=f(0)+hf'(0)+\frac{h^2}{2!}f''(0)+\ldots,$$

is known as **Maclaurin's Series.** The function R_n is known as **Lagrange's form of the remainder.**

Examples LVIII. 1. Let $f(x)=\sin x$. Then all the derivatives of $f(x)$ are continuous for all values of x. Also $|f^n(x)|\leqq 1$ for all values of x and n. Hence in this case $|R_n|\leqq h^n/n\,!$, which tends to zero as $n\to\infty$ (Ex. XXX. 12) whatever value h may have.

It follows that

$$\sin(x+h)=\sin x+h\cos x-\frac{h^2}{2!}\sin x-\frac{h^3}{3!}\cos x+\frac{h^4}{4!}\sin x+\ldots,$$

for all values of x and h. In particular

$$\sin h=h-\frac{h^3}{3!}+\frac{h^5}{5!}-\ldots,$$

for all values of h. Similarly we can prove that

$$\cos(x+h)=\cos x-h\sin x-\frac{h^2}{2!}\cos x+\frac{h^3}{3!}\sin x+\ldots,\quad \cos h=1-\frac{h^2}{2!}+\frac{h^4}{4!}-\ldots.$$

2. **The Binomial Series.** Let $f(x)=(1+x)^m$ where m is any rational number, positive or negative. Then $f^{(n)}(x)=m(m-1)\dots(m-n+1)(1+x)^{m-n}$, and Maclaurin's Series takes the form

$$(1+x)^m=1+\binom{m}{1}x+\binom{m}{2}x^2+\dots.$$

When m is a positive integer the series terminates, and we obtain the ordinary formula for the Binomial Theorem with a positive integral exponent. In the general case

$$R_n=\frac{x^n}{n!}f^{(n)}(\theta x)=\binom{m}{n}x^n(1+\theta x)^{m-n},$$

and in order to show that Maclaurin's Series really represents $(1+x)^m$, where m is not a positive integer, for any range of values of x, we must show that $R_n\to 0$ for every value of x in that range. This is so in fact if $-1<x<1$. When $0\leqq x<1$ this may be proved by means of the expression given above for R_n, since $(1+\theta x)^{m-n}<1$ if $n>m$, and $\binom{m}{n}x^n\to 0$ as $n\to\infty$ (Ch. IV, Misc. Ex. 5). But if $-1<x<0$ a difficulty arises because $1+\theta x<1$, and $(1+\theta x)^{m-n}>1$ if $n>m$; knowing only that $0<\theta<1$ we cannot be assured that $1+\theta x$ is not quite small and $(1+\theta x)^{m-n}$ quite large.

In fact, in order to establish the Binomial Theorem properly by means of Taylor's Theorem, we need some different form for R_n, such as will be given later (§ 146).

131. Applications of Taylor's Theorem. A. **Maxima and minima.** Taylor's Theorem may be applied to give greater theoretical completeness to the tests of Ch. VI, §§ 103–4, though the results are not of much practical importance. It will be remembered that, on the assumption that $\phi(x)$ and its first two differential coefficients are continuous for $x=\xi$, we stated the following as being sufficient conditions for a maximum or minimum of $\phi(x)$ at $x=\xi$: *for a Maximum*, $\phi'(\xi)=0$, $\phi''(\xi)<0$; *for a Minimum*, $\phi'(\xi)=0$, $\phi''(\xi)>0$. It is evident that these tests fail if $\phi''(\xi)$ as well as $\phi'(\xi)$ is zero.

Let us suppose that the first n derivatives

$$\phi'(x),\quad \phi''(x),\quad \dots,\quad \phi^{(n)}(x)$$

are all continuous at $x=\xi$, and that all save the last vanish there. Then for sufficiently small values of h,

$$\phi(\xi+h)-\phi(\xi)=\frac{h^n}{n!}\phi^{(n)}(\xi+\theta h).$$

In order that there should be a maximum or a minimum, this expression must be of constant sign for all sufficiently small values of h, positive or negative. This evidently requires that n should be even. And if n is even there will be a maximum or a minimum according as $\phi^{(n)}(\xi)$ is negative or positive.

Thus we obtain the test: *if there is to be a maximum or minimum the first derivative which does not vanish must be an even derivative, and there will be a maximum if it is negative, a minimum if it is positive.*

Examples LIX. 1. Verify the result when $\phi(x)=(x-a)^m$ and $\xi=a$, m being a positive integer.

2. Test the function $(x-a)^m(x-b)^n$, where m and n are positive integers, for maxima and minima at the points $x=a$, $x=b$. Draw graphs of the different possible forms of the curve $y=(x-a)^m(x-b)^n$.

3. Test the functions $\sin x-x$, $\sin x-x+\frac{x^3}{6}$, $\sin x-x+\frac{x^3}{6}-\frac{x^5}{120}$, …. $\cos x-1$, $\cos x-1+\frac{x^2}{2}$, $\cos x-1+\frac{x^2}{2}-\frac{x^4}{24}$, … for maxima or minima at $x=0$.

132. B. **The calculation of certain limits.** Suppose that $f(x)$ and $\phi(x)$ are two functions of x whose derivatives $f'(x)$ and $\phi'(x)$ are continuous for $x=\xi$: and that $f(\xi)$ and $\phi(\xi)$ are both equal to zero.

Then the function

$$\psi(x)=f(x)/\phi(x)$$

has no value for $x=\xi$. But of course it may well tend to a limit as $x\rightarrow\xi$.

Now $$f(x)=f(x)-f(\xi)=(x-\xi)f'(x_1),$$

where x_1 lies between ξ and x; and similarly $\phi(x)=(x-\xi)\phi'(x_2)$, where x_2 also lies between x and ξ. Thus $\psi(x)=f'(x_1)/\phi'(x_2)$. Then we must distinguish four cases:

(1) if neither $f'(\xi)$ nor $\phi'(\xi)$ is zero it is clear that

$$\{f(x)/\phi(x)\}\rightarrow f'(\xi)/\phi'(\xi);$$

(2) if $f'(\xi)=0$, $\phi'(\xi)\neq 0$, it is clear that $f(x)/\phi(x)\rightarrow 0$;

(3) if $f'(\xi)\neq 0$, $\phi'(\xi)=0$, it is clear that $f(x)/\phi(x)$ becomes numerically very large as $x\rightarrow\xi$; but whether $f(x)/\phi(x)\rightarrow\infty$ or $-\infty$ or is sometimes large and positive and sometimes large and negative we cannot say, without further information as to the way in which $\phi'(x)\rightarrow 0$ as $x\rightarrow\xi$ (if e.g. $f'(\xi)>0$, and $\phi'(x)>0$ when $x>\xi$, it is clear that $f(x)/\phi(x)\rightarrow+\infty$);

(4) if $f'(\xi)=0$, $\phi'(\xi)=0$, we can as yet say nothing about the behaviour of $f(x)/\phi(x)$ as $x\rightarrow 0$.

But in this case it may happen that $f(x)$ and $\phi(x)$ have continuous second derivatives. And then

$$f(x)=f(x)-f(\xi)-(x-\xi)f'(\xi)=\tfrac{1}{2}(x-\xi)^2 f''(x_1),$$
$$\phi(x)=\phi(x)-\phi(\xi)-(x-\xi)\phi'(\xi)=\tfrac{1}{2}(x-\xi)^2\phi''(x_2),$$

where again x_1 and x_2 lie between ξ and x; so that

$$\psi(x)=f''(x_1)/\phi''(x_2).$$

We can now distinguish a variety of cases similar to those considered above. In particular, if neither second derivative vanishes for $x=\xi$, we have

$$f(x)/\phi(x)\rightarrow f''(\xi)/\phi''(\xi).$$

It is obvious that this argument can be repeated indefinitely, and we obtain the following theorem: *suppose that $f(x)$ and $\phi(x)$ and their derivatives, so far as may be wanted, are continuous for $x=\xi$. Suppose further that $f^{(p)}(x)$ and $\phi^{(q)}(x)$ are the first derivatives of $f(x)$ and $\phi(x)$ respectively which do not vanish for $x=\xi$. Then*

(1) *if* $p=q$, $f(x)/\phi(x)\rightarrow f^{(p)}(\xi)/\phi^{(p)}(\xi)$;

(2) *if* $p>q$, $f(x)/\phi(x)\rightarrow 0$;

(3) *if* $p<q$, *and* $q-p$ *is even*, $f(x)/\phi(x)\rightarrow\pm\infty$, *the sign being the same as that of* $f^{(p)}(\xi)/\phi^{(q)}(\xi)$;

(4) *if* $p<q$ *and* $q-p$ *is odd*, $f(x)/\phi(x)\rightarrow\pm\infty$ *as* $x\rightarrow\xi+0$, *the sign being the same as that of* $f^{(p)}(\xi)/\phi^{(q)}(\xi)$, *while if* $x\rightarrow\xi-0$ *the ambiguous sign must be reversed.*

This theorem is in fact an immediate corollary from the equations

$$f(x)=\frac{(x-\xi)^p}{p!}f^{(p)}(x_1),\qquad \phi(x)=\frac{(x-\xi)^q}{q!}\phi^{(q)}(x_2).$$

Examples LX. 1. Find the limit of

$$\{x-(n+1)x^{n+1}+nx^{n+2}\}/(1-x)^2,$$

as $x\rightarrow 1$. [Here the functions and their first derivatives vanish for $x=1$, and $f''(1)=n(n+1)$, $\phi''(1)=2$.]

2. Find the limits as $x\rightarrow 0$ of

$$(\tan x-x)/(x-\sin x),\qquad (\tan nx-n\tan x)/(n\sin x-\sin nx).$$

3. Find the limit of $x\{\sqrt{(x^2+a^2)}-x\}$ as $x\rightarrow\infty$. [Put $x=1/y$.]

4. Prove that

$$\lim_{x\to n}(x-n)\operatorname{cosec} x\pi = \frac{(-1)^n}{\pi},\quad \lim_{x\to n}\frac{1}{x-n}\left\{\operatorname{cosec} x\pi - \frac{(-1)^n}{(x-n)\pi}\right\} = \frac{(-1)^n\pi}{6},$$

n being any integer, and evaluate the corresponding limits involving $\cot x\pi$.

5. Find the limits as $x \to 0$ of

$$\{\operatorname{cosec} x - (1/x) - (x/6)\}/x^3, \quad \{\cot x - (1/x) + (x/3)\}/x^3.$$

6. $(\sin x \arcsin x - x^2)/x^6 \to \frac{1}{18}$, $(\tan x \arctan x - x^2)/x^6 \to \frac{2}{9}$, as $x \to 0$.

133. C. **The contact of plane curves.** Let us suppose that $f(x)$, $\phi(x)$ are two functions which possess derivatives of all orders continuous for $x = \xi$, and consider the curves $y = f(x)$, $y = \phi(x)$. In general $f(\xi)$ and $\phi(\xi)$ will not be equal. In this case the abscissa $x = \xi$ does not correspond to a point of intersection of the curves. If however $f(\xi) = \phi(\xi)$, the curves intersect in the point $x = \xi$, $y = f(\xi) = \phi(\xi)$. Let us suppose this to be the case. Then in order that the curves should not only cut but *touch* at this point it is obviously necessary and sufficient that the first derivatives $f'(x)$, $\phi'(x)$ should also have the same value for $x = \xi$.

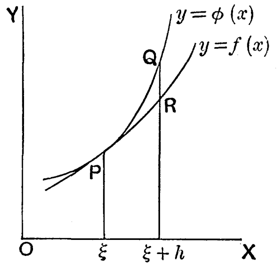

Fig. 55.

The contact of the curves in this case may be regarded from a different point of view. In the figure the two curves are drawn touching at P, and QR is equal to $\phi(\xi+h) - f(\xi+h)$, or, since $\phi(\xi) = f(\xi)$, $\phi'(\xi) = f'(\xi)$, to

$$\tfrac{1}{2}h^2[\phi''(\xi+\theta h) - f''(\xi+\theta_1 h)]$$

(where θ and θ_1 lie between 0 and 1).

Hence $$\lim_{h\to 0}\frac{QR}{h^2} = \tfrac{1}{2}\{\phi''(\xi) - f''(\xi)\}.$$

In other words, when the curves touch at the point for which $x = \xi$, *the difference of their ordinates for $x = \xi + h$, when h is small, is at least of the second order of smallness compared with h.*

The reader will easily verify that when the curves cut and do not touch $\lim (QR/h) = \phi'(\xi) - f'(\xi)$, so that QR is of the first order of smallness only.

It is evident that the degree of smallness of QR, as compared with h, may be taken as a kind of measure of the *closeness of the contact* of the curves. It is at once suggested that if the first

$n-1$ derivatives of f and ϕ all have equal values for $x=\xi$, then QR will be of the nth order of smallness: and the reader will have no difficulty in proving that this is so and that

$$\lim \frac{QR}{h^n} = \frac{1}{n!}\{\phi^{(n)}(\xi) - f^{(n)}(\xi)\}.$$

We are therefore led to frame the following definition:

Contact of the nth order. *If $f(\xi)=\phi(\xi)$, $f'(\xi)=\phi'(\xi)$, $\dots$ $f^{(n-1)}(\xi)=\phi^{(n-1)}(\xi)$, but $f^{(n)}(\xi)\neq\phi^{(n)}(\xi)$, the curves $y=f(x)$, $y=\phi(x)$ will be said to have contact of the nth order for $x=\xi$.*

Let us consider some particular cases of this definition in greater detail.

Examples LXI. 1. Let $\phi(x)=ax+b$, so that $y=\phi(x)$ is a straight line. The conditions for contact at the point for which $x=\xi$ are $f(\xi)=a\xi+b$, $f'(\xi)=a$. If we determine a and b to satisfy these equations we find $a=f'(\xi)$, $b=f(\xi)-\xi f'(\xi)$, and the equation of the tangent to $y=f(x)$ at the point $x=\xi$ is

$$y=xf'(\xi)+\{f(\xi)-\xi f'(\xi)\},$$

or $y-f(\xi)=(x-\xi)f'(\xi)$ (cf. Ex. XLI. 6).

2. The fact that the line is to have simple contact with the curve completely determines the line. In order that the tangent should have *contact of the second order* with the curve we must have $f''(\xi)=\phi''(\xi)$, i.e. $f''(\xi)=0$. A point at which the tangent to a curve has contact of the second order is called a **point of inflexion.**

3. Find the points of inflexion on the graphs of the functions $3x^4-6x^3+1$, $2x/(1+x^2)$, $a^2x/(x-a)^2$, $\sin x$, $\cos(x-a)$, $a\cos^2 x+b\sin^2 x$, $\tan x$, $\arctan x$.

4. Show that the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ cannot have a point of inflexion. [Here $ax+hy+g+(hx+by+f)y_1=0$ and

$$a+2hy_1+by_1{}^2+(hx+by+f)y_2=0,$$

suffixes denoting differentiations. Thus at a point of inflexion

$$a+2hy_1+by_1{}^2=0$$

or $$a(hx+by+f)^2-2h(ax+hy+g)(hx+by+f)+b(ax+hy+g)^2=0,$$

or $$(ab-h^2)\{ax^2+2hxy+by^2+2gx+2fy\}+af^2-2fgh+bg^2=0.$$

But this is inconsistent with the equation of the conic unless

$$af^2-2fgh+bg^2=c(ab-h^2)$$

or $abc+2fgh-af^2-bg^2-ch^2=0$; and this is the condition that the conic should degenerate into two straight lines.]

5. The curve $y=(ax^2+2bx+c)/(\alpha x^2+2\beta x+\gamma)$ has one or three points of inflexion according as the roots of $\alpha x^2+2\beta x+\gamma=0$ are real or complex. [The equation of the curve can, by a change of origin (cf. Ex. XLVIII. 22), be reduced to the form

$$\eta=\xi/(A\xi^2+2B\xi+C)=\xi/\{A(\xi-p)(\xi-q)\},$$

where p, q are real or conjugate. The condition for a point of inflexion will be found to be $\xi^3-3pq\xi+pq(p+q)=0$, which has one or three real roots according as $\{pq(p-q)\}^2$ is positive or negative; i.e. according as p and q are real or conjugate.]

6. Discuss in particular the curves $y=(1-x)/(1+x^2)$, $y=(1-x^2)/(1+x^2)$, $y=(1+x^2)/(1-x^2)$.

7. Show that when the curve of Ex. 5 has three points of inflexion, they lie on a straight line. [The equation $\xi^3-3pq\xi+pq(p+q)=0$ can be put in the form $(\xi-p)(\xi-q)(\xi+p+q)+(p-q)^2\xi=0$, so that the points of inflexion lie on the line $\xi+A(p-q)^2\eta+p+q=0$ or $A\xi-4(AC-B^2)\eta=2B$*.]

8. Show that the curves $y=x\sin x$, $y=(\sin x)/x$ have each infinitely many points of inflexion.

9. **Contact of a circle with a curve.** The general equation of a circle, viz.

$$(x-a)^2+(y-b)^2=r^2 \quad\text{.............................}(1),$$

contains three arbitrary constants. Let us attempt to determine them so that the circle has contact of as high an order as possible with the curve $y=f(x)$ at the point ξ, $\eta=f(\xi)$. We write η_1, η_2 for $f'(\xi)$, $f''(\xi)$. Differentiating the equation of the circle twice we obtain

$$(x-a)+(y-b)y_1=0 \quad\text{.............................}(2),$$

$$1+y_1^2+(y-b)y_2=0 \quad\text{.............................}(3).$$

If the circle touches the curve the equations (1), (2) are satisfied by $x=\xi$, $y=\eta$, $y_1=\eta_1$. This gives $(\xi-a)/\eta_1=-(\eta-b)=r/\sqrt{(1+\eta_1^2)}$.

If the contact is of the second order the equation (3) must also be satisfied by taking $y_2=\eta_2$. Then $b=\eta+\{(1+\eta_1^2)/\eta_2\}$; and hence we find

$$a=\xi-\{\eta_1(1+\eta_1^2)/\eta_2\},\quad b=\eta+\{(1+\eta_1^2)/\eta_2\},\quad r=(1+\eta_1^2)^{3/2}/\eta_2.$$

The circle which has contact of the second order with the curve at the point (ξ, η) is called the **circle of curvature**, and its radius the **radius of curvature.** The **measure of curvature** (or simply the *curvature*) is the reciprocal of the radius: thus the measure of curvature is $f''(\xi)/\{1+[f'(\xi)]^2\}^{3/2}$, or

$$\frac{d^2\eta}{d\xi^2}\Big/\left\{1+\left(\frac{d\eta}{d\xi}\right)^2\right\}^{3/2}.$$

10. Verify that the curvature of a circle is constant and equal to the reciprocal of the radius; and show that the circle is the only curve whose curvature is constant.

11. Find the centre and radius of curvature at any point of the conics $y^2=4ax$, $(x/a)^2+(y/b)^2=1$.

12. In an ellipse the radius of curvature at P is CD^3/ab, where CD is the semi-diameter conjugate to CP.

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* For more general results of this kind see a paper by Mr Bromwich, *Messenger of Mathematics*, vol. XXXII, p. 113.

13. Show that in general a conic can be drawn to have contact of the fourth order with the curve $y=f(x)$ at a given point P.

[Take the general equation of a conic, viz.

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

and differentiate four times with respect to x. Using suffixes to denote differentiation we obtain

$$ax+hy+g+(hx+by+f)\,y_1=0,$$
$$a+2hy_1+by_1{}^2+(hx+by+f)\,y_2=0,$$
$$3\,(h+by_1)\,y_2+(hx+by+f)\,y_3=0,$$
$$4\,(h+by_1)\,y_3+3by_2{}^2+(hx+by+f)\,y_4=0.$$

If the conic has contact of the fourth order these five equations must be satisfied by writing ξ, η, η_1, η_2, η_3, η_4 for x, y, ... We have thus just enough equations to determine the ratios $a:b:c:f:g:h$.]

14. An infinity of conics can be drawn having contact of the third order with the curve at P. Show that their centres all lie on a straight line.

[Take the tangent and normal as axes. Then the equation of the conic is of the form $2y=ax^2+2hxy+by^2$, and when x is small one value of y may be expressed (Ch. V, Misc. Ex. 9) in the form

$$y=\tfrac{1}{2}ax^2+\tfrac{1}{2}ahx^3(1+\epsilon_x),$$

where $\epsilon_x \to 0$ with x. But this expression must be the same as

$$y=\tfrac{1}{2}f''(0)\,x^2+\tfrac{1}{6}f'''(0)\,x^3(1+\epsilon_x'),$$

where $\epsilon_x' \to 0$ with x, and so (Ex. LVII. 15) $a=f''(0)$, $h=f'''(0)/3f''(0)$. But the centre lies on the line $ax+hy=0$.]

15. Determine a parabola which has contact of the third order with the ellipse $(x/a)^2+(y/b)^2=1$ at the extremity of the major axis.

16. The locus of the centres of conics which have contact of the third order with the ellipse $(x/a)^2+(y/b)^2=1$ at the point $(a\cos\alpha,\ b\sin\alpha)$ is the diameter $x/(a\cos\alpha)=y/(b\sin\alpha)$. [For the ellipse itself is one such conic.]

134. Differentiation of functions of several variables. Suppose that $f(x, y)$ is a function of two real variables x and y, and that the limits

$$\lim_{h\to 0}\{f(x+h,\ y)-f(x,\ y)\}/h, \quad \lim_{k\to 0}\{f(x,\ y+k)-f(x,\ y)\}/k$$

both exist; i.e. that $f(x, y)$ has a derivative with respect to x, which we have agreed to denote by df/dx or $D_x f(x, y)$, and also a derivative with respect to y, denoted similarly by df/dy or $D_y f(x, y)$. Another notation which it is natural to use is that of

$$f_x'(x,\ y), \quad f_y'(x,\ y)$$

or simply f_x', f_y' or f_x, f_y. It is usual to call these differential coefficients *partial differential coefficients*, and the process of forming them *partial differentiation*. We shall not, however, adopt this method of expression, which is apt to be misleading, and to lead the reader to imagine that there is some mystery about the process of 'partial differentiation.' In point of fact it is exactly the same process as 'ordinary differentiation.' The only novelty which occurs is the presence of a second variable y in $f(x, y)$. But the variation of y has nothing to do with that of x, and there is no possible ambiguity about the meaning of df/dx, df/dy, which are formed in precisely the same way as $f'(x)$ or df/dx in Ch. VI.

It is also usual to write

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}$$

for df/dx, df/dy, when f is a function of two variables: but, for the reasons indicated above, we shall not make any use of this notation.

In what precedes we have supposed x and y to be two real variables entirely independent of one another. If x and y were connected by a relation the state of affairs would be very different. In this case our definition of df/dx would fail entirely, as we could not change x into $x+h$ without at the same time changing y. But then $f(x, y)$ would not really be a function of two variables at all. A function of two variables, as we defined it in Ch. II, means essentially a function of two *independent* variables. If y depends on x, y is a function of x, say $y = \phi(x)$: and then

$$f(x, y) = f\{x, \phi(x)\}$$

is really a function of the single variable x. Of course we may also represent it as a function of the single variable y. Or, as is often most convenient, we may regard x and y as functions of a third variable t (Ch. II, § 23), and then $f(x, y)$, which is of the form $f\{\phi(t), \psi(t)\}$, is a function of the single variable t.

Examples LXII. 1. If $x = r\cos\theta$, $y = r\sin\theta$, so that $r = \surd(x^2+y^2)$, $\theta = \text{arc}\tan(y/x)$, prove that

$$\frac{dr}{dx} = \frac{x}{\surd(x^2+y^2)}, \quad \frac{dr}{dy} = \frac{y}{\surd(x^2+y^2)}, \quad \frac{d\theta}{dx} = -\frac{y}{x^2+y^2}, \quad \frac{d\theta}{dy} = \frac{x}{x^2+y^2},$$

$$\frac{dx}{dr} = \cos\theta, \quad \frac{dx}{d\theta} = -r\sin\theta, \quad \frac{dy}{dr} = \sin\theta, \quad \frac{dy}{d\theta} = r\cos\theta.$$

2. Account for the fact that $\frac{dr}{dx} \neq 1 \Big/ \left(\frac{dx}{dr}\right)$ and $\frac{d\theta}{dx} \neq 1 \Big/ \left(\frac{dx}{d\theta}\right)$. [When we were considering a function y of one variable x it followed from the definitions that dy/dx and dx/dy were reciprocals. This is no longer the case when we are dealing with functions of two variables. Let P (Fig. 56) be the point (x, y) or (r, θ). To find dr/dx we must increase x (say by an increment $MM_1 = \delta x$) while keeping y constant. This brings P to P_1. If along OP_1 we take $OP' = OP$, the increment of r is $P'P_1 = \delta r$, say; and $dr/dx = \lim (\delta r/\delta x)$. If on the other hand we want to calculate dx/dr, x and y being now regarded as functions of r and θ, we must increase r by Δr, say, keeping θ constant. This brings P to P_2, if $PP_2 = \Delta r$: the corresponding increment of x is $MM_1 = \Delta x$, say; and

$$dx/dr = \lim (\Delta x/\Delta r).$$

Now $\Delta x = \delta x$*: but $\Delta r \neq \delta r$. Indeed it is easy to see from the figure that

$$\lim (\delta r/\delta x) = \lim (P'P_1/PP_1) = \cos\theta,$$

but $\lim (\Delta r/\Delta x) = \lim (PP_2/PP_1) = \sec\theta$,

so that $\lim (\delta r/\Delta r) = \cos^2\theta$.

The fact is of course that *dx/dr and dr/dx are not formed upon the same hypothesis as to the variation of P.*]

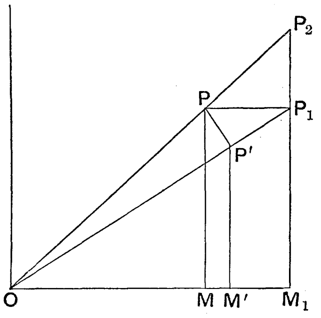

Fig. 56.

3. If $z = f(ax + by)$, prove that $b\,(dz/dx) = a\,(dz/dy)$.

[For $dz/dx = af'(ax + by)$, $dz/dy = bf'(ax + by)$.]

4. If $z = f(ax + by) + F(ax - by)$, then $b^2 (d^2z/dx^2) = a^2 (d^2z/dy^2)$.

5. Find dX/dx, dX/dy, etc. when $X + Y = x$, $Y = xy$. Express x, y as functions of X, Y and find dx/dX, dx/dY, etc.

6. Find dX/dx, etc. when $X + Y + Z = x$, $Y + Z = xy$, $Z = xyz$; express x, y, z in terms of X, Y, Z and find dx/dX, etc. [There is of course no difficulty in extending the ideas of the last section to functions of any number of variables.]

135. Differentiation of a function of two functions.

We shall not, in this volume, be much concerned with functions of two variables from the point of view of differentiation. But there is one theorem, concerning the differentiation of a function of *one* variable, and known generally, though not very happily, as the **Theorem of the Total Differential Coefficient,** which is of very great importance and depends on the notions explained in the preceding section regarding functions of *two* variables. This theorem gives us a rule for differentiating

---

* Of course the fact that $\Delta x = \delta x$ is due merely to the particular value of Δr that we have chosen (viz. PP_2). Any other choice would give us values of Δx, Δr proportional to those used here.

$$f\{\phi(t),\ \psi(t)\},$$

with respect to t.

Let us suppose, in the first instance, that $f(x, y)$ is a function of the two variables x and y, and that f_x', f_y' are continuous functions of both variables (§ 90) for all of their values which come in question.

And now let us suppose that the variation of x and y is restricted by supposing that (x, y) lies on a curve

$$x = \phi(t), \quad y = \psi(t),$$

where ϕ and ψ are functions of t with continuous differential coefficients $\phi'(t)$, $\psi'(t)$. Then, when x and y are thus restricted, $f(x, y)$ reduces to a function of the single variable t, say $F(t)$. The problem is to determine $F'(t)$.

Suppose that, when t changes to $t+\tau$, x and y change to $x+\xi$, $y+\eta$. Then by definition

$$\frac{dF(t)}{dt} = \lim_{\tau \to 0} \frac{1}{\tau}\left[f\{\phi(t+\tau),\ \psi(t+\tau)\} - f\{\phi(t),\ \psi(t)\}\right]$$

$$= \lim\ \{f(x+\xi,\ y+\eta) - f(x, y)\}/\tau$$

$$= \lim\left[\frac{f(x+\xi,\ y+\eta) - f(x,\ y+\eta)}{\xi}\frac{\xi}{\tau} + \frac{f(x,\ y+\eta) - f(x, y)}{\eta}\frac{\eta}{\tau}\right].$$

But by the Mean Value Theorem

$$\{f(x+\xi,\ y+\eta) - f(x,\ y+\eta)\}/\xi = f_x'(x+\theta\xi,\ y+\eta),$$

$$\{f(x,\ y+\eta) - f(x, y)\}/\eta = f_y'(x,\ y+\theta_1\eta),$$

where θ and θ_1 each lie between 0 and 1. As $\tau \to 0$, $\xi \to 0$ and $\eta \to 0$, and $\xi/\tau \to \phi'(t)$, $\eta/\tau \to \psi'(t)$: also

$$f_x'(x+\theta\xi,\ y+\eta) \to f_x'(x, y), \quad f_y'(x,\ y+\theta_1\eta) \to f_y'(x, y).$$

Hence

$$F'(t) = D_t f\{\phi(t),\ \psi(t)\} = f_x'(x, y)\,\phi'(t) + f_y'(x, y)\,\psi'(t),$$

where we are to put $x = \phi(t)$, $y = \psi(t)$ after carrying out the differentiations with respect to x and y.

This result may also be expressed in the form

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt}.$$

Examples LXIII. 1. Suppose $\phi(t)=(1-t^2)/(1+t^2)$, $\psi(t)=2t/(1+t^2)$, so that the locus of (x, y) is the circle $x^2+y^2=1$. Then

$$\phi'(t)=-4t/(1+t^2)^2, \quad \psi'(t)=2(1-t^2)/(1+t^2)^2,$$

$$F'(t)=D_x f(x, y)\{-4t/(1+t^2)^2\}+D_y f(x, y)\{2(1-t^2)/(1+t^2)^2\},$$

where x and y are to be put equal to $(1-t^2)/(1+t^2)$ and $2t/(1+t^2)$ after carrying out the differentiation.

We can easily verify this formula in particular cases. Suppose e.g. $f(x, y)\equiv x^2+y^2$. Then $f_x'=2x$, $f_y'=2y$, and it is easily verified that $F'(t)=2\{x\phi'(t)+y\psi'(t)\}=0$, which is obviously correct, since $F(t)\equiv 1$.

2. Verify the theorem in the same way when (*a*) $x=t^m$, $y=1-t^m$, $f(x, y)=x+y$; (*b*) $x=a\cos t$, $y=a\sin t$, $f(x, y)=x^2+y^2$.

3. One of the most important cases is that in which t is x itself. We then obtain

$$D_x f\{x, \psi(x)\}=D_x f(x, y)+D_y f(x, y)\,\psi'(x),$$

where y is to be replaced by $\psi(x)$ after differentiation.

It was this case which led to the introduction of the notation $\partial f/\partial x$, $\partial f/\partial y$. For it would seem natural to use the notation df/dx for *either* of the functions $D_x f\{x, \psi(x)\}$ and $D_x f(x, y)$, in one of which y is put equal to $\psi(x)$ before and in the other after differentiation. Suppose for example that $y=1-x$ and $f(x, y)=x+y$. Then $D_x f(x, 1-x)=D_x 1=0$, but $D_x f(x, y)=1$.

The distinction between the two functions is adequately shown by denoting the first by df/dx and the second by $\partial f/\partial x$, in which case the theorem takes the form

$$\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}.$$

But this notation is open to the objection that it is misleading to denote the functions $f\{x, \psi(x)\}$ and $f(x, y)$, whose forms as functions of x are quite different from one another, by the same letter f in df/dx, $\partial f/\partial x$. And if we prefer, as we shall do throughout this book, to regard x and y, when connected by a relation, as being each functions of an auxiliary variable t, no ambiguity can arise and the need for any such complication in our notation is not felt. Thus if $x=t$, $y=1-t$, $F(t)=f(x, y)=x+y=1$, the meaning of the equations

$$F'(t)=\frac{dF}{dt}=\frac{df}{dx}\frac{dx}{dt}+\frac{df}{dy}\frac{dy}{dt}=(1.1)+(1.-1)=0,$$

is perfectly straightforward and their truth obvious.

4. If the result of eliminating t between $x=\phi(t)$, $y=\psi(t)$ is $f(x, y)=0$, then

$$\frac{df}{dx}\frac{dx}{dt}+\frac{df}{dy}\frac{dy}{dt}=0 \quad \text{..............................(1).}$$

[Since $\left(\dfrac{dy}{dt}\right)\Big/\left(\dfrac{dx}{dt}\right)=\dfrac{dy}{dx}$, this is equivalent to

$$f_x'+f_y'\,(dy/dx)=0 \text{................................(2),}$$

a result which of course follows directly from the discussion of Ex. 3. The equation (1) is sometimes written in the form

$$f_x' \, dx + f_y' \, dy = 0 \quad \text{.................................(3).}$$

This formula as it stands means nothing, since dx and dy mean nothing by themselves: it is merely a convenient way of writing (1) or (2). There is however another point of view from which it may be regarded. Suppose that when t is changed to $t + \delta t$ the corresponding increments in x and y are δx, δy. Then

$$\lim \{f_x'(\delta x/\delta t) + f_y'(\delta y/\delta t)\} = 0;$$

in other words, when δt is small, $f_x' \delta x + f_y' \delta y$ is small compared with δt, instead of being, as an expression of the form $A\delta x + B\delta y$ *prima facie* appears to be, of the same order of smallness as δt. And the equation (3) may be regarded as a way of expressing this fact.]

5. If x and y are functions of t, and r and θ are the polar coordinates of (x, y), show that $r' = (xx' + yy')/r$, $\theta' = (xy' - yx')/r^2$, dashes denoting differentiations with respect to t.

136. The Mean Value Theorem for functions of two variables. Many of the results of the last chapter depended upon the Mean Value Theorem, expressed by the equation

$$\phi(x + h) - \phi(x) = hf'(x + \theta h),$$

or as it may be written, if $y = \phi(x)$,

$$\delta y = f'(x + \theta \delta x) \, \delta x.$$

Now suppose that $z = f(x, y)$ is a function of the two independent variables x and y, and that x and y receive increments h, k or δx, δy respectively; and let us attempt to express the corresponding increment of z, viz.

$$\delta z = f(x + h, y + k) - f(x, y),$$

in terms of h, k and the derivatives of z with respect to x and y.

Let $f(x + ht, y + kt) = F(t)$. Then

$$f(x+h, y+k) - f(x, y) = F(1) - F(0) = F'(\theta),$$

where $0 < \theta < 1$. But, by § 135,

$$\begin{aligned} F'(t) &= D_t f(x + ht, \ y + kt) \\ &= hf_x'(x + ht, \ y + kt) + kf_y'(x + ht, \ y + kt). \end{aligned}$$

Hence finally

$$\delta z = f(x+h, y+k) - f(x, y) = hf_x'(x + \theta h, y + \theta k) + kf_y'(x + \theta h, y + \theta k),$$

which is the formula desired. Since f_x', f_y' are supposed to be continuous functions of x and y, we have

$$\begin{aligned} f_x'(x + \theta h, y + \theta k) &= f_x'(x, y) + \epsilon_{h,k}, \\ f_y'(x + \theta h, y + \theta k) &= f_y'(x, y) + \epsilon'_{h,k}, \end{aligned}$$

where $\epsilon_{h,k}$ and $\epsilon'_{h,k} \to 0$ as h and $k \to 0$. Hence the theorem may be written in the form

$$\delta z = (f_x' + \epsilon)\,\delta x + (f_y' + \epsilon')\,\delta y,$$

where ϵ and ϵ' are small when δx and δy are small.

This may be expressed by saying that the equation

$$\delta z = f_x'\,\delta x + f_y'\,\delta y$$

is *approximately* true; i.e. that the difference between the two sides of the equation is small in comparison with the larger of δx and δy*. We must say '*the larger of* δx *and* δy' because one of them might be small in comparison with the other; one indeed might actually be zero.

It should be observed that if any equation of the form $\delta z = \lambda\delta x + \mu\delta y$ is 'approximately true' in this sense, we must have $\lambda = f_x'$, $\mu = f_y'$. For we have

$$\delta z - f_x'\,\delta x - f_y'\,\delta y = \epsilon\delta x + \epsilon'\delta y, \quad \delta z - \lambda\delta x - \mu\delta y = \epsilon_1\,\delta x + \epsilon_1'\delta y$$

where ϵ, ϵ', ϵ_1, ϵ_1' all tend to zero as δx and δy tend to zero; and so

$$(\lambda - f_x')\,\delta x + (\mu - f_y')\,\delta y = \rho\delta x + \rho'\delta y$$

where $\rho \to 0$ and $\rho' \to 0$. Hence if η is any assigned positive number, we can choose σ so that

$$|(\lambda - f_x')\,\delta x + (\mu - f_y')\,\delta y| < \eta\,(|\delta x| + |\delta y|)$$

for all values of δx and δy numerically less than σ. Taking $\delta y = 0$ we obtain $|(\lambda - f_x')\,\delta x| < \eta\,|\delta x|$, or $|\lambda - f_x'| < \eta$, and, as η may be as small as we please, this can only be the case if $\lambda = f_x'$. Similarly $\mu = f_y'$. This observation is of very frequent use, as will be seen from the examples which follow.

It is obvious that the theorems of this section are capable of immediate extension to functions of any number of variables.

Examples LXIV. 1. The area of an ellipse is given by $A = \pi ab$, where a, b are the semiaxes. Prove that if a, b receive small increments δa, δb, then $\delta A/A = (\delta a/a) + (\delta b/b)$, approximately.

2. Express Δ, the area of a triangle ABC, as a function of (in the usual notation) (i) a, B, C, (ii) A, b, c, and (iii) a, b, c, and establish the approximate formulae

$$\frac{\delta\Delta}{\Delta} = 2\frac{\delta a}{a} + \frac{c\,\delta B}{a \sin B} + \frac{b\,\delta C}{a \sin C}, \quad \frac{\delta\Delta}{\Delta} = \cot A\,\delta A + \frac{\delta b}{b} + \frac{\delta c}{c},$$

$$\delta\Delta = R\,(\cos A\,\delta a + \cos B\,\delta b + \cos C\,\delta c).$$

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* Or with $|\delta x| + |\delta y|$ or $\surd(\delta x^2 + \delta y^2)$.

3. The sides of a triangle vary in such a way that the area remains constant. They are then connected by one relation, and a may be regarded as a function of b and c. Prove that

$$da/db = -\cos B/\cos A, \quad da/dc = -\cos C/\cos A.$$

[This follows from the fact that both of the equations

$$\delta a = \frac{da}{db}\delta b + \frac{da}{dc}\delta c, \quad \cos A\,\delta a + \cos B\,\delta b + \cos C\,\delta c = 0,$$

are approximately true.]

4. If a, b, c vary so that R remains constant, then $\Sigma(\delta a/\cos A) = 0$, approximately, and so $da/db = -\cos A/\cos B$, $da/dc = -\cos A/\cos C$. [Use the formulae $a = 2R\sin A$, etc., and the facts that R and $A+B+C$ are constant.]

5. If $f(x, y) = 0$, and f_x', f_y' are continuous and $f_y' \neq 0$, then

$$dy/dx = -f_x'/f_y'.$$

6. The equation of the tangent to the curve whose equation is $f(x, y) = 0$, at the point (x_0, y_0), is

$$(x - x_0)f_x'(x_0, y_0) + (y - y_0)f_y'(x_0, y_0) = 0.$$

7. If z is a function of u and v, which are functions of x and y, prove that

$$\frac{dz}{dx} = \frac{dz}{du}\frac{du}{dx} + \frac{dz}{dv}\frac{dv}{dx}, \quad \frac{dz}{dy} = \frac{dz}{du}\frac{du}{dy} + \frac{dz}{dv}\frac{dv}{dy}.$$

[We have

$$\delta z = z_u\delta u + z_v\delta v, \quad \delta u = u_x\delta x + u_y\delta y, \quad \delta v = v_x\delta x + v_y\delta y,$$

approximately, suffixes denoting differentiations. Substitute for δu and δv in the first equation and compare the result with the equation $\delta z = z_x\delta x + z_y\delta y$.]

8. Let z be a function of x and y, and let X, Y, Z be defined by the equations

$$x = a_1X + b_1Y + c_1Z, \quad y = a_2X + b_2Y + c_2Z, \quad z = a_3X + b_3Y + c_3Z.$$

Then Z may be expressed as a function of X and Y. Express dZ/dX, dZ/dY in terms of dz/dx, dz/dy. [Let these differential coefficients be denoted by P, Q and p, q. Then $\delta z - p\delta x - q\delta y = 0$, approximately, or

$$(c_1p + c_2q - c_3)\,\delta Z + (a_1p + a_2q - a_3)\,\delta X + (b_1p + b_2q - b_3)\,\delta Y = 0.$$

Comparing this equation with $\delta Z - P\delta X - Q\delta Y = 0$ we see that

$$P = -\frac{a_1p + a_2q - a_3}{c_1p + c_2q - c_3}, \quad Q = -\frac{b_1p + b_2q - b_3}{c_1p + c_2q - c_3}.]$$

9. If $(a_1x + b_1y + c_1z)\,p + (a_2x + b_2y + c_2z)\,q = a_3x + b_3y + c_3z$,

then $(a_1X + b_1Y + c_1Z)\,P + (a_2X + b_2Y + c_2Z)\,Q = a_3X + b_3Y + c_3Z$.

(*Math. Trip.* 1899.)

10. If u and v are functions of x and y, x and y may be expressed as functions of u and v. Establish the formulae

$$Jx_u = v_y, \quad Jx_v = -u_y, \quad Jy_u = -v_x, \quad Jy_v = u_x,$$

where $J = u_xv_y - u_yv_x$. Verify these formulae when $u = r$, $v = \theta$, by means of the results of Ex. LXII. 1.

11. **Independent and non-independent functions.** Let u and v be functions of x and y: and suppose that x and y vary in such a way that u remains constant. In general, of course, v will not remain constant; in this case we shall say that u and v are *independent* functions. If on the other hand the constancy of u does involve the constancy of v, then to a given value of u corresponds one (or several) definite values of v, and v may be regarded as a function of u.

In this case we shall say that u and v are *non-independent* functions of x and y, or functions of x and y connected by a functional relation. Show that the necessary and sufficient condition that u and v should be connected by a functional relation is that

$$J = u_x v_y - u_y v_x$$

should vanish for all values of x and y.

[We have the approximate equations

$$\delta u = u_x \delta x + u_y \delta y, \quad \delta v = v_x \delta x + v_y \delta y.$$

That the condition $J=0$ is *necessary* follows from the fact that $v_x \delta x + v_y \delta y$ must vanish for all values of the ratio $\delta x : \delta y$ for which $u_x \delta x + u_y \delta y$ vanishes. Moreover

$$\text{(i)} \qquad \delta v = (\delta u - u_y \delta y)(v_x/u_x) + v_y \delta y = (v_x/u_x)\,\delta u,$$

approximately, if $J=0$ and $u_x \neq 0$. But, if we express v as a function of u and y, eliminating x by means of the relation between x, y and u, we must have

$$\text{(ii)} \qquad \delta v = v_y \delta y + v_u \delta u$$

approximately. Comparing (i) and (ii) we see that $v_y = 0$, so that v is a function of u only.

If $u_x = 0$ for all values of x and y, then either u_y or v_x must also vanish for all values of x and y. The first alternative would show that u was a mere constant, the second that u and v are both functions of y only, and therefore v a function of u.

It is usual to call J the *Jacobian* of u and v with respect to x and y. It can be shown by an extension of this argument that three functions u, v, w of three independent variables x, y, z are connected by a functional relation if and only if

$$J = \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix}$$

vanishes for all values of x, y, z: and so generally for any number of variables.]

12. Show that $ax^2 + 2hxy + by^2$ and $Ax^2 + 2Hxy + By^2$ are independent functions unless $a/A = h/H = b/B$.

13. Show that $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ can be expressed as a function of two linear functions of x, y, and z if and only if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

[Let $px+qy+rz$, $p'x+q'y+r'z$ be the linear functions. Then we must have

$$\begin{vmatrix} ax+hy+gz & hx+by+fz & gx+fy+cz \\ p & q & r \\ p' & q' & r' \end{vmatrix} \equiv 0,$$

or $aP+hQ+gR=0$, $hP+bQ+fR=0$, $gP+fQ+cR=0$, where $P=qr'-q'r$, etc. Eliminating P, Q, R we have the condition required.]

137. Definite Integrals and Areas. It will be remembered that, in Ch. VI, § 127, we assumed that if $f(x)$ is a continuous function of x the region $PpqQ$ shown in Fig. 57 has associated with it a definite number which we call its *area*. If we fix $Op=a$, but let $Oq=x$ vary, it is clear that this area is a function of x, which, as in § 127, we denote by $S(x)$.

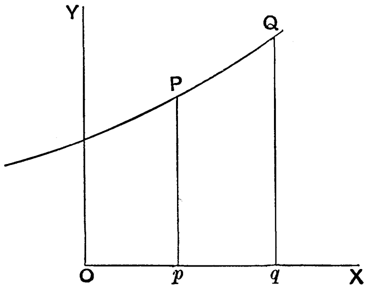

FIG. 57.

Making this assumption, we proved in § 127 that $S'(x)=f(x)$, and we showed how this result might be used in the calculation of the areas of particular curves. But we have still to justify the fundamental assumption that there is such a number as the area $S(x)$.

We know indeed, what is meant by the area of a *rectangle*, and that it is measured by the product of its sides. Also the properties of triangles, parallelograms, and polygons proved by Euclid enable us to attach a definite meaning to the areas of such figures. But nothing which we know so far provides us with a direct definition of the area of a figure bounded by curved lines.

We shall now show, in the case of a particular class of functions $f(x)$, how to give a definition of $S(x)$ by means of which we can *prove* its existence*.

We suppose that $f(x)$ is not only continuous, but is positive and increases steadily (as in Fig. 57) from $x=a$ to $x=b$. Let P, Q be the points whose abscissae Op, Oq are a and b respectively. Draw parallels to the axes of coordinates as shown in Fig. 58 a;

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* I have followed very closely the argument used by Mr Bromwich, *Infinite Series*, Appendix II, pp. 407–9.

and denote the area of the rectangle pQ bounded by a thick line (which we call the *outer* rectangle) by S_0, and the area of the shaded rectangle Pq (which we call the *inner* rectangle) by s_0. The outer rectangle includes and the inner is included by the curvilinear region $PpqQ$: and

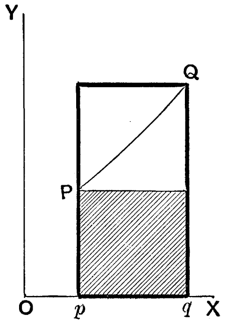

FIG. 58 *a*.

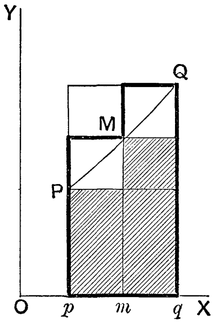

FIG. 58 *b*.

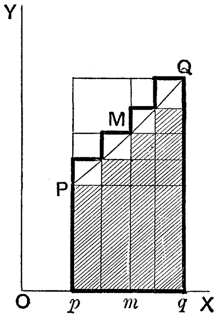

FIG. 58 *c*.

$$S_0 - s_0 = \text{rect. } PQ = (b-a)\{f(b) - f(a)\}.$$

Now bisect pq at m, and draw mM parallel to OY, and other lines parallel to OX as shown in Fig. 58 *b*. And now let us denote the sum of the rectangles pM, mQ, which are bounded by a thick line in Fig. 58 *b*, and which we again call the outer rectangles, by S_1, and the sum of the shaded rectangles Pm, Mq (which we again call the inner rectangles) by s_1. Then again the outer rectangles include and the inner are included by $PpqQ$; also

$$S_1 < S_0, \quad s_1 > s_0$$

and $$S_1 - s_1 = \text{rect. } PM + \text{rect. } MQ = \tfrac{1}{2}(b-a)\{f(b) - f(a)\}.$$

We now carry the construction one stage further by bisecting pm and mq and forming outer and inner rectangles, of total areas S_2 and s_2 respectively, as in Fig. 58 *c*. The outer rectangles again include and the inner are again included by $PpqQ$; and

$$S_2 < S_1 < S_0, \quad s_2 > s_1 > s_0.$$

Also $S_2 - s_2$ is the sum of four rectangles, each of breadth $\frac{1}{4}(b-a)$, and of total height $f(b) - f(a)$, so that

$$S_2 - s_2 = \frac{1}{2^2}(b-a)\{f(b) - f(a)\}.$$

It is evident that we may proceed as far as we like in this way by continually bisecting segments of the line pq: and so we define two sequences $S_0, S_1, S_2, \ldots$; $s_0, s_1, s_2, \ldots$ such that

(1) S_n is the sum of the areas of a set of rectangles which include the curvilinear region $PpqQ$, and s_n is the sum of the areas of a set of rectangles included by it;

(2) $$S_0 > S_1 > S_2 > S_3 > \ldots; \quad s_0 < s_1 < s_2 < \ldots;$$

(3) $$S_n - s_n = \frac{1}{2^n}(b-a)\{f(b)-f(a)\}.$$

From (2) and the fact that every S_n is greater than the corresponding s_n, it follows that S_n and s_n, as $n \to \infty$, each tend to limits. Also (3) shows that $S_n - s_n \to 0$, and therefore that S_n and s_n tend to the *same* limit, which we may call S. And (1) naturally leads us to define the **area** of the region $PpqQ$ as being equal to this number S. That is to say, our definition of this area is as follows: *divide pq into 2^n equal parts, and form the two sets of rectangles, of total areas S_n, s_n, as indicated in Fig. 58 a b c; then S_n and s_n tend, as $n \to \infty$, to a common limit S, which we call the area of the region.*

138. This definition, however, may be generalised. Suppose that instead of dividing pq into 2^n equal parts we divide it into ν parts of any kind, subject only to the condition that each part is less than δ_ν, where δ_ν is a number which tends to 0 as $\nu \to \infty$. We then construct an outer and an inner area Σ_ν, σ_ν as shown in Fig. 59. If S_n and s_n are defined as in § 137, it is easy to see* that $S_n > \sigma_\nu$ and $s_n < \Sigma_\nu$ for all values of n and ν; and, as S is the limit of S_n and s_n, it follows that $S \geqq \sigma_\nu$, $S \leqq \Sigma_\nu$ for all values of ν. Hence

$$S - \sigma_\nu \leqq \Sigma_\nu - \sigma_\nu, \quad \Sigma_\nu - S \leqq \Sigma_\nu - \sigma_\nu.$$

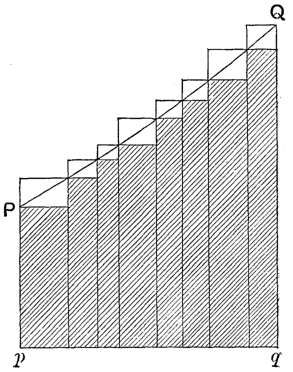

FIG. 59.

But it is obvious from the figure that

$$\Sigma_\nu - \sigma_\nu < \delta_\nu \{f(b)-f(a)\},$$

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* The reader should draw a figure to illustrate these inequalities.

and so $\Sigma_\nu - \sigma_\nu \to 0$: hence $S - \sigma_\nu \to 0$, or $\lim \sigma_\nu = S$; and similarly $\lim \Sigma_\nu = S$.

139. We can give an analytical form to the definition of the area as follows: *divide up the interval* (a, b) *by the points*

$$a = x_0,\ x_1,\ x_2,\ \dots,\ x_\nu = b$$

and let

$$\Sigma_\nu = (x_1 - x_0) f(x_1) + (x_2 - x_1) f(x_2) + \dots + (x_\nu - x_{\nu-1}) f(x_\nu),$$
$$\sigma_\nu = (x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1) + \dots + (x_\nu - x_{\nu-1}) f(x_{\nu-1}).$$

Then if, as $\nu \to \infty$, *all the sub-intervals* $x_{s+1} - x_s$ *tend to zero, i.e. if, given* δ, *we can choose* n *so that for* $\nu > n$ *all of*

$$x_1 - x_0,\quad x_2 - x_1,\quad \dots,\quad x_\nu - x_{\nu-1}$$

are less than δ, *the two sums* Σ_ν, σ_ν *tend to a common limit* s, *which we define to be the area of the region* $PpqQ$.

140. So far we have supposed that $f(x)$ is a *positive and increasing* function. It remains to generalize our definitions so as to apply to a number of other cases.

(i) The preceding analysis is in no way affected if we suppose that $f(x)$ is a *decreasing* instead of an increasing function.

(ii) Now let us suppose that $f(x)$, while increasing or decreasing steadily throughout (a, b), changes its sign (as at R in Fig. 60). In this case we may still use the analytical definition of § 139. It should however be observed that, along the arc PR, $f(x)$ is negative. If we apply the definition of § 139 to the regions PpR, RqQ separately, the first has a negative and the second a positive area; and the number furnished by the definition as the area of the whole region $PpqQ$ is the difference and not the sum of these sub-regions. If we want the sum we must calculate the area of each sub-region separately and change the sign of the area of the first before adding it to that of the second.

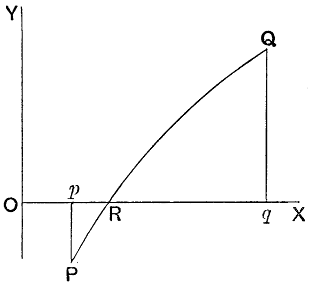

Fig. 60.

(iii) Next we may suppose that $f(x)$ does not either increase or decrease steadily throughout the *whole* interval (a, b), but that

(a, b) can be divided into a finite number of regions throughout each of which $f(x)$ increases or decreases steadily (as in Fig. 61).

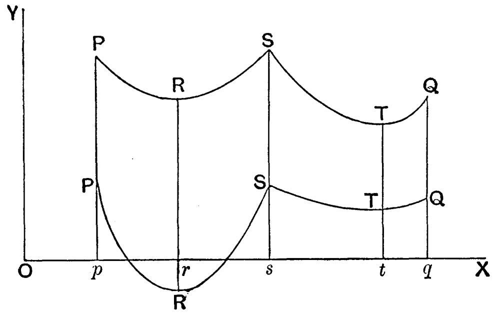

FIG. 61.

Then we define the areas of each of the regions $PprR$, $RrsS$, $SstT$, $TtqQ$ as in § 139, and that of $PpqQ$ as their sum. Here again it is to be observed that if $f(x)$ changes sign some parts of the region will be reckoned as having negative areas.

(iv) The reader will now find no difficulty in seeing how to define the area of any such curvilinear region as is likely to occur in elementary work. Thus in the case of the region shown in Fig. 62 we add the areas of the regions bounded by dotted lines

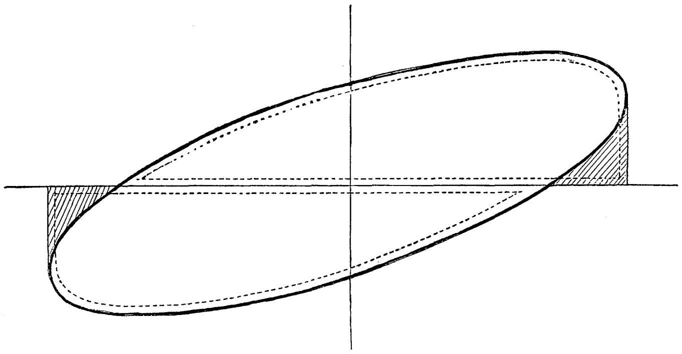

FIG. 62

and subtract those of the shaded regions. This gives the total area, counting the part below the axis of x negatively. To find the area of the closed curve, in the ordinary geometrical sense, we must, before carrying out this process, change the sign of the areas of the two negative sub-regions.

141. Lengths of Curves. The notion of the *length* of a curve is capable of precise mathematical analysis just as much as that of an area, but the analysis is rather more difficult, and we shall not give any general treatment of the question here. We shall however consider briefly the case in which the curve is a circle, since the assumption that a circular arc has a length is fundamental in elementary Trigonometry (see for example p. 169, footnote).

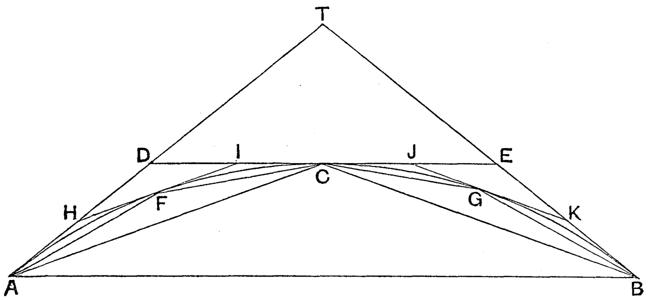

Fig. 63.

Let ACB (Fig. 63) be an arc of a circle, C its middle point (i.e. the point equidistant from A and B), and AT, BT the tangents at A and B. Then it is easy to prove that the tangent at C meets AT, BT in points D, E between A, T and B, T respectively[*]. Then $AC+CB>AB$ and

$$AT+TB=AD+DT+TE+EB>AD+DE+EB.$$

Thus if $AB=s_0$, $AC+CB=s_1$, $AT+TB=S_0$ and $AD+DE+EB=S_1$, we have $s_0<s_1$, $S_0>S_1$.

Now let F and G be the middle points of the arcs AC, CB, and HFI, JGK the corresponding tangents. If we put

$$AF+FC+CG+GB=s_2, \quad AH+HI+IJ+JK+KB=S_2,$$

it is easy to prove as above that $s_0<s_1<s_2$ and $S_0>S_1>S_2$.

Proceeding in this way we define two sequence of numbers s_0, s_1, s_2, ..., S_0, S_1, S_2, ... such that s_n is the perimeter of part of a polygon inscribed in the circle and S_n that of part of a polygon circumscribed to the circle, while

$$s_0<s_1<s_2<\dots<s_n<\dots, \quad S_0>S_1>S_2>\dots>S_n>\dots.$$

It is moreover easy to see that every s_n is less than every S_n. Hence s_n and S_n tend to limits as $n\rightarrow\infty$. But if α is the angle subtended by the whole arc at the centre of the circle, that subtended by one of the chords by which s_n is defined is $\alpha/2^n$, and

$$S_n-s_n=S_n\{1-\cos(\alpha/2^n)\}<S_0\{1-\cos(\alpha/2^n)\},$$

and therefore tends to zero as $n\rightarrow\infty$. Hence s_n and S_n tend to the same limit l, which we call the **length** of the arc AB.

---

[*] For the distance of every point of the tangent, other than C, from the centre of the circle is greater than that of any point on the circumference.

It is easy to generalise the foregoing reasoning to cover the case in which the arc is subdivided in any manner instead of by repeated bisection (cf. § 138). Similar reasoning may be applied to curves other than circles, the only difficulty being that of defining the precise conditions under which such a point as D always falls between A and T*.

142. The definite integral. Let us now suppose that $f(x)$ is a function which satisfies the conditions of the preceding sections, so that the region bounded by the curve $y=f(x)$, the ordinates $x=a$ and $x=b$, and the axis of x has a definite area. Then we proved in Ch. VI (§ 127) that if we could find an 'integral function' of $f(x)$, i.e. a function $F(x)$ such that

$$F'(x)=f(x), \quad F(x)=\int f(x)\,dx,$$

then the area in question is $F(b)-F(a)$.

As we saw in Ch. VI, however, to actually determine the form of $F(x)$ is not always practicable. It is therefore convenient to have a formula which represents the area $PpqQ$ and contains no explicit reference to $F(x)$. We shall write

$$\text{area } PpqQ=\int_a^b f(x)\,dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The expression on the right-hand side of this equation may then be regarded as being defined in either of two ways. We may regard it as simply an abbreviation for $F(b)-F(a)$, where $F(x)$ is some integral function of $f(x)$, whether an actual formula expressing it is known or not; or we may regard it as the value of the area $PpqQ$, as directly defined in § 139. It follows from § 127 that these two definitions are equivalent in all cases to which we have shown that the direct definition is applicable.

An example of a case in which the direct definition (so far as we know at present) cannot be applied is obtained by supposing

$$f(x)=3x^2\sin(1/x)-x\cos(1/x) \ (x\neq 0), \quad f(x)=0 \ (x=0),$$

and
$$F(x)=x^3\sin(1/x) \ (x\neq 0), \quad F(x)=0 \ (x=0).$$

Then $F'(x)=f(x)$ and

$$F(b)-F(a)=b^3\sin(1/b)-a^3\sin(1/a).$$

But since $f(x)$ changes its sign infinitely often as x approaches 0, and has an infinity of maxima and minima near $x=0$, our proof of the applicability of the direct method of definition fails if a and b have opposite signs.

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* I am indebted to Mr Bromwich for the substance of this section. It enables us to complete the proof of the inequalities $\sin x < x < \tan x$ assumed in Ex. XXXVII. 14.

As a matter of fact the scope of the direct definition may be greatly extended. In particular it may be proved applicable to all *continuous* functions (such as $f(x)$ above). But the restricted proof which we have given will be sufficient for our present purposes.

The number $$\int_a^b f(x)\,dx \qquad\qquad(1),$$

is called a **definite integral**; a and b are called its **lower and upper limits**; $f(x)$ is called the **subject of integration** or **integrand**, and the interval (a, b) the **range of integration.** The definite integral depends on a and b and the form of the function $f(x)$ only. The reader should be careful to observe that it is *not a function of x.* On the other hand the integral function

$$F(x) = \int f(x)\,dx$$

is sometimes called the **indefinite integral** of $f(x)$.

The distinction between the definite and the indefinite integral is merely one of point of view. The definite integral $\int_a^b f(x)\,dx = F(b) - F(a)$ is a function of b, and may be regarded as a particular integral function of $f(b)$. On the other hand the indefinite integral $F(x)$ can always be expressed by means of a definite integral, since

$$F(x) = F(a) + \int_a^x f(t)\,dt.$$

But when we are considering 'indefinite integrals' or 'integral functions' we are usually thinking of *a relation between two functions*, in virtue of which one is the derivative of the other. And when we are considering a 'definite integral' we are not as a rule concerned with any possible variation of the limits; usually the limits are constants such as 0 and 1: and

$$\int_0^1 f(x)\,dx = F(1) - F(0)$$

is not a function at all, but a mere number.

If, in the definition of § 139, we write δx_ν for $x_{\nu+1} - x_\nu$, we obtain the formulae

$$\int_a^b f(x)\,dx = \lim \Sigma f(x_{\nu+1})\,\delta x_\nu = \lim \Sigma f(x_\nu)\,\delta x_\nu.$$

It was this expression of the definite integral as the limit of a sum which suggested the notation which is now invariably used.

Since $1/x$ is a continuous and steadily decreasing function of x for all positive values of x, the investigations of the preceding paragraphs supply us with a proof of the actual existence of the function $\log x$, which we agreed to assume provisionally in § 112.

Examples LXV. Calculation of the definite from the indefinite integral. 1. Show that

$$\int_a^b x^n\,dx=\frac{b^{n+1}-a^{n+1}}{n+1},$$

and in particular
$$\int_0^1 x^n\,dx=\frac{1}{n+1}.$$

2. $\displaystyle\int_a^b \cos mx\,dx=(\sin mb-\sin ma)/m,\quad \int_a^b \sin mx\,dx=(\cos ma-\cos mb)/m.$

3. $\displaystyle\int_a^b \frac{dx}{1+x^2}=\operatorname{arc}\tan b-\operatorname{arc}\tan a,\quad \int_0^1 \frac{dx}{1+x^2}=\tfrac{1}{4}\pi.$

[There is an apparent difficulty here owing to the fact that $\operatorname{arc}\tan x$ is a many valued function. The difficulty may be avoided by agreeing that in formulae of this kind $\operatorname{arc}\tan x$ is always to denote an angle lying between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. With this convention the equation

$$\int_0^x \frac{dt}{1+t^2}=\operatorname{arc}\tan x$$

is true for all values of x, for the integral vanishes for $x=0$ and increases steadily and continuously towards $+\frac{1}{2}\pi$ as x increases towards $+\infty$, decreases steadily and continuously towards $-\frac{1}{2}\pi$ as x decreases towards $-\infty$. Similarly in the equation

$$\int_0^x \frac{dt}{\sqrt{(1-t^2)}}=\operatorname{arc}\sin x,$$

where $-1<x<1$, $\operatorname{arc}\sin x$ denotes an angle always lying between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.]

4. If a and b are both numerically less than unity,

$$\int_a^b \frac{dx}{\sqrt{(1-x^2)}}=\operatorname{arc}\sin b-\operatorname{arc}\sin a.$$

5. $\displaystyle\int_0^1 \frac{dx}{1-x+x^2}=\frac{2\pi}{3\sqrt{3}},\quad \int_0^1 \frac{dx}{1+x+x^2}=\frac{\pi}{3\sqrt{3}}.$

6. If $-\pi<a<\pi$, $\displaystyle\int_0^1 \frac{dx}{1+2x\cos a+x^2}=\frac{a}{2\sin a}$, except when $a=0$, when the value of the integral is $\frac{1}{2}$, which is the limit of $\frac{1}{2}a\operatorname{cosec}a$ as $a\rightarrow 0$.

7. $\displaystyle\int_0^1 \sqrt{(1-x^2)}\,dx=\tfrac{1}{4}\pi,\quad \int_0^a \sqrt{(a^2-x^2)}\,dx=\tfrac{1}{4}\pi a^2\quad (a>0).$

8. $\displaystyle\int_0^\pi \frac{dx}{a+b\cos x}=\frac{\pi}{\sqrt{(a^2-b^2)}}$, if $a>|b|$. [For the form of the indefinite integral see Ex. LV. 3. If $|a|<|b|$ the subject of integration has an infinity between 0 and π. What is the value of the integral when a is negative and $-a>|b|$?]

9. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x}=\frac{\pi}{2ab}$, if a and b are positive. What is the value of the integral when a and b have opposite signs, or when both are negative?

10. **Fourier's integrals.** Prove that if m and n are integers

$$\int_0^{2\pi} \cos mx \sin nx\, dx$$

is always equal to zero, and

$$\int_0^{2\pi} \cos mx \cos nx\, dx, \quad \int_0^{2\pi} \sin mx \sin nx\, dx$$

are equal to zero unless $m=n$, when each is equal to π.

11. Prove that $\int_0^{\pi} \cos mx \cos nx\, dx$, $\int_0^{\pi} \sin mx \sin nx\, dx$, are each equal to zero except when $m=n$, when each is equal to $\frac{1}{2}\pi$; and that

$$\int_0^{\pi} \cos mx \sin nx\, dx = \frac{2n}{n^2-m^2}$$

or zero, according as $n \sim m$ is odd or even.

143. Calculation of the definite integral from its definition as the limit of a sum. In a few cases we can evaluate a definite integral by a direct calculation, starting from the definition of § 139. As a rule this is by no means as easy as by using the indefinite integral, but the reader will find it instructive to work through a few examples.

Examples LXVI. 1. Evaluate $\int_a^b x\, dx$ by dividing (a, b) into n equal parts by the points of division $a=x_0, x_1, x_2, \dots, x_n=b$, and calculating the limit as $n \to \infty$ of

$$(x_1-x_0)f(x_0)+(x_2-x_1)f(x_1)+\dots+(x_n-x_{n-1})f(x_{n-1}).$$

[This sum is

$$\frac{b-a}{n}\left[a+\left(a+\frac{b-a}{n}\right)+\left(a+2\frac{b-a}{n}\right)+\dots+\left\{a+(n-1)\frac{b-a}{n}\right\}\right]$$

$$=\frac{b-a}{n}\left[na+\frac{b-a}{n}\{1+2+\dots+(n-1)\}\right]=(b-a)\left\{a+(b-a)\frac{n(n-1)}{2n^2}\right\},$$

which tends to the limit $\frac{1}{2}(b^2-a^2)$ as $n \to \infty$. Verify the result by graphical reasoning.]

2. Calculate $\int_a^b x^2\, dx$ in the same way.

3. Calculate $\int_a^b x\, dx$, where $0 < a < b$, by dividing (a, b) into n parts by the points of division $a, ar, ar^2, \dots ar^{n-1}$, where $r^n = b/a$. Extend the result to the more general integral $\int_a^b x^m\, dx$.

4. Calculate $\int_a^b \cos mx\, dx$ and $\int_a^b \sin mx\, dx$ by the method of Ex. 1.

5. Prove that $n\sum_{r=0}^{n-1}\frac{1}{n^2+r^2}\rightarrow\frac{1}{4}\pi$ as $n\rightarrow\infty$.

[This follows from the fact that

$$\frac{n}{n^2}+\frac{n}{n^2+1^2}+\dots+\frac{n}{n^2+(n-1)^2}=\sum_{r=0}^{n-1}\frac{(1/n)}{1+(r/n)^2},$$

which tends to the limit $\int_0^1\frac{dx}{1+x^2}$ as $n\rightarrow\infty$, by the direct definition of the integral.]

6. Prove that $\frac{1}{n^2}\sum_{r=0}^{n-1}\surd(n^2-r^2)\rightarrow\frac{1}{4}\pi$. [The limit is $\int_0^1\surd(1-x^2)\,dx$.]

144. General properties of the definite integral. The definite integral possesses the important properties expressed by the equations:

$$(1)\qquad \int_a^b f(x)\,dx=-\int_b^a f(x)\,dx.$$

This follows at once from the definition of the integral by means of the integral function $F(x)$, since $F(b)-F(a)=-\{F(a)-F(b)\}$. It should be observed that in the *direct* definition it was essentially supposed that the upper limit was greater than the lower; thus this method of definition does not apply to the integral $\int_b^a f(x)\,dx$ when $a<b$. If we adopt this definition as fundamental we must extend it to such cases by regarding the equation (1) as a definition of its right-hand side.

$$(2)\qquad \int_a^a f(x)\,dx=0.$$

$$(3)\qquad \int_a^b f(x)\,dx+\int_b^c f(x)\,dx=\int_a^c f(x)\,dx.$$

$$(4)\qquad \int_a^b \kappa f(x)\,dx=\kappa\int_a^b f(x)\,dx.$$

$$(5)\qquad \int_a^b\{f(x)+\phi(x)\}\,dx=\int_a^b f(x)\,dx+\int_a^b\phi(x)\,dx.$$

The reader will find it an instructive exercise to write out formal proofs of these properties, in each case giving a proof starting from (α) the definition by means of the integral function (β) the direct definition of § 139.

The following theorems are also important.

(6) *If* $f(x)\geqq 0$ *for* $a\leqq x\leqq b$, *then* $\int_a^b f(x)\,dx\geqq 0$.

(i) If we start from the definition by means of $F(x)$ this is merely another way of stating the result of § 102, Ch. VI.

(ii) If we start from the direct definition the result is geometrically intuitive. To supply a formal proof we have only to observe that σ_ν (§ 138) cannot be negative.

(7) *If, throughout the interval* (a, b), $L \leqq F(x) \leqq G$, *then*

$$L(b-a) \leqq \int_a^b f(x)\,dx \leqq G(b-a).$$

This follows at once if we apply (6) to $f(x)-L$ and $G-f(x)$.

$$(8) \qquad \int_a^b f(x)\,dx = (b-a)f(\xi),$$

where ξ *lies between* a *and* b.

This follows from (7). For we can take L to be the least and G the greatest value of $f(x)$ in (a, b). Then the integral is equal to $H(b-a)$, where H lies between L and G. But since $f(x)$ is continuous there must be a value of ξ for which $f(\xi)=H$ (§ 86).

If $F(x)$ is the integral function we can write the result of (8) in the form

$$F(b)-F(a)=(b-a)F'(\xi),$$

so that (8) appears now to be only another way of stating the Mean Value Theorem of Ch. VI, § 106. We may call (8) the **First Mean Value Theorem for Integrals.**

(9) **The Generalised Mean Value Theorem for integrals.** *If* $\phi(x)$ *is positive and* L *and* G *are defined as in* (7), *then*

$$L\int_a^b \phi(x)\,dx \leqq \int_a^b f(x)\,\phi(x)\,dx \leqq G\int_a^b \phi(x)\,dx;$$

and

$$\int_a^b f(x)\,\phi(x)\,dx = f(\xi)\int_a^b \phi(x)\,dx,$$

where ξ *is defined as in* (8).

This follows at once by applying (6) to the integrals

$$\int_a^b \{f(x)-L\}\,\phi(x)\,dx, \quad \int_a^b \{G-f(x)\}\,\phi(x)\,dx.$$

The reader should formulate for himself the corresponding result which holds when $\phi(x)$ is always negative.

Examples LXVII. 1. Show, by means of the direct definition of the definite integral, and equations (1)–(5) above, that

(i) $\int_{-a}^{a} \phi(x^2)\,dx = 2\int_0^a \phi(x^2)\,dx, \quad \int_{-a}^{a} x\phi(x^2)\,dx = 0.$

(ii) $\int_0^{\frac{1}{2}\pi} \phi(\cos x)\,dx = \int_0^{\frac{1}{2}\pi} \phi(\sin x)\,dx = \frac{1}{2}\int_0^{\pi} \phi(\sin x)\,dx.$

(iii) $\int_0^{m\pi} \phi(\cos^2 x)\,dx = m\int_0^{\pi} \phi(\cos^2 x)\,dx,$

m being an integer. [The truth of these equations will appear geometrically intuitive, if the graphs of the functions under the sign of integration are sketched.]

2. Prove that $\int_0^{\pi} \frac{\sin n\theta}{\sin\theta}\, d\theta = \pi$ or 0 according as n is odd or even. [Use the formula $(\sin n\theta)/(\sin\theta) = 2\cos(n-1)\theta + 2\cos(n-3)\theta + \dots$, the last term being 1 or $2\cos\theta$.]

3. Prove that $\int_0^{\pi} \sin n\theta \cot\theta\, d\theta = 0$ or π according as n is odd or even.

4. If $\phi(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + \dots + a_n\cos n\theta + b_n\sin n\theta$, then

$$\int_0^{2\pi} \phi(\theta)\, d\theta = 2\pi a_0, \quad \int_0^{2\pi} \cos k\theta\, \phi(\theta)\, d\theta = \pi a_k, \quad \int_0^{2\pi} \sin k\theta\, \phi(\theta)\, d\theta = \pi b_k,$$

if k is a positive integer less than n. If $k > n$ the value of each of the last two integrals is zero. [Use Ex. LXV. 10.]

5. If $\phi(\theta) = a_0 + a_1\cos\theta + a_2\cos 2\theta + \dots + a_n\cos n\theta$ then

$$\int_0^{\pi} \phi(\theta)\, d\theta = \pi a_0, \quad \int_0^{\pi} \cos k\theta\, \phi(\theta)\, d\theta = \tfrac{1}{2}\pi a_k,$$

if k is a positive integer less than n. If $k > n$ the value of the last integral is zero. [Use Ex. LXV. 11.]

6. If $f(x) \leqq \phi(x)$ for $a \leqq x \leqq b$, then $\int_a^b f\, dx \leqq \int_a^b \phi\, dx$.

7. Prove that

$$0 < \int_0^{\frac{1}{2}\pi} (\sin\theta)^{n+1}\, d\theta < \int_0^{\frac{1}{2}\pi} (\sin\theta)^n\, d\theta, \quad 0 < \int_0^{\frac{1}{4}\pi} (\tan\theta)^{n+1}\, d\theta < \int_0^{\frac{1}{4}\pi} (\tan\theta)^n\, d\theta.$$

8[*]. If $n > 1$, $\cdot 5 < \int_0^{1/2} \frac{dx}{\sqrt{(1-x^{2n})}} < \cdot 524$. [The first inequality follows from the fact that $\sqrt{(1-x^{2n})} < 1$, the second from the fact that $\sqrt{(1-x^{2n})} > \sqrt{(1-x^2)}$.]

9. Prove that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{(4-x^2+x^3)}} < \frac{1}{6}\pi$.

10. Prove that $(3x+8)/16 < 1/\sqrt{(4-3x+x^3)} < 1/\sqrt{(4-3x)}$ if $0 < x < 1$, and hence that $\frac{19}{32} < \int_0^1 \frac{dx}{\sqrt{(4-3x+x^3)}} < \frac{2}{3}$.

11. Prove that $\cdot 573 < \int_1^2 \frac{dx}{\sqrt{(4-3x+x^3)}} < \cdot 595$. [Put $x = 1+u$: then replace $2+3u^2+u^3$ by $2+4u^2$ and by $2+3u^2$.]

12. If a and ϕ are positive acute angles

$$\phi < \int_0^{\phi} \frac{dx}{\sqrt{(1-\sin^2 a \sin^2 x)}} < \frac{\phi}{\sqrt{(1-\sin^2 a \sin^2 \phi)}}.$$

If $a = \phi = \frac{1}{6}\pi$, the integral lies between $\cdot 523$ and $\cdot 541$.

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[*] Exs. 8–12 are taken from Prof. Gibson's *Elementary Treatise on the Calculus.*

13. Prove that $\left|\int_a^b \phi(x)\,dx\right| \leqq \int_a^b |\phi(x)|\,dx$. [Let β, γ, ... κ be the values of x for which $\phi(x)$ changes sign. Then

$$\int_a^b \phi(x)\,dx \leqq \left|\int_a^\beta \phi(x)\,dx\right| + \left|\int_\beta^\gamma \phi(x)\,dx\right| + \ldots + \left|\int_\kappa^b \phi(x)\,dx\right|$$

$$= \left(\int_a^\beta + \int_\beta^\gamma + \ldots + \int_\kappa^b\right) |\phi(x)|\,dx = \int_a^b |\phi(x)|\,dx.$$

The reader should also attempt to prove this result directly from the definition of the integral as the limit of a sum Σ_ν (§ 138), showing that, if Σ_ν' is the corresponding sum for the function $|\phi(x)|$, then $|\Sigma_\nu| \leqq \Sigma_\nu'$.]

14. If $|f(x)| \leqq M$, then $\left|\int_a^b f(x)\,\phi(x)\,dx\right| \leqq M\int_a^b |\phi(x)|\,dx$.

145. Integration by parts and by substitution. It follows from Theorem (3) of § 94 that

$$\int_a^b f(x)\,\phi'(x)\,dx = f(b)\,\phi(b) - f(a)\,\phi(a) - \int_a^b f'(x)\,\phi(x)\,dx.$$

This formula is known as the formula for **integration by parts**, as applied to definite integrals. Again we know (§ 108) that if $F(t)$ is the integral function of $f(t)$, then

$$\int f\{\phi(x)\}\,\phi'(x)\,dx = F\{\phi(x)\}.$$

Hence if $\phi(a) = c$, $\phi(b) = d$, we have

$$\int_c^d f(t)\,dt = F(d) - F(c) = F\{\phi(b)\} - F\{\phi(a)\} = \int_a^b f\{\phi(x)\}\,\phi'(x)\,dx;$$

which is the formula for the transformation of a definite integral by **substitution**.

The formulae for integration by parts and for transformation often enable us to evaluate a definite integral without the labour of actually finding the integral function of the subject of integration, and sometimes even when the integral function cannot be found. Some instances of this will be found in the following examples. That the value of a definite integral may sometimes be found without a knowledge of the integral function is only to be expected, for the fact that we cannot determine the general form of a function $F(x)$ in no way precludes the possibility that we may be able to determine the difference $F(b) - F(a)$ between two of its particular values. But as a rule this can only be effected by the use of more advanced methods than are at present at our disposal.

Examples LXVIII. 1. Prove that

$$\int_a^b xf''(x)\,dx = \{bf'(b) - f(b)\} - \{af'(a) - f(a)\}.$$

2. More generally, $\int_a^b x^m f^{(m+1)}(x)\,dx = F(b) - F(a)$, where

$$F(x) = x^m f^{(m)}(x) - mx^{m-1} f^{(m-1)} x + m(m-1) x^{m-2} f^{(m-2)} x - \dots + (-1)^m m!\,f(x).$$

3. Prove that $\int_0^1 \operatorname{arc\,sin} x\,dx = \frac{1}{2}\pi - 1$, $\int_0^1 x \operatorname{arc\,tan} x\,dx = \frac{1}{4}\pi - \frac{1}{2}$.

4. Prove that

$$\int_0^{\frac{1}{2}\pi} \frac{x\cos x\sin x\,dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \frac{\pi}{4ab^2(a+b)}.$$

[Integrate by parts and use Ex. LV. 9.]

5. If $f_1(x) = \int_0^x f(t)\,dt$, $f_2(x) = \int_0^x f_1(t)\,dt$, $\dots f_k(x) = \int_0^x f_{k-1}(t)\,dt$, then

$$f_k(x) = \frac{1}{(k-1)!}\int_0^x f(t)(x-t)^{k-1}\,dt.$$

[Integrate repeatedly by parts.]

6. If $u_{m,n} = \int_0^1 x^m(1-x)^n\,dx$, where m and n are positive integers, prove by integration by parts that $(m+n+1)\,u_{m,n} = nu_{m,n-1}$, and deduce that

$$u_{m,n} = \frac{m!\,n!}{(m+n+1)!}.$$

7. If $u_n = \int_0^{\frac{1}{4}\pi} (\tan\theta)^n\,d\theta$, prove that $u_n + u_{n-2} = 1/(n-1)$. Hence evaluate the integral for all positive integral values of n.

[Put $(\tan\theta)^n = (\tan\theta)^{n-2}(\sec^2\theta - 1)$ and integrate by parts.]

8. Deduce from the last example that u_n lies between $1/\{2(n-1)\}$ and $1/\{2(n+1)\}$.

9. If $u_n = \int_0^{\frac{1}{2}\pi} (\sin x)^n\,dx$, prove that $u_n = \{(n-1)/n\}\,u_{n-2}$. [Write $(\sin x)^{n-1}\sin x$ for $(\sin x)^n$ and integrate by parts.]

10. Deduce that u_n is equal to

$$\frac{2.4.6..(n-1)}{3.5.7..n}, \qquad \frac{1}{2}\pi\,\frac{1.3.5..(n-1)}{2.4.6..n}$$

according as n is odd or even.

11. **The Second Mean Value Theorem.** If $f(x)$ is a function of x which has a differential coefficient of constant sign for all values of x from $x=a$ to $x=b$, then there is a number ξ between a and b such that

$$\int_a^b f(x)\,\phi(x)\,dx = f(a)\int_a^\xi \phi(x)\,dx + f(b)\int_\xi^b \phi(x)\,dx.$$

[Let $\int_a^x \phi(t)\,dt = \Phi(x)$. Then

$$\int_a^b f(x)\phi(x)\,dx = \int_a^b f(x)\Phi'(x)\,dx = f(b)\Phi(b) - \int_a^b f'(x)\Phi(x)\,dx$$
$$= f(b)\Phi(b) - \Phi(\xi)\int_a^b f'(x)\,dx,$$

by the generalised Mean-value Theorem of § 144: i.e.

$$\int_a^b f(x)\phi(x)\,dx = f(b)\Phi(b) - \{f(b) - f(a)\}\Phi(\xi),$$

which is equivalent to the result given.]

12. **Bonnet's form of the Second Mean Value Theorem.** If $f'(x)$ is of constant sign, and $f(b)$ and $f(a) - f(b)$ have the same sign, then

$$\int_a^b f(x)\phi(x)\,dx = f(a)\int_a^X \phi(x)\,dx$$

where X lies between a and b. [For $f(b)\Phi(b) + \{f(a) - f(b)\}\Phi(\xi) = \mu f(a)$, where μ lies between $\Phi(\xi)$ and $\Phi(b)$, and so is the value of $\Phi(x)$ for a value of x such as X. The important case is that in which $0 \leqq f(b) \leqq f(x) \leqq f(a)$.]

Prove similarly that if $f(a)$ and $f(b) - f(a)$ have the same sign

$$\int_a^b f(x)\phi(x)\,dx = f(b)\int_X^b \phi(x)\,dx$$

where X lies between a and b. [Use the function $\psi(\xi) = \int_\xi^b \phi(x)\,dx$. It will be found that the integral can be expressed in the form

$$f(a)\psi(a) + \{f(b) - f(a)\}\psi(\xi).$$

The important case is that in which $0 \leqq f(a) \leqq f(x) \leqq f(b)$.]

13. Prove that $\left|\int_X^{X'} \frac{\sin x}{x}\,dx\right| < \frac{2}{X}$, if $X' > X > 0$. [Apply the first formula of Ex. 12, and note that the integral of $\sin x$ over any interval whatever is numerically less than 2.]

14. Establish the results of Ex. LXVII. 1 by means of the rule for substitution.

[In (i) divide the range of integration into the two parts $(-a, 0)$, $(0, a)$, and put $x = -y$ in the first. In (ii) use the substitution $x = \frac{1}{2}\pi - y$ to obtain the first equation: to obtain the second divide the range $(0, \pi)$ into two equal parts and use the substitution $x = \frac{1}{2}\pi + y$. In (iii) divide the range into m equal parts and use the substitutions $x = \pi + y$, $2\pi + y$, etc.]

15. Prove that $\int_a^b F(x)\,dx = \int_a^b F(a + b - x)\,dx$.

16. Prove that $\int_0^{\frac{1}{2}\pi} (\cos x \sin x)^m\,dx = 2^{-m}\int_0^{\frac{1}{2}\pi} (\cos x)^m\,dx$.

17. Prove that $\int_0^\pi x\phi(\sin x)\,dx = \frac{1}{2}\pi\int_0^\pi \phi(\sin x)\,dx$. [Put $x = \pi - y$.]

18. Prove that $\int_0^\pi \frac{x\sin x}{1 + \cos^2 x}\,dx = \frac{1}{4}\pi^2$.

19. Show by means of the transformation $x=a\cos^2\theta+b\sin^2\theta$ that $\int_a^b \sqrt{\{(x-a)(b-x)\}}\,dx=\frac{1}{8}\pi(b-a)^2$.

20. Show by means of the substitution $(a+b\cos x)(a-b\cos y)=a^2-b^2$ that

$$\int_0^\pi (a+b\cos x)^{-n}\,dx=(a^2-b^2)^{-(n-\frac{1}{2})}\int_0^\pi (a-b\cos y)^{n-1}\,dy,$$

if n is a positive integer and $a>|b|$, and evaluate the integral when $n=1$, 2, 3.

21. If m and n are positive integers, then

$$\int_a^b (x-a)^m(b-x)^n\,dx=(b-a)^{m+n+1}\frac{m!\,n!}{(m+n+1)!}.$$

[Put $x=a+(b-a)y$, and use Ex. 6.]

146. Proof of Taylor's Theorem by Integration by Parts. We shall conclude this chapter by giving the alternative form of the proof of Taylor's Theorem to which we alluded in § 129.

Let $f(x)$ be a function whose first n derivatives are continuous, and consider the integral

$$R_n=\frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1}f^{(n)}(a+th)\,dt.$$

Integrating repeatedly by parts we find that

$$\begin{aligned}R_n&=\frac{h^{n-1}}{(n-1)!}\int_0^1 (1-t)^{n-1}\frac{d}{dt}\{f^{(n-1)}(a+th)\}\,dt\\ &=-\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^{n-1}}{(n-2)!}\int_0^1 (1-t)^{n-2}f^{(n-1)}(a+th)\,dt\\ &=-\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)-\frac{h^{n-2}}{(n-2)!}f^{(n-2)}(a)-\dots\\ &\qquad\qquad -hf'(a)+\int_0^1 f'(a+th)\,dt\\ &=f(a+h)-f(a)-hf'(a)-\dots-\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a).\end{aligned}$$

Hence

$$\begin{aligned}f(a+h)&=f(a)+hf'(a)+\dots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)\\ &\qquad+\frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1}f^{(n)}(a+th)\,dt \quad\dots\dots\dots(1),\end{aligned}$$

which may be regarded as another form of Taylor's Theorem.

Now, by Theorem (9) of § 144,

$$\int_0^1 (1-t)^{n-1} f^{(n)}(a+th)\,dt = \int_0^1 (1-t)^{n-p}(1-t)^{p-1} f^{(n)}(a+th)\,dt$$
$$= (1-\theta)^{n-p} f^{(n)}(a+\theta h)\int_0^1 (1-t)^{p-1}\,dt,$$

where $0<\theta<1$, p being any positive integer not greater than n. Hence

$$R_n = (1-\theta)^{n-p} f^{(n)}(a+\theta h)\, h^n/p\,(n-1)!\ \ldots\ldots\ldots(2).$$

If we take $p=n$ we obtain Lagrange's form of R_n. If on the other hand we take $p=1$ we obtain **Cauchy's form**, viz.[*]

$$R_n = (1-\theta)^{n-1} f^{(n)}(a+\theta h)\, h^n/(n-1)!\ \ldots\ldots\ldots\ldots(3).$$

The truth of the equation (1) may also be proved as follows.

Let (in the notation of § 129)

$$F_n(x) = f(b) - f(x) - (b-x)f'(x) - \ldots - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x).$$

Then

$$F_n'(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x),$$

by differentiation, and so

$$F_n(a) = F_n(b) - \int_a^b F_n'(x)\,dx = \frac{1}{(n-1)!}\int_a^b (b-x)^{n-1} f^{(n)}(x)\,dx;$$

and the equation (1) above follows at once on writing $b=a+h$, $x=a+th$.

147. Application of Cauchy's form to the binomial series. If $f(x)=(1+x)^m$, where m is not a positive integer, Cauchy's form of the remainder is

$$R_n = \frac{m(m-1)\ldots(m-n+1)}{1.2\ldots(n-1)}\,\frac{(1-\theta)^{n-1}x^n}{(1+\theta x)^{n-m}}.$$

Now $(1-\theta)/(1+\theta x)$ is less than unity, so long as $-1<x<1$, whether x is positive or negative (it will be remembered that the difficulty in using Lagrange's form, in Ex. LVIII. 2, arose in connection with negative values of x); and $(1+\theta x)^{m-1}$ is less than a constant K for all values of n, being in fact less than $(1+|x|)^{m-1}$ if $m>1$ and than $(1-|x|)^{m-1}$ if $m<1$. Hence

$$|R_n| < K\,|m|\left|\binom{m-1}{n-1}\right| |x^n| = \rho_n,$$

say. But (Ch. IV, Misc. Ex. 5) $\rho_n \rightarrow 0$, and so $R_n \rightarrow 0$, as $n \rightarrow \infty$. The truth of the Binomial Theorem is thus established for all real values of m and all values of x between -1 and $+1$.

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[*] The method used in § 129 can also be modified so as to obtain these alternative forms.

MISCELLANEOUS EXAMPLES ON CHAPTER VII.

1. Verify the terms given of the following Taylor's Series:

(1) $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots,$

(2) $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \dots,$

(3) $x \operatorname{cosec} x = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \dots,$

(4) $x \cot x = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 - \dots.$

2. If $f(x)$ and its first $n+2$ derivatives are continuous, and $f^{(n+1)}(0) \neq 0$, and θ_n is the value of θ which occurs in Lagrange's form of the remainder after n terms of Taylor's Series, show that

$$\theta_n = \frac{1}{n+1} + \frac{n}{2(n+1)^2(n+2)} \frac{f^{(n+2)}(0)}{f^{(n+1)}(0)} x (1 + \epsilon_x),$$

where $\epsilon_x \to 0$ with x. [Follow the method of Ex. LVII. 12.]

3. Verify the last result when $f(x) = 1/(1+x)$. [Here $(1+\theta_n x)^{n+1} = 1 + x$.]

4. Show that if $f(x)$ has derivatives of the first three orders

$$f(b) = f(a) + \tfrac{1}{2}(b-a)\{f'(a) + f'(b)\} - \tfrac{1}{12}(b-a)^3 f'''(\alpha),$$

where $a < \alpha < b$. [Apply to the function

$$f(x) - f(a) - \tfrac{1}{2}(x-a)\{f'(a) + f'(x)\} - \left(\frac{x-a}{b-a}\right)^3 [f(b) - f(a) - \tfrac{1}{2}(b-a)\{f'(a) + f'(b)\}]$$

arguments similar to those of § 129.]

5. Show that under the same conditions

$$f(b) = f(a) + (b-a) f'\{\tfrac{1}{2}(a+b)\} + \tfrac{1}{24}(b-a)^3 f'''(\alpha).$$

6. Show that if $f(x)$ has derivatives of the first five orders

$$f(b) = f(a) + \tfrac{1}{6}(b-a)\left\{f'(a) + f'(b) + 4f'\left(\frac{a+b}{2}\right)\right\} - \tfrac{1}{2880}(b-a)^5 f^{(5)}(\alpha).$$

7. Show that under the same conditions

$$f(b) = f(a) + \tfrac{1}{2}(b-a)\{f'(a) + f'(b)\} - \tfrac{1}{12}(b-a)^2\{f''(b) - f''(a)\} + \tfrac{1}{720}(b-a)^5 f^{(5)}(\alpha).$$

8. Establish the formulae

(i) $$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (a-b) \begin{vmatrix} f(a) & f'(\beta) \\ g(a) & g'(\beta) \end{vmatrix},$$

where β lies between a and b, and

(ii) $$\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix} = \tfrac{1}{2}(b-c)(c-a)(a-b) \begin{vmatrix} f(a) & f'(\beta) & f''(\gamma) \\ g(a) & g'(\beta) & g''(\gamma) \\ h(a) & h'(\beta) & h''(\gamma) \end{vmatrix},$$

where β and γ lie between the least and greatest of a, b, c. [To prove (ii) consider the function

$$\phi(x)=\begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix} - \frac{(x-a)(x-b)}{(c-a)(c-b)} \begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix}$$

which vanishes for $x=a$, b, c. Its first derivative, by Th. B of § 102, must vanish for two distinct values of x lying between the least and greatest of a, b, c; and its second derivative must therefore vanish for a value γ of x satisfying the same condition. We thus obtain the formula

$$\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix} = \tfrac{1}{2}(c-a)(c-b) \begin{vmatrix} f(a) & f(b) & f''(\gamma) \\ g(a) & g(b) & g''(\gamma) \\ h(a) & h(b) & h''(\gamma) \end{vmatrix}.$$

The reader will now complete the proof without difficulty.]

9. If $F(x)$ is a function which has continuous derivatives of the first n orders, of which the first $n-1$ vanish for $x=0$, and $A \leqq F^{(n)}(x) \leqq B$ for $0 \leqq x \leqq h$, then $A(x^n/n!) \leqq F(x) \leqq B(x^n/n!)$ for $0 \leqq x \leqq h$.

Apply this result to

$$f(x)-f(0)-xf'(0)-\dots-\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0),$$

and deduce Taylor's Theorem.

10. If $\Delta_h\phi(x)=\phi(x)-\phi(x+h)$, $\Delta_h^2\phi(x)=\Delta_h\{\Delta_h\phi(x)\}$, and so on, and $\phi(x)$ has derivatives of the first n orders, show that

$$\Delta_h^n\phi(x)=\sum_{r=0}^{n}(-1)^r\binom{n}{r}\phi(x+rh)=(-h)^n\phi^{(n)}(\xi),$$

where ξ lies between x and $x+nh$. Deduce that, if $\phi^{(n)}(x)$ is continuous, $\{\Delta_h^n\phi(x)\}/h^n \rightarrow \phi^{(n)}(x)$ as $h \rightarrow 0$. [This has already been proved when $n=2$ (Ex. LVII. 13).]

11. Deduce from Ex. 10 that $x^{n-m}\Delta_h^n x^m \rightarrow m(m-1)\dots(m-n+1)h^n$ as $x \rightarrow +\infty$, m being any rational number and n any positive integer. In particular prove that

$$x\sqrt{x}\{\sqrt{x}-2\sqrt{(x+1)}+\sqrt{(x+2)}\} \rightarrow -\tfrac{1}{4}.$$

12. Suppose that $y=\phi(x)$ is a function of x with continuous derivatives of at least the first four orders, and that $\phi(0)=0$, $\phi'(0)=1$, so that

$$y=\phi(x)=x+a_2x^2+a_3x^3+(a_4+\epsilon_x)x^4,$$

where $\epsilon_x \rightarrow 0$ with x. Establish the formula

$$x=\psi(y)=y-a_2y^2+(2a_2^2-a_3)y^3-(5a_2^3-5a_2a_3+a_4+\epsilon_y)y^4,$$

where $\epsilon_y \rightarrow 0$ with y, for that value of x which vanishes with y; and prove that, as $x \rightarrow 0$,

$$\frac{\phi(x)\psi(x)-x^2}{x^4} \rightarrow a_2^2.$$

13. The coordinates (ξ, η) of the centre of curvature, at the point (x, y), of the curve $x=f(t)$, $y=F(t)$ are given by

$$-(\xi-x)/y'=(\eta-y)/x'=(x'^2+y'^2)/(x'y''-x''y'),$$

and its radius of curvature is

$$(x'^2+y'^2)^{3/2}/(x'y''-x''y'),$$

dashes denoting differentiations with respect to t.

14. The coordinates (ξ, η) of the centre of curvature, at the point (x, y), of the curve $27ay^2=4x^3$ are given by

$$3a(\xi+x)+2x^2=0, \quad \eta=4y+(9ay)/x. \quad (\textit{Math. Trip.}\ 1899.)$$

15. Prove that the circle of curvature at a point (x, y) will have contact of the third order with the curve if at that point $y_3(1+y_1^2)=3y_1y_2^2$.

Also prove that the circle is the only curve which possesses this property at every point; and that the only points on a conic which possess the property are the extremities of the axes.

16. The conic of closest contact with the curve $y=ax^2+bx^3+cx^4+\dots+kx^n$, at the origin, is $a^3y=a^4x^2+a^2bxy+(ac-b^2)y^2$. Deduce that the conic of closest contact at the point (ξ, η) of the curve $y=f(x)$ is

$$18\eta_2^3T=9\eta_2^4(x-\xi)^2+6\eta_2^2\eta_3(x-\xi)T+(3\eta_2\eta_4-4\eta_3^2)T^2,$$

where $T=(y-\eta)-\eta_1(x-\xi)$. (*Math. Trip.* 1907.)

17. **Homogeneous functions***. If $u=x^nf(y/x, z/x, \dots)$, u is unaltered, save for a factor λ^n, if $x, y, z, \dots$ are all increased in the ratio $\lambda : 1$. In these circumstances u is called a *homogeneous function of degree n* in the variables $x, y, z, \dots$. Prove that if u is homogeneous and of degree n, then

$$x\frac{du}{dx}+y\frac{du}{dy}+z\frac{du}{dz}+\dots=nu.$$

This result is known as **Euler's Theorem** on homogeneous functions.

18. If u is homogeneous and of degree n, du/dx, du/dy, ... are homogeneous and of degree $n-1$.

19. Let $f(x, y)=0$ be an equation in x and y (e.g. $x^n+y^n-x=0$), and let $F(x, y, z)=0$ be the form it assumes when made homogeneous by the introduction of a third variable z in place of unity (e.g. $x^n+y^n-xz^{n-1}=0$). Show that the equation of the tangent at the point (ξ, η) of the curve $f(x, y)=0$ is

$$xF_\xi+yF_\eta+zF_\zeta=0,$$

where F_ξ, F_η, F_ζ denote the values of F_x, F_y, F_z when $x=\xi$, $y=\eta$, $z=\zeta=1$.

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* In this and the following examples the reader is to assume the continuity of all the derivatives which occur.

20. If u and v are functions of ξ and η, which are themselves functions of x and y, and the Jacobian (Ex. LXIV. 11) of u and v with respect to ξ and η is denoted by $\dfrac{d(u, v)}{d(\xi, \eta)}$, then

$$\frac{d(u, v)}{d(x, y)} = \frac{d(u, v)}{d(\xi, \eta)} \cdot \frac{d(\xi, \eta)}{d(x, y)}.$$

Extend the result to any number of variables.

21. Let $f(x)$ be a function of x whose derivative is $1/x$ and which vanishes for $x=1$. Show that if $u=f(x)+f(y)$, $v=xy$, then $u_x v_y - u_y v_x = 0$, and hence that u and v are connected by a functional relation. By putting $y=1$, show that this relation must be $f(x)+f(y)=f(xy)$.

22. Prove in a similar manner that if the derivative of $f(x)$ is $1/(1+x^2)$, and $f(0)=0$, then $f(x)$ must satisfy the equation

$$f(x)+f(y)=f\{(x+y)/(1-xy)\}.$$

23. Show that if a functional relation exists between

$$u=f(x)+f(y)+f(z), \quad v=f(y)f(z)+f(z)f(x)+f(x)f(y), \quad w=f(x)f(y)f(z),$$

then f must reduce to a constant. [The condition for a functional relation will be found to be

$$f'(x)f'(y)f'(z)\{f(y)-f(z)\}\{f(z)-f(x)\}\{f(x)-f(y)\}=0.]$$

24. Suppose that x and y are functions of two variables α and β, and that the equation of a curve is given by a relation between α and β, say $f(\alpha, \beta)=0$. Then we may regard α, β (and therefore x, y) as functions of an auxiliary variable t. If dashes denote differentiations with respect to t, we have $f_\alpha \alpha' + f_\beta \beta' = 0$, so that α' and β' are proportional to $-f_\beta$ and f_α; and by choosing t appropriately we can reduce this proportionality to equality, so that $\alpha' = -f_\beta$, $\beta' = f_\alpha$. Then

$$x' = f_\alpha x_\beta - f_\beta x_\alpha = J, \quad y' = f_\alpha y_\beta - f_\beta y_\alpha = K,$$

say. Show that the measure of curvature of the curve is

$$\left\{J\frac{d(K, f)}{d(\alpha, \beta)} - K\frac{d(J, f)}{d(\alpha, \beta)}\right\} \Big/ (J^2+K^2)^{3/2}. \text{ (*Math. Trip.* 1899.)}$$

25. If $u=0$, $v=0$, $w=0$ are the equations of three circles, rendered homogeneous as in Ex. 19, the equation

$$\frac{d(u, v, w)}{d(x, y, z)}=0,$$

represents the circle which cuts them all orthogonally. (*Math. Trip.* 1900.)

26. If A, B, C are three functions of x such that

$$\begin{vmatrix} A & A' & A'' \\ B & B' & B'' \\ C & C' & C'' \end{vmatrix}$$

vanishes identically, we can find constants λ, μ, ν such that $\lambda A + \mu B + \nu C$ vanishes identically: and conversely. [The converse is almost obvious. To prove the direct theorem let $\alpha = BC' - B'C$, etc. Then $\alpha' = BC'' - B''C$, etc., and in virtue of the vanishing of the determinant $\beta\gamma'-\beta'\gamma$, $\gamma\alpha'-\alpha'\gamma$, $\alpha\beta'-\alpha'\beta$ all vanish; from which it follows that the ratios $\alpha:\beta:\gamma$ are constant. But $\alpha A+\beta B+\gamma C=0$.]

27. Let z be a function of x and y, and z_x, z_y its two first derivatives. Now let x be expressed as a function of y and z, and let x_y, x_z denote its first derivatives. Prove that

$$x_y=-z_y/z_x, \quad x_z=1/z_x.$$

[We have the approximate equations

$$\delta z=z_x\,\delta x+z_y\,\delta y, \quad \delta x=x_y\,\delta y+x_z\,\delta z.$$

The result of substituting for δx in the first is

$$\delta z=(z_x x_y+z_y)\,\delta y+z_x x_z\,\delta z.$$

Since δy and δz are independent this can be true only if $z_x x_y+z_y=0$, $z_x x_z=1$.]

28. Four variables x, y, z, u are connected by two relations. Show that

$$y_z{}^u z_x{}^u x_y{}^u=-y_z{}^x z_x{}^y x_y{}^z=1, \quad x_z{}^u z_x{}^y+y_z{}^u z_y{}^x=1,$$

where $y_z{}^u$ denotes the derivative of y, when expressed as a function of z and u, with respect to z. (*Math. Trip.* 1897.)

29. Prove the formulae

$$\frac{d}{dx}\int_a^x f(t)\,dt=f(x), \quad \frac{d}{dx}\int_a^{\phi(x)} f(t)\,dt=f(x)\,\phi'(x).$$

30. Find A, B, C, λ so that the first four derivatives of

$$\int_a^{a+x} f(t)\,dt-x\,[Af(a)+Bf(a+\lambda x)+Cf(a+x)],$$

vanish for $x=0$; and A, B, C, D, λ, μ so that the first six derivatives of

$$\int_a^{a+x} f(t)\,dt-x\,[Af(a)+Bf(a+\lambda x)+Cf(a+\mu x)+Df(a+x)]$$

vanish for $x=0$.

31. If $a>0$, $ac-b^2>0$, and $x_1>x_0$, then

$$\int_{x_0}^{x_1}\frac{dx}{ax^2+2bx+c}=\frac{1}{\sqrt{(ac-b^2)}}\arctan\left\{\frac{(x_1-x_0)\sqrt{(ac-b^2)}}{ax_1x_0+b\,(x_1+x_0)+c}\right\},$$

the inverse tangent lying between 0 and π*.

32. Evaluate the integral $\displaystyle\int_{-1}^{1}\frac{\sin\alpha\,dx}{1-2x\cos\alpha+x^2}$. For what values of α is the integral a discontinuous function of α? (*Math. Trip.* 1904.)

[The value of the integral is $\frac{1}{2}\pi$ if $2n\pi<\alpha<(2n+1)\pi$, and $-\frac{1}{2}\pi$ if $(2n-1)\pi<\alpha<2n\pi$, n being any integer; and 0 if α is a multiple of π.]

33. If $ax^2+2bx+c>0$ for $x_0\leqq x\leqq x_1$, $f(x)=\sqrt{(ax^2+2bx+c)}$, and

$$y=f(x), \quad y_0=f(x_0), \quad y_1=f(x_1), \quad X=(x_1-x_0)/(y_1+y_0),$$

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* In connection with Exs. 31–33 see Mr Bromwich's paper quoted on p. 236.

then $$\int_{x_0}^{x_1}\frac{dx}{y}=\frac{1}{\surd a}\log\frac{1+X\surd a}{1-X\surd a},\quad \frac{2}{\surd(-a)}\arctan\{X\surd(-a)\},$$
according as a is positive or negative. In the latter case the inverse tangent lies between 0 and $\frac{1}{2}\pi$. [It will be found that the substitution $t=(x-x_0)/(y+y_0)$ reduces the integral to the form $2\int_0^X\frac{dt}{1-at^2}$.]

34. Prove that $$\int_0^1\sqrt{\left(\frac{1-x}{1+x}\right)}\,dx=\tfrac{1}{2}\pi-1.$$

35. If $a>1$, then $$\int_{-1}^1\frac{\surd(1-x^2)}{a-x}\,dx=\pi\{a-\surd(a^2-1)\}.$$

36. If $p>1$, $0<q<1$, then
$$\int_0^1\frac{dx}{\surd[\{1+(p^2-1)x\}\{1-(1-q^2)x\}]}=\frac{2\omega}{(p+q)\sin\omega}$$
where ω is the positive acute angle whose cosine is $(1+pq)/(p+q)$.

37. If $a>b>0$, then $$\int_0^{2\pi}\frac{\sin^2\theta\,d\theta}{a-b\cos\theta}=\frac{2\pi}{b^2}\{a-\surd(a^2-b^2)\}.$$
(*Math. Trip.* 1904.)

38. Prove that, if $a>\surd(b^2+c^2)$ and $c>0$, then
$$\int_0^\pi\frac{d\theta}{a+b\cos\theta+c\sin\theta}=\frac{2}{\surd(a^2-b^2-c^2)}\arctan\{(1/c)\surd(a^2-b^2-c^2)\},$$
while if $c<0$ we must add $2\pi/\surd(a^2-b^2-c^2)$ to the result, the inverse tangent in each case being in absolute value less than $\frac{1}{2}\pi$.

39. If $f(x)$ is continuous and never negative, and $\int_a^b f(x)\,dx=0$, then $f(x)=0$ for $a\leqq x\leqq b$. [If $f(x)$ were equal to $k\,(k>0)$ for $x=\xi$, say, we could, in virtue of the continuity of f, find an interval $(\xi-\delta,\ \xi+\delta)$ throughout which $f(x)>\frac{1}{2}k$; and then the value of the integral would be greater than δk.]

40. If $P_n(x)=\dfrac{1}{(\beta-\alpha)^n n!}\left(\dfrac{d}{dx}\right)^n\{(x-\alpha)(\beta-x)\}^n$, $P_n(x)$ is a polynomial of degree n, which possesses the property that
$$\int_\alpha^\beta P_n(x)\,\theta(x)\,dx=0,$$
if $\theta(x)$ is any polynomial of degree less than n. [Integrate by parts $m+1$ times, if m is the degree of $\theta(x)$, and observe that $\theta^{(m+1)}(x)\equiv 0$.]

41. Prove that $\int_\alpha^\beta P_m(x)\,P_n(x)\,dx=0$, if $m\neq n$, but that if $m=n$ the value of the integral is $(\beta-\alpha)/(2n+1)$.

42. If $Q_n(x)$ is a polynomial of degree n, which possesses the property that $\int_\alpha^\beta Q_n(x)\,\theta(x)\,dx=0$ if $\theta(x)$ is any polynomial of degree less than n, then $Q_n(x)$ is a constant multiple of $P_n(x)$.

[We can choose κ so that $Q_n - \kappa P_n$ is of degree $n-1$: then

$$\int_\alpha^\beta Q_n(Q_n - \kappa P_n)\,dx = 0, \quad \int_\alpha^\beta P_n(Q_n - \kappa P_n)\,dx = 0,$$

and so

$$\int_\alpha^\beta (Q_n - \kappa P_n)^2\,dx = 0.$$

Now apply Ex. 39.]

43. **Approximate Values of definite integrals.** Show that the error in taking $\frac{1}{2}(b-a)\{\phi(a)+\phi(b)\}$ as the value of the integral $\int_a^b \phi(x)\,dx$ is less than $\frac{1}{12}M(b-a)^3$, where M is the maximum of $|\phi''(x)|$ in the interval (a, b); and the error in taking $(b-a)\,\phi\{\frac{1}{2}(a+b)\}$ is less than $\frac{1}{24}M(b-a)^3$.

[Write $f'(x) = \phi(x)$ in Exs. 4, 5.]

Show that the error in taking $\frac{1}{6}(b-a)\left\{\phi(a)+\phi(b)+4\phi\left(\frac{a+b}{2}\right)\right\}$ as the value is less than $\frac{1}{2880}M(b-a)^5$, where M is the maximum of $\phi^{(4)}(x)$.

[Use Ex. 6. This rule, which gives a very good approximation, is known as **Simpson's Rule.** It amounts to taking one-third of the first approximation given above and two-thirds of the second.]

Show that the approximation assigned by Simpson's Rule is the area bounded by the lines $x=a$, $x=b$, $y=0$ and a parabola with its axis parallel to OY and passing through the three points on the curve $y=\phi(x)$ whose abscissae are a, $\frac{1}{2}(a+b)$, b.

It should be observed that if $\phi(x)$ is any cubic polynomial, $\phi^{(4)}(x) \equiv 0$, and Simpson's Rule is exact. That is to say, given three points whose abscissae are a, $\frac{1}{2}(a+b)$, b, we can draw through them an infinity of curves of the type $y = \alpha + \beta x + \gamma x^2 + \delta x^3$; and *all such curves give the same area.* For one curve $\delta = 0$, and this curve is a parabola.

44. Apply Simpson's Rule to the calculation of π from the formula $\frac{1}{4}\pi = \int_0^1 \frac{dx}{1+x^2}$. [The result is ·7833... If we divide the integral into two, from 0 to $\frac{1}{2}$ and $\frac{1}{2}$ to 1, and apply Simpson's Rule to the two integrals separately, we obtain ·7853916.... The correct value is ·7853981....]

45. Show that $8{\cdot}9 < \int_3^5 \sqrt{(4+x^2)}\,dx < 9$. (*Math. Trip.* 1903.)

46. Calculate the integrals

$$\int_0^1 \frac{dx}{1+x}, \quad \int_0^1 \frac{dx}{\sqrt{(1+x^4)}}, \quad \int_0^\pi \sqrt{(\sin x)}\,dx, \quad \int_0^\pi \frac{\sin x}{x}\,dx,$$

to two places of decimals. [In the last integral the subject of integration is not defined for $x=0$: but if we assign to it, when $x=0$, the value 1, it becomes continuous throughout the range of integration.]

CHAPTER VIII.

THE CONVERGENCE OF INFINITE SERIES AND INFINITE INTEGRALS.

148. In Ch. IV we explained what was meant by saying that an infinite series is *convergent*, *divergent*, or *oscillatory*, and illustrated our definitions by a few simple examples, mainly derived from the geometrical series

$$1 + x + x^2 + \dots$$

and other series closely connected with it. In this chapter we shall pursue the subject in a more systematic manner, and prove a number of theorems which enable us to determine when the simplest series which commonly occur in analysis are convergent.

149. Series of Positive Terms. We pointed out in Ch. IV (§ 70) that the easiest type of series to consider from the point of view of its convergence is that in which all the terms are *positive* (using positive to include zero). We shall consider such series first, not only because they are the easiest to deal with, but also because the discussion of the convergence of a series containing negative or complex terms can often be made to depend upon a similar discussion for a series of positive terms only.

When we are discussing the convergence or divergence of any series we may disregard any finite number of terms. Thus if a series contains a finite number only of negative or complex terms, we may omit them, and apply the theorems which follow to the remainder.

150. It will be well to recall the following fundamental theorems established in § 70.

A. *A series of positive terms must be convergent or diverge to* $+\infty$, *and cannot oscillate.*

B. *The necessary and sufficient condition that* Σu_n *should be convergent is that there should be a number* G *such that*

$$u_0 + u_1 + \dots + u_n < G$$

for all values of n*.

C. **The comparison theorem.** *If* Σu_n *is convergent, and* $v_n \leqq u_n$ *for all values of* n, *then* Σv_n *is convergent, and* $\Sigma v_n \leqq \Sigma u_n$. *More generally*†, *if* $v_n \leqq K u_n$, *where* K *is a constant,* Σv_n *is convergent and* $\Sigma v_n \leqq K \Sigma u_n$. *And if* Σu_n *is divergent, and* $v_n \geqq K u_n$, *then* Σv_n *is divergent.*

Moreover, in inferring the convergence or divergence of Σv_n by means of one of these tests, it is sufficient to know that the test is satisfied for *sufficiently large* values of n (i.e. for all values of n greater than a definite value n_0). But of course the conclusion that $\Sigma v_n \leqq K \Sigma u_n$ does not necessarily hold in this case.

A particularly useful case of this theorem is

D. *If* Σu_n *is convergent (divergent) and* u_n/v_n *tends to a limit (other than zero) as* $n \rightarrow \infty$, *then* Σv_n *is convergent (divergent).*

151. First applications of these tests. The one important fact which we know at present, as regards the convergence of any special class of series, is that Σr^n is convergent if $r < 1$ and divergent if $r \geqq 1$. It is therefore natural to try to apply Theorem C, taking $u_n = r^n$. We at once find

1. *The series* Σv_n *is convergent if* $v_n \leqq K r^n$, *where* $r < 1$, *for all sufficiently large values of* n.

If $K = 1$, this condition may be written in the form $v_n^{1/n} \leqq r$. Hence we obtain what is known as **Cauchy's test** for the convergence of a series of positive terms; viz.

2. *The series* Σv_n *is convergent if* $v_n^{1/n} \leqq r$, *where* $r < 1$, *for all sufficiently large values of* n.

There is a corresponding test for divergence, viz.

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* It is of course a matter of indifference whether we denote our series by $u_1 + u_2 + \dots$ (as in Ch. IV) or by $u_0 + u_1 + \dots$ (as here). Later in this chapter we shall be concerned with series of the type $a_0 + a_1 x + a_2 x^2 + \dots$; for these the latter notation is clearly more convenient.

† Only the first part of this theorem was actually stated in § 70. The reader will be able to supply the proof of the second part.

2 *a*. *The series* Σv_n *is divergent if* $v_n^{1/n} \geqq 1$ *for an infinity of values of* n.

This hardly requires proof, for $v_n^{1/n} \geqq 1$ involves $v_n \geqq 1$. The two theorems 2 and 2 *a* are of very wide application, but for some purposes it is more convenient to use a different test of convergence, viz.

3. *The series* Σv_n *is convergent if* $v_{n+1}/v_n \leqq r$, *where* $r < 1$, *for all sufficiently large values of* n.

To prove this we observe that if $v_{n+1}/v_n \leqq r$ for $n \geqq n_0$, then

$$v_n = \frac{v_n}{v_{n-1}} \frac{v_{n-1}}{v_{n-2}} \dots \frac{v_{n_0+1}}{v_{n_0}} v_{n_0} \leqq r^{n-n_0} v_{n_0};$$

and the result follows by comparison with the convergent series Σr^n.

This is known as **d'Alembert's test**. We shall see later that it is less general, theoretically, than Cauchy's, in that Cauchy's test can be applied whenever d'Alembert's can, and sometimes when the latter cannot. Moreover the test for divergence which corresponds to d'Alembert's test for convergence is much less general than the test given by Theorem 2 *a*. It is true, as the reader will easily prove for himself, that if $v_{n+1}/v_n \geqq r > 1$ for *all* values of n, or all sufficiently large values, then Σv_n is divergent. But it is not true (see Ex. LXIX. 10) that this is so if only $v_{n+1}/v_n \geqq r > 1$ for *an infinity of values of* n, whereas in Theorem 2 *a* our test had only to be satisfied for such an infinity of values. None the less d'Alembert's test is very useful in practice, because when v_n is a complicated function v_{n+1}/v_n is often much less complicated, and so easier to work with.

In the simplest cases which occur in analysis it often happens that v_{n+1}/v_n or $v_n^{1/n}$ tends to a limit as $n \rightarrow \infty$ *. If this limit is less than 1 it is evident that the Theorems 2 or 3 above are satisfied. Thus

4. *If* $v_n^{1/n}$ *or* v_{n+1}/v_n *tends to a limit less than unity as* $n \rightarrow \infty$, *the series* Σv_n *is convergent.*

It is almost obvious that if either function tend to a limit greater than unity, then Σv_n is divergent. We leave the formal proof of this as an exercise to the reader. But when $v_n^{1/n}$ or v_{n+1}/v_n tends to 1 all these tests fail completely, and they fail also when $v_n^{1/n}$ or v_{n+1}/v_n oscillates in such a way that, while always less than 1, it assumes for an infinity of values of n values approaching indefinitely near to 1. And the tests which involve v_{n+1}/v_n fail even when that ratio oscillates so as to be sometimes less than and sometimes greater than 1. When $v_n^{1/n}$ behaves in this way Theorem 2 *a* is sufficient to prove the divergence of the series. But it is clear that there is a wide margin of cases in which some more subtle tests will be needed.

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* It will be proved in Ch. IX (Ex. LXXXIX. 33) that if $v_{n+1}/v_n \rightarrow l$, then $v_n^{1/n} \rightarrow l$, but that the converse is not true.

Examples LXIX. 1. Apply Cauchy's and d'Alembert's tests (as specialised in Theorem 4 above) to the series Σnr^n.

[Here $v_{n+1}/v_n = \{(n+1)/n\}r \to r$, so that d'Alembert's test shows at once that the series is convergent if $r<1$, divergent if $r>1$. The test fails if $r=1$: but the series is then obviously divergent. Since $\lim n^{1/n}=1$ (Ex. XXX. 11) Cauchy's test leads at once to the same conclusions.]

2. Apply these tests to $\Sigma n^2 r^n$, $\Sigma r^n/n$, $\Sigma r^n/n^2$.

3. Consider the series $\Sigma\,(An^k+Bn^{k-1}+\dots+K)\,r^n$.

[Let the coefficient of r^n be denoted by $P(n)$. Then if A is positive, as we may suppose, $P(n)$ is positive for all sufficiently large values of n; and since

$$\lim\{P(n+1)/P(n)\}=1$$

(Ex. XXIX. 4), d'Alembert's test gives convergence or divergence according as $r<1$ or $r>1$. When $r=1$ the series is obviously divergent.]

4. Consider $$\Sigma\frac{An^k+Bn^{k-1}+\dots+K}{\alpha n^l+\beta n^{l-1}+\dots+\lambda}r^n.$$

[D'Alembert's test gives the same results as in Ex. 3, when $r \neq 1$. The case in which $r=1$ requires further consideration. It is evident that the series is divergent if $r=1$ and $k \geqq l$.]

5. We have seen (Ch. IV, Misc. Ex. 18) that the series

$$\Sigma\,1/\{n(n+1)\},\quad \Sigma\,1/\{n(n+1)\dots(n+p)\}$$

are convergent. Show that Cauchy's and d'Alembert's tests both fail when applied to them. [For $\lim u_n^{1/n}=\lim(u_{n+1}/u_n)=1$.]

6. Show that the series Σn^{-p}, where p is an integer not less than 2, is convergent.

[Since $\lim\{n(n+1)\dots(n+p-1)\}/n^p=1$, this follows from the convergence of the series considered in Ex. 5. It has already been shown (§ 70) that the series is divergent if $p=1$.]

7. Show that the series

$$\Sigma\frac{An^k+Bn^{k-1}+\dots+K}{\alpha n^l+\beta n^{l-1}+\dots+\lambda},$$

is convergent if $l \geqq k+2$, but divergent if $l=k+1$. [If v_n is the general term of the series, $\lim n^{l-k}v_n=A/\alpha$.]

8. If m_n is a positive integer, and $m_{n+1} > m_n$, the series Σr^{m_n} is convergent if $r < 1$, divergent if $r \geqq 1$. For example the series $1 + r + r^4 + r^9 + \dots$ is convergent if $r < 1$, divergent if $r \geqq 1$.

9. Sum the series $1 + 2q + 2q^4 + \dots$ to 24 places of decimals when $q = \cdot 1$ and to 2 places when $q = \cdot 9$. [If $q = \cdot 1$ the first 5 terms give the sum $1 \cdot 200200002000000200000000$ and the error is

$$2q^{25} + 2q^{36} + \dots < 2q^{25} + 2q^{36} + 2q^{47} + \dots = 2q^{25}/(1 - q^{11}) < 3/10^{25}.$$

If $q = \cdot 9$ the first 8 terms give the sum $5 \cdot 458\dots$, and the error is less than $2q^{64}/(1 - q^{17}) < \cdot 003$.]

10. If $0 < a < b < 1$, the series $a + b + a^2 + b^2 + a^3 + \dots$ is convergent. Show that Cauchy's test may be applied to this series, but that d'Alembert's test fails. [For

$$v_{2n+1}/v_{2n} = (b/a)^{n+1} \to +\infty, \quad v_{2n+2}/v_{2n+1} = b\,(a/b)^{n+2} \to 0.]$$

11. The series $1 + x + \dfrac{x^2}{2\,!} + \dfrac{x^3}{3\,!} + \dots$ and $1 + x + \dfrac{x^2}{2^2} + \dfrac{x^3}{3^3} + \dots$ are convergent for all positive values of x.

12. If Σu_n is convergent so is $\Sigma u_n/(1 + u_n)$: and $\Sigma u_n/(1 - u_n)$ is also convergent if $u_n < 1$.

13. If Σu_n is convergent so is Σu_n^2.

14. If Σu_n^2 is convergent so is $\Sigma u_n/n$. [For $u_n^2 + (1/n^2) \geqq 2u_n/n$ and $\Sigma(1/n^2)$ is convergent.]

15. Show that $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dots = \dfrac{3}{4}\left(1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dots\right)$ and

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots = \frac{15}{16}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right).$$

[To prove the first result we note that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \left(1 + \frac{1}{2^2}\right) + \left(\frac{1}{3^2} + \frac{1}{4^2}\right) + \dots$$

$$= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right),$$

by Theorems (8) and (6) of § 70.]

16. Prove by a *reductio ad absurdum* that $\Sigma(1/n)$ is divergent. [If the series were convergent we should have, by the argument used in Ex. 15,

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots = (1 + \tfrac{1}{3} + \tfrac{1}{5} + \dots) + \tfrac{1}{2}(1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots),$$

or

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \dots = 1 + \tfrac{1}{3} + \tfrac{1}{5} + \dots$$

which is obviously absurd, since every term of the first series is less than the corresponding term of the second.]

152. Before proceeding further in the investigation of tests of convergence and divergence, we shall prove an important general theorem concerning series of positive terms.

Dirichlet's Theorem*. *The sum of a series of positive terms is the same in whatever order the terms are taken.*

This theorem asserts that if we have a convergent series of positive terms, $u_0 + u_1 + u_2 + \dots$ say, and form any other series

$$v_0 + v_1 + v_2 + \dots$$

out of *the same terms*, by taking them in any new order, the second series is convergent and has the same sum as the first. Of course no terms must be omitted: i.e. every u must come somewhere among the v's, and *vice versa*.

The proof is extremely simple. Let s be the sum of the series of u's. Then the sum of any number of terms, selected from the u's, is not greater than s. But every v is a u, and therefore the sum of any number of terms selected from the v's is not greater than s. Hence Σv_n is convergent, and its sum t is not greater than s. But we can show in exactly the same way that $s \leqq t$. Thus $s = t$.

153. Multiplication of Series of Positive Terms. An immediate corollary from Dirichlet's Theorem is the following theorem: *if* $u_0 + u_1 + u_2 + \dots$ *and* $v_0 + v_1 + v_2 + \dots$ *are two convergent series of positive terms, and* s, t *are their respective sums, then the series*

$$u_0v_0 + (u_1v_0 + u_0v_1) + (u_2v_0 + u_1v_1 + u_0v_2) + \dots$$

is convergent and has the sum st.

Arrange all the possible products of pairs u_mv_n in the form of a doubly infinite array

$$\begin{array}{ccccc} u_0v_0 & u_1v_0 & u_2v_0 & u_3v_0 & \dots \\ u_0v_1 & u_1v_1 & u_2v_1 & u_3v_1 & \dots \\ u_0v_2 & u_1v_2 & u_2v_2 & u_3v_2 & \dots \\ u_0v_3 & u_1v_3 & u_2v_3 & u_3v_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

We can arrange these terms in the form of a simply infinite series in a variety of ways. Among these are the following.

(1) We begin with the single term u_0v_0 for which $m + n = 0$; then we take the two terms u_1v_0, u_0v_1 for which $m + n = 1$; then

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* This theorem seems to have first been stated explicitly by Dirichlet in 1837. It was no doubt known to earlier writers, in particular to Cauchy.

the three terms u_2v_0, u_1v_1, u_0v_2 for which $m+n=2$; and so on. We thus obtain the series

$$u_0v_0 + (u_1v_0 + u_0v_1) + (u_2v_0 + u_1v_1 + u_0v_2) + \dots$$

of the theorem.

(2) We begin with the single term u_0v_0 for which both suffixes are zero; then we take the terms u_1v_0, u_1v_1, u_0v_1 which involve a suffix 1 but no higher suffix; then the terms u_2v_0, u_2v_1, u_2v_2, u_1v_2, u_0v_2 which involve a suffix 2, but no higher suffix; and so on. The sums of these groups of terms are respectively equal to

$$u_0v_0,\ (u_0+u_1)(v_0+v_1) - u_0v_0,$$
$$(u_0+u_1+u_2)(v_0+v_1+v_2) - (u_0+u_1)(u_0+u_2),\ \dots$$

and the sum of the first $n+1$ groups is

$$(u_0+u_1+\dots+u_n)(v_0+v_1+\dots+v_n),$$

and tends to *st* as $n \to \infty$. When the sum of the series is formed in this manner the sum of the first one, two, three, ... groups comprises all the terms in the first, second, third, ... rectangles indicated in the diagram on p. 301.

The sum of the series formed in the second manner is *st*. But the two series are rearrangements of one another; and therefore, by Dirichlet's Theorem, the sum of the first series is also *st*. Thus the theorem is proved.

Examples LXX. 1. Verify that if $r<1$

$$1+r^2+r+r^4+r^6+r^3+\dots = 1+r+r^3+r^2+r^5+r^7+\dots = 1/(1-r).$$

2. If either of the series $u_0+u_1+\dots$, $v_0+v_1+\dots$ is divergent, so is the series $u_0v_0+(u_1v_0+u_0v_1)+(u_2v_0+u_1v_1+u_0v_2)+\dots$

3. If the series $u_0+u_1+\dots$, $v_0+v_1+\dots$, $w_0+w_1+\dots$ (all of whose terms are positive) converge to sums r, s, t, the series $\sum\limits_{k=0}^{\infty}\lambda_k$, where $\lambda_k = \Sigma u_m v_n w_p$, the summation being extended to all sets of values of m, n, p such that $m+n+p=k$, converges to the sum rst.

154. Further tests for convergence and divergence. The examples which precede will suffice to show how many simple and interesting types of series of positive terms cannot be dealt with by the general tests of § 151. In fact, if we consider the simplest type of series, in which u_{n+1}/u_n tends to a limit as $n \to \infty$, *the tests of* § 151 *fail completely when this limit is* 1. Thus in Ex. LXIX. 6, 7 these tests failed, and we had to fall back upon a special device, which was in essence that of using the series $\Sigma 1/\{n(n+1)\dots(n+p)\}$ as our comparison series, instead of the geometric series.

The fact is that the geometric series, by comparison with which the tests of § 151 were obtained, is not only convergent but *very rapidly* convergent, far more rapidly than is necessary in order to ensure convergence. The tests derived from comparison with it are therefore naturally very crude, and far more delicate tests are often wanted.

It was proved in Ex. XXX. 7 that $n^k r^n \rightarrow 0$ as $n \rightarrow \infty$, provided $r<1$, whatever value k may have; we saw indeed in Ex. LXIX. 3 that not only does $n^k r^n$ tend to zero but the series $\Sigma n^k r^n$ is convergent. It follows that the sequence $1, r, r^2, \dots, r^n, \dots$, where $r<1$, diminishes more rapidly than the sequence $1, 2^{-k}, 3^{-k}, \dots, n^{-k}, \dots$. This seems at first paradoxical if r is not much less than unity, and k is large. Thus of the two sequences

$$\tfrac{2}{3}, \tfrac{4}{9}, \tfrac{8}{27}, \tfrac{16}{81}, \dots; \quad 1, \tfrac{1}{4096}, \tfrac{1}{531441}, \dots$$

whose general terms are $(\frac{2}{3})^n$ and n^{-12}, the second seems at first sight to decrease far more rapidly. But this is far from being the case; if only we go far enough into the sequences we shall find the terms of the first sequence very much the smaller. Thus

$$(2/3)^4 = 16/81 < 1/5, \quad (2/3)^{12} < (1/5)^3 < (1/10)^2, \quad (2/3)^{1000} < (1/10)^{166},$$

while

$$1000^{-12} = 10^{-36}.$$

Thus the 1000th term of the first sequence is less than the 10^{130}th part of the corresponding term of the second sequence, and so the series $\Sigma(2/3)^n$ is far more rapidly convergent than the series Σn^{-12}, and even this series is very much more rapidly convergent than Σn^{-2}*.

155. We shall proceed to establish two further tests for the convergence or divergence of series of positive terms, **Maclaurin's (or Cauchy's) Integral Test** and **Cauchy's Condensation Test**, which, though very far from being completely general, are sufficiently general for our needs in this chapter.

In applying either of these tests we make a further assumption as to the nature of the function u_n, about which we have so far assumed only that it is positive. We assume that *u_n decreases with n*: i.e. $u_{n+1} \leqq u_n$ for all values of n (or at any rate all sufficiently large values).

This condition is satisfied in all the most important cases. From one point of view it may be regarded as no restriction at all, so long as we are dealing with series of positive terms—for in virtue of Dirichlet's theorem above we may rearrange the terms without affecting the question of convergence or divergence: and there is nothing to prevent us rearranging the terms in *descending order of magnitude*, and applying our tests to the series of decreasing terms thus obtained.

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* Five terms suffice to give the sum of Σn^{-12} correctly to 7 places of decimals, whereas some 10,000,000 are needed to give an equally good approximation to Σn^{-2}.

But before we proceed to the statement of these two tests, we shall state and prove a very simple and important theorem, **Pringsheim's Theorem.** This is a *one-sided* theorem in that it gives a sufficient test for divergence only and not for convergence, but it is essentially of a more elementary character than the two theorems mentioned above.

156. Pringsheim's Theorem. *If* Σu_n *is a convergent series of positive and decreasing terms, then* $\lim nu_n = 0$.

This is easily proved by a *reductio ad absurdum*. To say that $nu_n \to 0$ is the same thing as to say that, given any positive number δ, however small, we can find n_0 so that $nu_n < \delta$ for $n \geqq n_0$. If this is not the case there must be *some* such positive number δ for which *no* choice of n_0 will ensure the satisfaction of this inequality; i.e. such that there is an infinite sequence of values of n for which $nu_n \geqq \delta$. Let n_1 be the first such value of n; n_2 the next such value of n which is more than twice as large as n_1; n_3 the next such value of n which is more than twice as large as n_2; and so on. Then we have a sequence of numbers n_1, n_2, n_3, ... such that $n_2 > 2n_1$, $n_3 > 2n_2$, ... and so $n_2 - n_1 > \frac{1}{2}n_2$, $n_3 - n_1 > \frac{1}{2}n_3$, ...; and also

$$n_1 u_{n_1} \geqq \delta, \quad n_2 u_{n_2} \geqq \delta, \quad \ldots.$$

But since u_n decreases as n increases

$$\begin{aligned} u_0 + u_1 + \ldots + u_{n_1 - 1} &\geqq n_1 u_{n_1} \geqq \delta, \\ u_{n_1} + \ldots + u_{n_2 - 1} &\geqq (n_2 - n_1)\, u_{n_2} > \tfrac{1}{2} n_2 u_{n_2} > \tfrac{1}{2}\delta, \\ u_{n_2} + \ldots + u_{n_3 - 1} &\geqq (n_3 - n_2)\, u_{n_3} > \tfrac{1}{2} n_3 u_{n_3} > \tfrac{1}{2}\delta, \end{aligned}$$

and so on; and we can bracket the terms of the series Σu_n so as to obtain a new series whose terms are severally greater than those of the divergent series

$$\delta + \tfrac{1}{2}\delta + \tfrac{1}{2}\delta + \ldots:$$

and therefore Σu_n is divergent.

An alternative proof is as follows. Since $u_0 + u_1 + \ldots + u_n$ tends to a limit s, we can choose n_0 so that

$$s - \delta < u_0 + u_1 + \ldots + u_n \leqq s$$

for $n \geqq n_0$, however small be the positive number δ. Hence

$$u_{n+1} + u_{n+2} + \ldots < \delta,$$

and *a fortiori*

$$u_{n+1}+u_{n+2}+\dots+u_{2n}<\delta;$$

and still more $nu_{2n}<\delta$, for $n \geqq n_0$. Therefore $nu_{2n}\to 0$, and therefore $2nu_{2n}\to 0$. And since

$$(2n+1)\,u_{2n+1}<(2n+1)\,u_{2n}<\{(2n+1)/2n\}\,2n\,u_{2n},$$

it is clear also that $(2n+1)\,u_{2n+1}\to 0$. Hence $nu_n\to 0$ as $n\to\infty$ through either odd or even values, and therefore as $n\to\infty$ in any manner.

Examples LXXI. 1. Use Pringsheim's theorem to show that $\Sigma(1/n)$ and $\Sigma\{1/(an+b)\}$ are divergent. [Here $nu_n\to 1$ or $1/a$.]

2. Show that $\Sigma(1/n^s)$ is divergent if $s \leqq 1$. [Compare with $\Sigma(1/n)$: or use Pringsheim's theorem directly.]

3. Show that Pringsheim's theorem is not true if we omit the condition that u_n decreases as n increases. [The series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{9}+\frac{1}{10^2}+\dots,$$

in which $u_n=1/n$ or $1/n^2$, according as n is or is not a perfect square, is convergent, since it may be rearranged in the form

$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{10^2}+\dots+\left(1+\frac{1}{4}+\frac{1}{9}+\dots\right),$$

and each of these series is convergent. But since $nu_n=1$ whenever n is a perfect square, it is clearly not true that $nu_n\to 0$.]

4. *The converse of Pringsheim's theorem is not true*, i.e. it is not true that, if u_n decreases with n and $\lim nu_n=0$, then Σu_n is convergent.

[Take the series $\Sigma(1/n)$ and multiply the first term by 1, the second by $\frac{1}{2}$, the next two by $\frac{1}{3}$, the next four by $\frac{1}{4}$, the next eight by $\frac{1}{5}$, and so on. On grouping in brackets the terms of the new series thus formed we obtain

$$1+\tfrac{1}{2}\cdot\tfrac{1}{2}+\tfrac{1}{3}(\tfrac{1}{3}+\tfrac{1}{4})+\tfrac{1}{4}(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8})+\dots;$$

and this series is divergent, since its terms are greater than those of

$$1+\tfrac{1}{2}\cdot\tfrac{1}{2}+\tfrac{1}{3}\cdot\tfrac{1}{2}+\tfrac{1}{4}\cdot\tfrac{1}{2}+\dots,$$

which is divergent. But it is easy to see that, in the series

$$1+\tfrac{1}{2}\cdot\tfrac{1}{2}+\tfrac{1}{3}\cdot\tfrac{1}{3}+\tfrac{1}{3}\cdot\tfrac{1}{4}+\tfrac{1}{4}\cdot\tfrac{1}{5}+\tfrac{1}{4}\cdot\tfrac{1}{6}+\dots,$$

$nu_n\to 0$. In fact $nu_n=1/\nu$ if $2^\nu<n\leqq 2^{\nu+1}$, and $\nu\to\infty$ with n.]

157. Maclaurin's (or Cauchy's) Integral Test*. If u_n decreases as n increases, we can write $u_n=\phi(n)$, and can suppose that $\phi(n)$ is the value assumed when $x=n$ by a continuous and decreasing function $\phi(x)$ of the continuous variable x. Then

$$\phi(1)\geqq\int_1^2\phi(x)\,dx\geqq\phi(2),$$

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* The test was discovered by Maclaurin but forgotten, and rediscovered by Cauchy, to whom it is usually attributed.

$$\phi(2) \geqq \int_2^3 \phi(x)\,dx \geqq \phi(3),$$

……………………………………

$$\phi(n-1) \geqq \int_{n-1}^{n} \phi(x)\,dx \geqq \phi(n),$$

since $\phi(n-1)$ is the greatest and $\phi(n)$ the least of the values assumed by $\phi(x)$ in the interval $(n-1, n)$.

We have to use the sign $\geqq$, and not simply the sign $>$, in the inequalities above, because $\phi(\nu)$ may be equal to $\phi(\nu+1)$ for some values of ν, in which case $\phi(x)$ will obviously be constant in the interval $(\nu, \nu+1)$.

Hence by addition

$$\phi(1) + \phi(2) + \dots + \phi(n-1) \geqq \int_1^n \phi(x)\,dx$$
$$\geqq \phi(2) + \phi(3) + \dots + \phi(n).$$

Let us write

$$\Phi(\xi) = \int_1^{\xi} \phi(x)\,dx.$$

Then, since $\phi(x) \geqq 0$, $\Phi(\xi)$ is an increasing function of ξ, and so either (i) $\Phi(\xi)$ tends to a limit l as $\xi \to +\infty$, or (ii) $\Phi(\xi) \to +\infty$.

In the first case

$$\phi(2) + \phi(3) + \dots + \phi(n) \leqq l$$

and so $\phi(2) + \phi(3) + \dots$ converges to a sum not exceeding l; and therefore $\phi(1) + \phi(2) + \dots$ converges to a sum not exceeding $\phi(1) + l$.

In the second case, since

$$\phi(1) + \phi(2) + \dots + \phi(n-1) \geqq \Phi(n),$$

it is evident that the series is divergent. Hence *if $\phi(x)$ is a function of x which is positive and continuous for all values of x greater than unity, and steadily decreases as x increases, the series*

$$\phi(1) + \phi(2) + \dots$$

will or will not converge according as

$$\Phi(\xi) = \int_1^{\xi} \phi(x)\,dx$$

does or does not tend to a limit l as $\xi \to +\infty$: and, in the first case, the sum of the series will be not greater than $\phi(1) + l$.

Examples LXXII. 1. This argument may be put in a geometrical form. In the figure $\Phi(n)$ is the area $PpqQ$: while $\phi(1)+\phi(2)+\dots+\phi(n-1)$ is the sum of the exterior rectangles (bounded by a thick line) and $\phi(2)+\phi(3)+\dots+\phi(n)$ is the sum of the interior rectangles (shaded). The inequalities used above follow from a glance at the figure.

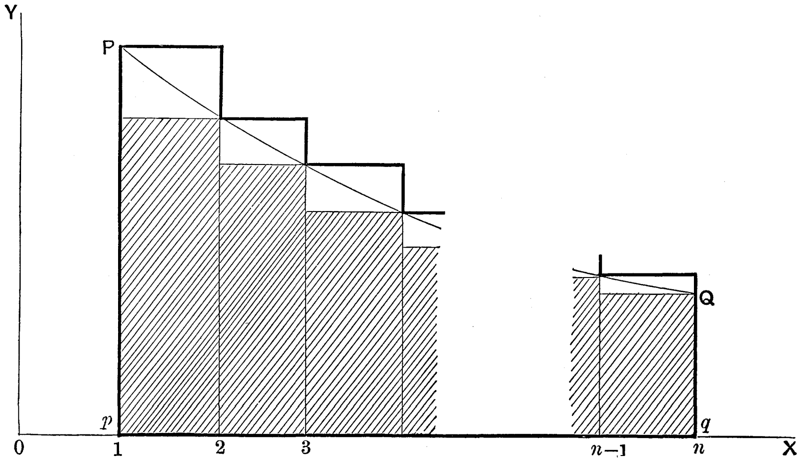

Fig. 64.

2. The convergence of either of the series

$$\phi(1)+\phi(3)+\phi(5)+\dots, \quad \phi(2)+\phi(4)+\phi(6)+\dots$$

is a necessary and sufficient condition for that of $\phi(1)+\phi(2)+\phi(3)+\dots$.

[By using (1, 3), (3, 5), ... as intervals of integration we can show as above that $\phi(1)+\phi(3)+\dots$ will or will not converge according as $\Phi(\xi)\to l$ or $\Phi(\xi)\to+\infty$. Or we may use the inequalities

$$2\phi(1) > \phi(1)+\phi(2), \quad \phi(2)+\phi(3) > 2\phi(3) > \phi(3)+\phi(4), \dots.$$

For a more general result see Ex. LXXVI. 3.]

3. Prove that $\displaystyle\sum_1^\infty \frac{1}{n^2+1} < 1+\tfrac{1}{4}\pi.$

158. The series Σn^{-s}. By far the most important application of the Integral Test is to the series

$$1^{-s}+2^{-s}+3^{-s}+\dots+n^{-s}+\dots,$$

where s is any rational number.

If $s \leqq 0$ it is obvious that the series is divergent. If $s > 0$ u_n decreases as n increases, and we can apply the test. Here (unless $s = 1$)

$$\Phi(\xi) = \int_1^\xi \frac{dx}{x^s} = \frac{\xi^{1-s}-1}{1-s}.$$

If $s > 1$, $\xi^{1-s} \to 0$ as $\xi \to +\infty$, and $\Phi(\xi) \to 1/(s-1) = l$, say. But if $s < 1$, $\xi^{1-s} \to +\infty$ with x, and so $\Phi(\xi) \to +\infty$. Hence we derive the result

The series Σn^{-s} *is convergent if* $s > 1$, *divergent if* $s < 1$: *if* $s > 1$ *its sum is less than* $s/(s-1)$.

So far as the divergence for $s < 1$ is concerned, this result might (as in Ex. LXXI. 2) have been derived at once from comparison with $\Sigma(1/n)$, which we already know to be divergent.

It is however interesting to see how the Integral Test may be applied to the series $\Sigma(1/n)$ (when the preceding work fails). In this case

$$\Phi(\xi) = \int_1^{\xi} \frac{dx}{x},$$

and it is easy to see that $\Phi(\xi) \to +\infty$ with ξ. For, if $\xi > 2^n$, then

$$\Phi(\xi) > \int_1^{2^n} \frac{dx}{x} = \int_1^2 \frac{dx}{x} + \int_2^4 \frac{dx}{x} + \dots + \int_{2^{n-1}}^{2^n} \frac{dx}{x};$$

and since (putting $x = 2^r u$)

$$\int_{2^r}^{2^{r+1}} \frac{dx}{x} = \int_1^2 \frac{du}{u},$$

we obtain $\Phi(\xi) > n \displaystyle\int_1^2 \frac{du}{u}$, which shows that $\Phi(\xi) \to +\infty$ with ξ.

Examples LXXIII. 1. Prove by an argument similar to that used above, and without integration, that $\Phi(\xi) = \displaystyle\int_1^{\xi} \frac{dx}{x^s}$, where $s < 1$, tends to $+\infty$ with ξ.

2. The series Σn^{-2}, $\Sigma n^{-3/2}$, $\Sigma n^{-11/10}$ are convergent, and their sums are not greater than 2, 3, 11 respectively. The series $\Sigma n^{-1/2}$, $\Sigma n^{-10/11}$ are divergent.

3. The series $\Sigma n^s/(n^t + a)$, where $a > 0$, is convergent or divergent according as $t > 1 + s$ or $t \leqq 1 + s$. [Compare with Σn^{s-t}.]

4. Discuss the convergence or divergence of the series

$$\Sigma(a_1 n^{s_1} + a_2 n^{s_2} + \dots + a_k n^{s_k})/(b_1 n^{t_1} + b_2 n^{t_2} + \dots + b_l n^{t_l}),$$

where all the letters denote positive numbers and the s's and t's are arranged in descending order of magnitude.

5. If $\phi(n) \to l > 1$ the series $\Sigma n^{-\phi(n)}$ is convergent. If $\phi(n) \to l < 1$ it is divergent.

159. Cauchy's Condensation Test. The second of the two tests mentioned in § 155 is as follows: *if* $u_n = \phi(n)$ *is a decreasing function of* n, *the series* $\Sigma\phi(n)$ *converges or diverges according as* $\Sigma 2^n \phi(2^n)$ *converges or diverges.*

We can prove this by an argument which we have already used (Ch. IV, § 70) in the special case of the series $\Sigma(1/n)$.

In the first place

$$\phi(3) + \phi(4) \geqq 2\phi(4),$$
$$\phi(5) + \phi(6) + \dots + \phi(8) \geqq 4\phi(8),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\phi(2^n + 1) + \phi(2^n + 2) + \dots + \phi(2^{n+1}) \geqq 2^n \phi(2^{n+1}),$$

inequalities which make it obvious that if $\Sigma 2^n \phi(2^{n+1})$ or, what is the same thing, $\Sigma 2^{n+1} \phi(2^{n+1})$ or $\Sigma 2^n \phi(2^n)$ diverges so does $\Sigma\phi(n)$.

On the other hand

$$\phi(2) + \phi(3) \leqq 2\phi(2), \quad \phi(4) + \phi(5) + \dots + \phi(7) \leqq 4\phi(4),$$

and so on. And from this set of inequalities it follows that if $\Sigma 2^n \phi(2^n)$ converges, so does $\Sigma\phi(n)$. Thus the theorem is established.

For our present purposes the field of application of this test is practically the same as that of the Integral Test. It enables us to discuss the series Σn^{-s} with equal ease. For Σn^{-s} will converge or diverge according as $\Sigma 2^n 2^{-ns}$ converges or diverges: i.e. according as $s > 1$ or $s \leqq 1$.

Examples LXXIV. 1. Show that if a is any positive integer greater than 1, the series $\Sigma\phi(n)$ converges or diverges with $\Sigma a^n \phi(a^n)$. [Use the same arguments as above, taking groups of a, a^2, a^3, ... terms.]

2. If $\Sigma\phi(n)$ converges it is obvious that $\lim 2^n \phi(2^n) = 0$. Hence deduce Pringsheim's Theorem.

160. Infinite Integrals. The Integral Test of § 157 shows that if $\phi(x)$ is a positive and decreasing function of x, the series $\Sigma\phi(n)$ is convergent or divergent according as the integral function $\Phi(x)$ does or does not tend to a limit as $x \to +\infty$. Let us suppose that it does tend to a limit, and that

$$\lim_{x \to +\infty} \int_1^x \phi(t)\,dt = l.$$

Then we shall say that *the integral*

$$\int_1^\infty \phi(t)\,dt$$

is **convergent,** *and has the value* l; and we shall call the integral an **infinite integral.**

So far we have supposed $\phi(t)$ positive and decreasing. But it is natural to extend our definition to other cases. Nor is there any special point in supposing the lower limit to be unity. We are accordingly led to formulate the following definition:

If $\phi(t)$ is a function of t which satisfies the conditions of §§ 137 et seq. throughout the interval (a, x), where x is any number greater than or equal to a, and if

$$\lim_{x \to +\infty} \int_a^x \phi(t)\,dt = l,$$

we shall say that the infinite integral

$$\int_a^\infty \phi(t)\,dt \qquad \text{..........................(1)}$$

is convergent and has the value l.

The ordinary integral between limits a and A, as defined in Ch. VII, we shall sometimes in contrast call a **finite** integral.

On the other hand when

$$\int_a^x \phi(t)\,dt \to +\infty$$

we shall say that the integral *diverges* to $+\infty$. Similarly we define divergence to $-\infty$. Finally, when none of these alternatives occur, we shall say that as $x \to +\infty$ the integral *oscillates* (finitely or infinitely). The following remarks suggest themselves.

(i) If we write
$$\int_a^x \phi(t)\,dt = \Phi(x),$$
the definitions of convergence, divergence, and oscillation of the integral are precisely the same as those already formulated (Ch. V, § 77) for the behaviour of $\Phi(x)$ as $x \to +\infty$.

(ii) In the special case in which $\phi(t)$ is always positive (or zero) it is clear that $\Phi(x)$ is an increasing function of x. Hence the only alternatives are convergence and divergence to $+\infty$.

(iii) The integral (1) of course depends on a, but is quite independent of t, and is in no way altered by the substitution of any other letter for t. In this respect the infinite integral only resembles the finite integral (cf. Ch. VII, § 142).

(iv) Of course the reader will not be puzzled by the use of the term *infinite integral* to denote something whose value is a definite number such as 2 or $\frac{1}{2}\pi$. The word *infinite* is used to distinguish the infinite integral, in which the variable of integration ranges through all values greater than a

or, as it is often expressed, whose 'upper limit is infinite,' from the finite integral, just as it is used to distinguish an infinite series from a finite series: no one supposes that an infinite series is necessarily divergent.

(v) The integral $\int_a^x \phi(t)\,dt$ was defined in Ch. VII, § 139, as a *simple limit*, the limit of a finite sum $\Sigma f(x_\nu)(x_{\nu+1}-x_\nu)$. The infinite integral is therefore *the limit of a limit*, or what is known as *a repeated limit*. The notion of the infinite integral is in fact an essentially more complex notion than that of the finite integral, of which it is a development.

(vi) The Integral Test of § 157 may now be stated in the form: *if $\phi(x)$ is positive and steadily decreases as x increases, the infinite series $\Sigma\phi(n)$ and the infinite integral $\int_1^\infty \phi(x)\,dx$ converge or diverge together.*

161. The case in which $\phi(x)$ is positive. It is natural to consider what are the general theorems concerning the convergence or divergence of the infinite integral (1) and analogous to theorems A—D (§ 150 above). That A is true of integrals as well as of series we have already seen (§ 160 (ii)). Corresponding to B we obviously have the theorem that *the necessary and sufficient condition for the convergence of the integral* (1) *is that it should be possible to find a constant K such that*

$$\int_a^x \phi(x)\,dx < K$$

for all values of x greater than a.

Similarly, corresponding to C, we have the theorem: *if $\int_a^\infty \phi(x)\,dx$ is convergent, and $\psi(x) \leqq K\phi(x)$ for all values of x greater than a, then $\int_a^\infty \psi(x)\,dx$ is convergent, and*

$$\int_a^\infty \psi(x)\,dx \leqq K\int_a^\infty \phi(x)\,dx.$$

We leave it to the reader to formulate the corresponding test for divergence.

We may observe that D'Alembert's test (§ 151, 3), depending as it does on the notion of successive terms, has no analogue for integrals: and the analogue of Cauchy's test (§ 151, 1) is not of much importance, and in any case could only be formulated when we have investigated in greater detail the theory of the function

$\phi(x) = r^x$, as we shall do in Ch. IX. And the most important particular tests are the following, which are derived at once by comparison with the integral

$$\int_a^\infty \frac{dx}{x^s} \quad (a > 0)$$

(whose convergence or divergence we have investigated in § 158): viz.

If, for $x \geqq a$, $\phi(x) < Kx^{-s}$, *where* $s > 1$, *then* $\int_a^\infty \phi(x)\,dx$ *is convergent: if on the other hand, for* $x \geqq a$, $\phi(x) > Kx^{-s}$, *where* $s \leqq 1$, *the integral is divergent. In particular, if* $\lim x^s\phi(x) = l$, *where* $l > 0$, *the integral is convergent or divergent according as* $s > 1$ *or* $s \leqq 1$.

There is one exceedingly fundamental and important property of a convergent infinite series, the obvious analogue of which is not necessarily true of a convergent infinite integral. If $\Sigma\phi(n)$ is convergent then $\phi(n) \to 0$; but it is *not* always true, even when $\phi(x)$ is always positive, that if $\int_a^\infty \phi(x)\,dx$ is convergent then $\phi(x) \to 0$.

Consider for example the function $\phi(x)$ whose graph is indicated by the thick line in the figure. Here the height of the peaks corresponding to the points $x=1, 2, 3, \dots$ is in each case unity, and the breadth of the peak corresponding to $x=n$ is $2/(n+1)^2$. The area of the peak is $1/(n+1)^2$, and it is evident that, for any value of ξ,

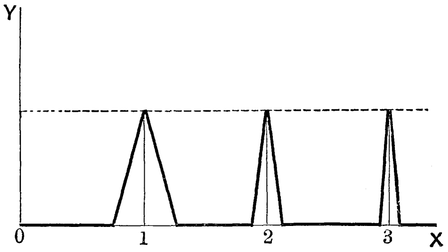

FIG. 65.

$$\int_0^\xi \phi(x)\,dx < \sum_0^\infty \frac{1}{(n+1)^2},$$

and so $\int_0^\infty \phi(x)\,dx$ is convergent; but it is not true that $\phi(x) \to 0$.

Examples LXXV. 1. If $\int_a^\infty \phi(x)\,dx$ is convergent then $\int_A^\infty \phi(x)\,dx$ is convergent for any value of A greater than a, and

$$\int_a^\infty \phi(x)\,dx = \int_a^A \phi(x)\,dx + \int_A^\infty \phi(x)\,dx.$$

2. The integral $\displaystyle\int_a^\infty \frac{\alpha x^r+\beta x^{r-1}+\dots+\lambda}{Ax^s+Bx^{s-1}+\dots+L}\,dx,$

where α and A are positive and a is greater than the greatest root of the denominator, is convergent if $s>r+1$ and otherwise divergent.

3. Which of the integrals

$$\int_a^\infty \frac{dx}{\sqrt{x}},\quad \int_a^\infty \frac{dx}{x\sqrt[3]{x}},\quad \int_a^\infty \frac{dx}{c^2+x^2},\quad \int_a^\infty \frac{x\,dx}{c^2+x^2},\quad \int_a^\infty \frac{x^2\,dx}{c^2+x^2},\quad \int_a^\infty \frac{x^2\,dx}{\alpha+2\beta x^2+\gamma x^4},$$

are convergent?

4. The integrals

$$\int_a^\xi \cos x\,dx,\quad \int_a^\xi \sin x\,dx,\quad \int_a^\xi \cos(\alpha x+\beta)\,dx,$$

oscillate finitely as $\xi\to+\infty$.

5. The integrals

$$\int_a^\xi x\cos x\,dx,\quad \int_a^\xi x^2\sin x\,dx,\quad \int_a^\xi x^n\cos(\alpha x+\beta)\,dx$$

(where n is any positive integer) oscillate infinitely as $\xi\to+\infty$.

6. **Integrals to $-\infty$.** If $\displaystyle\int_\xi^a \phi(x)\,dx$ tends to a limit l as $\xi\to-\infty$, we say $\displaystyle\int_{-\infty}^a \phi(x)\,dx$ is convergent and equal to l. Such integrals possess properties in every respect analogous to those of the integrals discussed in the preceding sections: the reader will find no difficulty in formulating them.

7. **Integrals from $-\infty$ to $+\infty$.** If the integrals

$$\int_{-\infty}^a \phi(x)\,dx,\quad \int_a^\infty \phi(x)\,dx,$$

are both convergent, and have the values k, l respectively, we say that

$$\int_{-\infty}^\infty \phi(x)\,dx$$

is convergent and has the value $k+l$.

8. Prove that

$$\int_{-\infty}^0 \frac{dx}{1+x^2}=\int_0^\infty \frac{dx}{1+x^2}=\tfrac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^2}=\tfrac{1}{2}\pi.$$

9. Prove generally that

$$\int_{-\infty}^\infty \phi(x^2)\,dx=2\int_0^\infty \phi(x^2)\,dx,$$

provided the integral $\displaystyle\int_0^\infty \phi(x^2)\,dx$ is convergent.

10. Prove that if $\displaystyle\int_0^\infty x\phi(x^2)\,dx$ is convergent, then $\displaystyle\int_{-\infty}^\infty x\phi(x^2)\,dx=0$.

11. **Analogue of Pringsheim's Theorem.** *If* $\phi(x)$ *is positive and steadily decreases, and* $\int_a^\infty \phi(x)\,dx$ *is convergent, then* $x\phi(x)\rightarrow 0$. Prove this (*a*) by means of Pringsheim's Theorem and the Integral Test, (*b*) directly, by arguments analogous to those of § 156.

12. If $a=x_0<x_1<x_2<\dots$ and $x_n\rightarrow+\infty$, and $u_n=\int_{x_n}^{x_{n+1}}\phi(x)\,dx$, then the convergence of $\int_a^\infty \phi(x)\,dx$ involves that of Σu_n. If $\phi(x)$ is always positive the converse statement is also true. [That the converse is not true in general is shown by the example in which $\phi(x)=\cos x$, $x_n=n\pi$.]

162. Application to infinite integrals of the rules for substitution and integration by parts. The rules for the transformation of a definite integral which were discussed in § 145 may be extended so as to apply to infinite integrals.

(1) **Transformation by substitution.** Suppose that

$$\int_a^\infty \phi(x)\,dx \quad \dots\dots\dots\dots(1)$$

is convergent. Further suppose that, for any value of ξ greater than a, we have, as in § 145,

$$\int_a^\xi \phi(x)\,dx=\int_b^\tau \phi\{f(t)\}f'(t)\,dt \quad \dots\dots\dots\dots(2),$$

where $a=f(b)$, $\xi=f(\tau)$. Finally suppose that the functional relation $x=f(t)$ is such that $t\rightarrow+\infty$ with x. Then, making ξ and so τ tend to $+\infty$ in (2) we see that the integral

$$\int_b^\infty \phi\{f(t)\}f'(t)\,dt \quad \dots\dots\dots\dots(3)$$

is convergent and has the same value as (1).

On the other hand it may happen that as $\xi\rightarrow+\infty$, $\tau\rightarrow c$ or $\tau\rightarrow-\infty$. In these cases we have the equations

$$\int_a^\infty \phi(x)\,dx=\int_b^c \phi\{f(t)\}f'(t)\,dt,$$

$$\int_a^\infty \phi(x)\,dx=\int_b^{-\infty} \phi\{f(t)\}f'(t)\,dt=-\int_{-\infty}^b \phi\{f(t)\}f'(t)\,dt.$$

In the first of these two cases we observe that the method of substitution transforms an infinite into a finite integral. There are of course corresponding results for the integrals

$$\int_{-\infty}^a \phi(x)\,dx,\quad \int_{-\infty}^\infty \phi(x)\,dx,$$

which it is not worth while to set out in detail: the reader will be able to formulate them for himself.

Examples LXXVI. 1. Show, by means of the substitution $x=t^{\alpha}$, that if $s>1$ and $\alpha>0$ then

$$\int_1^{\infty} x^{-s}\,dx=\alpha\int_1^{\infty} t^{\alpha(1-s)-1}\,dt;$$

and verify the result by actually calculating the value of each side.

2. If $\int_a^{\infty}\phi(x)\,dx$ is convergent, it is equal to one or other of the integrals

$$\alpha\int_{(a-\beta)/\alpha}^{\infty}\phi(\alpha t+\beta)\,dt,\qquad -\alpha\int_{-\infty}^{(a-\beta)/\alpha}\phi(\alpha t+\beta)\,dt,$$

according as α is positive or negative.

3. If $\phi(x)$ is positive for all positive values of x, and α and β are any positive numbers, the convergence of the series $\Sigma\phi(n)$ involves and is involved by that of the series $\Sigma\phi(\alpha n+\beta)$.

[It follows at once, on making the substitution $x=\alpha t+\beta$, that the integrals

$$\int_0^{\infty}\phi(x)\,dx,\qquad \int_{(a-\beta)/\alpha}^{\infty}\phi(\alpha t+\beta)\,dt$$

converge or diverge together. Now use the Integral Test.]

4. Show that $\int_1^{\infty}\dfrac{dx}{(1+x)\sqrt{x}}=\frac{1}{2}\pi$. [Put $x=t^2$.]

5. Show, by means of the substitution $x=t/(1-t)$, that if l and m are both positive then

$$\int_0^{\infty}\frac{x^{l-1}}{(1+x)^{l+m}}\,dx=\int_0^1 t^{l-1}(1-t)^{m-1}\,dt.$$

6. Show by means of the substitution $x=pt/(p+1-t)$, that if a, b, and p are positive then

$$\int_0^1 x^{a-1}(1-x)^{b-1}\frac{dx}{(x+p)^{a+b}}=\frac{1}{(1+p)^a p^b}\int_0^1 t^{a-1}(1-t)^{b-1}\,dt.$$

7. If $\phi(x)\to h$ as $x\to+\infty$ and $\phi(x)\to k$ as $x\to-\infty$, then

$$\int_{-\infty}^{\infty}\{\phi(x-a)-\phi(x-b)\}\,dx=-(a-b)(h-k).$$

[For $\displaystyle\int_{-\xi'}^{\xi}\{\phi(x-a)-\phi(x-b)\}\,dx=\int_{-\xi'}^{\xi}\phi(x-a)\,dx-\int_{-\xi'}^{\xi}\phi(x-b)\,dx$

$$=\int_{-\xi'-a}^{\xi-a}\phi(t)\,dt-\int_{-\xi'-b}^{\xi-b}\phi(t)\,dt=\int_{-\xi'-a}^{-\xi'-b}\phi(t)\,dt-\int_{\xi-a}^{\xi-b}\phi(t)\,dt.$$

The first of these two integrals may be expressed in the form

$$(a-b)k+\int_{-\xi'-a}^{-\xi'-b}\rho\,dt,$$

where $\rho \to 0$ as $\xi' \to +\infty$, and the modulus of the last integral is less than or equal to $|a-b|\kappa$, where κ is the greatest value of ρ throughout the interval $(-\xi'-a, -\xi'-b)$. Hence $\int_{-\xi'-a}^{-\xi'-b} \phi(t)\,dt \to (a-b)k$. Similarly for the second integral.]

(2) **Integration by parts.** The formula for integration by parts (§ 145) is

$$\int_a^\xi f(x)\,\phi'(x)\,dx = f(\xi)\,\phi(\xi) - f(a)\,\phi(a) - \int_a^\xi f'(x)\,\phi(x)\,dx.$$

Suppose now that $\xi \to +\infty$. Then if any two of the three terms in the above equation which involve ξ tend to limits, so does the third, and we obtain the result

$$\int_a^\infty f(x)\,\phi'(x)\,dx = \lim_{\xi \to +\infty} f(\xi)\,\phi(\xi) - f(a)\,\phi(a) - \int_a^\infty f'(x)\,\phi(x)\,dx.$$

There are of course similar results for integrals to $-\infty$, or from $-\infty$ to $+\infty$.

Examples LXXVII. 1. Show that $\int_0^\infty \frac{x}{(1+x)^3}\,dx = \frac{1}{2}\int_0^\infty \frac{dx}{(1+x)^2} = \frac{1}{2}$.

2. $\int_0^\infty \frac{x^2}{(1+x)^4}\,dx = \frac{2}{3}\int_0^\infty \frac{x}{(1+x)^3}\,dx = \frac{1}{3}$.

3. If m and n are positive integers and $I_{m,n} = \int_0^\infty \frac{x^m\,dx}{(1+x)^{m+n}}$, then $I_{m,n} = \{m/(m+n-1)\}\,I_{m-1,n}$. Hence prove that $I_{m,n} = m!\,(n-2)!/(m+n-1)!$.

4. Show similarly that if $I_{m,n} = \int_0^\infty \frac{x^{2m+1}\,dx}{(1+x^2)^{m+n}}$, then

$$I_{m,n} = \{m/(m+n-1)\}\,I_{m-1,n}, \quad 2I_{m,n} = m!\,(n-2)!/(m+n-1)!.$$

Verify the result by applying the substitution $x = t^2$ to the result of Ex. 3.

163. Other types of infinite integrals. It was assumed, in the definition of the ordinary or finite integral given in Ch. VII, that (1) the range of integration was finite, (2) the subject of integration was continuous.

It is possible so to modify this definition that it applies to many cases in which these conditions are not satisfied. But this can only be done by introducing considerations of a somewhat more advanced character than those used in Ch. VII. And as a matter of fact, when condition (2) is satisfied, but not condition (1), as was the case with the infinite integrals of the preceding sections, we found it best not to attempt to give a direct definition on the lines of Ch. VII, but to adhere to that definition and, so to say, superimpose another on it; and (as was pointed out in § 160, (v)) to define the infinite integral not as a simple limit but as a repeated limit.

We shall now suppose that it is the second of the conditions (1), (2), that is not satisfied. It is natural to try to obtain definitions which enable us to extend the notion of an integral to some such cases. There is only one such case which we shall consider here. We shall suppose that $\phi(x)$ is continuous throughout the range of integration (a, A) except for a finite number of values of x, $x = \xi_1, \xi_2, \dots$, but that $\phi(x) \rightarrow +\infty$ (or $-\infty$) as $x \rightarrow \xi_r$ from either side.

It is evident that we need only consider the case in which (a, A) contains *one* such point ξ. If there is more than one such point we can divide up (a, A) into a finite number of sub-intervals each of which contains only one; and if the value of the integral over each of these sub-intervals has been defined we naturally define the value of the integral over the whole interval as being the sum of the integrals over each sub-interval. Further, we can suppose that the one point ξ in (a, A) comes at one or other of the limits a, A. For if it comes between a and A we naturally define $\int_a^A \phi(x)\,dx$ as

$$\int_a^\xi \phi(x)\,dx + \int_\xi^A \phi(x)\,dx,$$

assuming these integrals to have been satisfactorily defined. We shall suppose, then, that $\xi = a$; it is evident that the definitions to which we are led will apply, with trifling changes, to the case in which $\xi = A$.

Let us then suppose $\phi(x)$ to be continuous throughout (a, A) except for $x = a$, while $\phi(x) \rightarrow +\infty$ as $x \rightarrow a$ through values greater than a. A typical example of such a function is given by

$$\phi(x) = (x-a)^{-s} \quad (s > 0),$$

or, in particular, if $a = 0$, by $\phi(x) = x^{-s}$ $(s > 0)$. Let us therefore consider how we can define

$$\int_0^A \frac{dx}{x^s} \quad (s > 0) \quad \dots\dots\dots\dots\dots\dots\dots\dots (1).$$

The integral $\int_{1/A}^{\infty} y^{s-2}\,dy$ is convergent if $s<1$ (§ 158) and means $\lim_{\eta\to+\infty}\int_{1/A}^{\eta} y^{s-2}\,dy$. But if we make the substitution $y=1/x$, we obtain

$$\int_{1/A}^{\eta} y^{s-2}\,dy=\int_{1/\eta}^{A} x^{-s}\,dx.$$

Thus $\lim_{\eta\to+\infty}\int_{1/\eta}^{A} x^{-s}\,dx$, or, what is the same thing, $\lim_{\epsilon\to+0}\int_{\epsilon}^{A} x^{-s}\,dx$ exists provided $s<1$; and it is natural to define the value of the integral (1) as being equal to this limit. Similar considerations lead us to define $\int_a^A (x-a)^{-s}\,dx$ by the equation

$$\int_a^A (x-a)^{-s}\,dx=\lim_{\epsilon\to+0}\int_{a+\epsilon}^{A}(x-a)^{-s}\,dx.$$

We are thus led to the following general definition: *if the integral*

$$\int_{a+\epsilon}^{A}\phi(x)\,dx$$

tends to a limit l as $\epsilon\to+0$, we shall say that the integral

$$\int_a^A \phi(x)\,dx$$

is convergent, and has the value l.

Similarly, of course, we should define $\int_a^A \phi(x)\,dx$, if $\phi(x)\to+\infty$ as x tends to the upper limit A, as being

$$\lim_{\epsilon\to+0}\int_a^{A-\epsilon}\phi(x)\,dx:$$

and then, as explained above, we can extend our definitions to cover the case in which the interval (a, A) contains any finite number of points near which $\phi(x)\to+\infty$ or $-\infty$.

An integral in which the subject of integration tends to $+\infty$ or $-\infty$ as x tends to some value or values included in the range of integration will be called an *infinite integral of the second kind*: the *first kind* of infinite integrals being the class discussed in § 160 *et seq.*

Examples LXXVIII. 1. If $\phi(x)$ is continuous except for $x=a$, while, as $x\to a$, $\phi(x)\to+\infty$, the necessary and sufficient condition that $\int_a^A \phi(x)\,dx$ should be convergent is that we can find a constant K such that

$$\int_{a+\epsilon}^{A}\phi(x)\,dx<K$$

for all values of ϵ, however small (compare § 161).

It is clear that we can choose a number A' between a and A, such that $\phi(x)$ is positive in (a, A'). If $\phi(x)$ is positive in the whole interval (a, A) we can of course identify A' and A. Then

$$\int_{a-\epsilon}^{A} \phi(x)\,dx = \int_{a-\epsilon}^{A'} \phi(x)\,dx + \int_{A'}^{A} \phi(x)\,dx.$$

The first integral on the right-hand side of the above equation increases as ϵ decreases, and therefore tends to a limit or to $+\infty$; and the truth of the result stated becomes evident.

If the condition is not satisfied, $\int_{a-\epsilon}^{A} \phi(x)\,dx \to +\infty$. We shall then say that the integral $\int_{a}^{A} \phi(x)\,dx$ **diverges** to $+\infty$. It is clear that, if $\phi(x) \to +\infty$ as $x \to a+0$, convergence and divergence to $+\infty$ are the only alternatives for the integral. Similarly we may discuss the case in which $\phi(x) \to -\infty$.

2. Prove directly from the definition that

$$\int_{a}^{A} (x-a)^{-s}\,dx = (A-a)^{1-s}/(1-s)$$

if $s < 1$, while the integral is divergent if $s \geqq 1$.

3. If $\phi(x) \to +\infty$ as $x \to a+0$ and $\phi(x) < K(x-a)^{-s}$, where $s < 1$, then $\int_{a}^{A} \phi(x)\,dx$ is convergent: but if $\phi(x) > K(x-a)^{-s}$, where $s \geqq 1$, the integral is divergent. [This is merely a particular case of a general comparison theorem analogous to that stated in § 161.]

4. Are the integrals

$$\int_{a}^{A} \frac{dx}{\sqrt{(x-a)}},\quad \int_{a}^{A} \frac{dx}{\sqrt[3]{(A-x)}},\quad \int_{a}^{A} \frac{dx}{(x-a)\sqrt{(x-a)}},\quad \int_{a}^{A} \frac{dx}{(A-x)\sqrt[3]{(A-x)}},$$

$$\int_{a}^{A} \frac{dx}{\sqrt{\{(x-a)(A-x)\}}},\quad \int_{a}^{A} \frac{dx}{(x-a)\sqrt{(A-x)}},\quad \int_{a}^{A} \frac{dx}{(A-x)\sqrt[3]{(x-a)}},$$

$$\int_{a}^{A} \frac{dx}{\sqrt{(x^2-a^2)}},\quad \int_{a}^{A} \frac{dx}{\sqrt[3]{(A^3-x^3)}},\quad \int_{a}^{A} \frac{dx}{x^2-a^2},\quad \int_{a}^{A} \frac{dx}{A^3-x^3}$$

convergent or divergent?

5. The integrals $\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}}$, $\int_{a-1}^{a+1} \frac{dx}{\sqrt[3]{(x-a)}}$ are convergent, and the value of each is zero.

6. The integral $\int_{0}^{\pi} \frac{dx}{\sqrt{(\sin x)}}$ is convergent. [The subject of integration tends to $+\infty$ as x tends to either limit: like $1/\sqrt{x}$ as $x \to 0$, and like $1/\sqrt{(\pi-x)}$ as $x \to \pi$.]

7. The integral $\int_{0}^{\pi} \frac{dx}{(\sin x)^s}$ is convergent if and only if $s < 1$.

8. The integral $\int_{0}^{\frac{1}{2}\pi} \frac{x^s}{(\sin x)^t}\,dx$ is convergent if $t < s+1$.

9. The integral $\int_0^{\frac{1}{2}\pi} (\cos x)^l (\sin x)^m dx$ is convergent if and only if $l > -1$, $m > -1$.

10. Such an integral as $\int_0^\infty \frac{x^{s-1}dx}{1+x}$ does not fall directly under any of our previous definitions, if $s < 1$. For the range of integration is infinite, and the subject of integration tends to $+\infty$ as $x \rightarrow +0$. It is natural to define it as equal to the sum

$$\int_0^1 \frac{x^{s-1}dx}{1+x} + \int_1^\infty \frac{x^{s-1}dx}{1+x},$$

provided these two integrals are both convergent.

The first integral is a convergent infinite integral of the second kind if $0 < s < 1$. The second is a convergent infinite integral of the first kind if $s < 1$. It should be noted that if $s > 1$ the first integral is an ordinary finite integral: but then the second is divergent. Thus the integral from 0 to ∞ is convergent if and only if $0 < s < 1$.

11. Prove that $\int_0^\infty \frac{x^{s-1}}{1+x^t} dx$ is convergent if and only if $0 < s < t$.

12. The integral $\int_0^\infty \frac{x^{s-1} - x^{t-1}}{1-x} dx$ is convergent if and only if $0 < s < 1$, $0 < t < 1$. [It should be noticed that the subject of integration is undefined for $x = 1$; but $(x^{s-1} - x^{t-1})/(1-x) \rightarrow t - s$ as $x \rightarrow 1$ from either side; so that if we agree to regard the subject of integration as being equal to $t - s$ for $x = 1$, it becomes a continuous function of x, and there is no difficulty as regards its behaviour for $x = 1$.

It often happens that the subject of integration has a discontinuity which is due simply to a failure in its definition at a particular point in the range of integration, and can be removed by attaching a particular value to it at that point. In this case it is usual to suppose the definition of the subject of integration completed in this way. Thus the integrals

$$\int_0^{\frac{1}{2}\pi} \frac{\sin mx}{x} dx, \quad \int_0^{\frac{1}{2}\pi} \frac{\sin mx}{\sin x} dx$$

are ordinary finite integrals, the subjects of integration being regarded as having for $x = 0$ the value m.]

13. **Substitution and integration by parts.** The formulae for transformation by substitution and integration by parts may of course be extended to infinite integrals of the second as well as of the first kind. The reader should formulate the general theorems for himself, on the lines of § 162.

14. If $s > 0$, $t > 1$, prove by integration by parts that

$$\int_0^1 x^{s-1}(1-x)^{t-1}dx = \frac{t-1}{s}\int_0^1 x^s(1-x)^{t-2}dx.$$

15. If $s > 0$, $\int_0^1 \frac{x^{s-1}dx}{1+x} = \int_1^\infty \frac{t^{-s}dt}{1+t}$. [Put $x = 1/t$.]

16. If $0<s<1$, then $\displaystyle\int_0^1 \frac{x^{s-1}+x^{-s}}{1+x}\,dx=\int_0^\infty \frac{t^{-s}\,dt}{1+t}=\int_0^\infty \frac{t^{s-1}\,dt}{1+t}$.

17. $\displaystyle\int_0^1 \frac{dx}{(1+x)\sqrt{x}}=\tfrac{1}{2}\pi,\quad \int_0^\infty \frac{dx}{(1+x)\sqrt{x}}=\pi.$ [Put $x=t^2$.]

18. Prove that $\displaystyle\int_a^b \frac{dx}{\sqrt{\{(x-a)(b-x)\}}}=\pi$ and $\displaystyle\int_a^b \frac{x\,dx}{\sqrt{\{(x-a)(b-x)\}}}=\tfrac{1}{2}\pi(a+b)$, (i) by means of the substitution $x=a+(b-a)t^2$, (ii) by means of the substitution $(b-x)/(x-a)=t$, and (iii) by means of the substitution $x=a\cos^2 t+b\sin^2 t$.

19. If $s>-1$, then

$$\int_0^{\frac{1}{2}\pi}(\sin\theta)^s\,d\theta=\int_0^1 \frac{x^s\,dx}{\sqrt{(1-x^2)}}=\tfrac{1}{2}\int_0^1 \frac{x^{\frac{1}{2}(s-1)}\,dx}{\sqrt{(1-x)}}=\tfrac{1}{2}\int_0^1 (1-x)^{\frac{1}{2}(s-1)}\frac{dx}{\sqrt{x}}.$$

20. Establish the formulae

$$\int_0^1 \frac{f(x)\,dx}{\sqrt{(1-x^2)}}=\int_0^{\frac{1}{2}\pi} f(\sin\theta)\,d\theta,$$

$$\int_a^b \frac{f(x)\,dx}{\sqrt{\{(x-a)(b-x)\}}}=2\int_0^{\frac{1}{2}\pi} f(a\cos^2\theta+b\sin^2\theta)\,d\theta,$$

$$\int_{-a}^a f\left\{\sqrt{\left(\frac{a-x}{a+x}\right)}\right\}dx=4a\int_0^{\frac{1}{2}\pi} f(\tan\theta)\sin\theta\cos\theta\,d\theta.$$

164. Some care has occasionally to be exercised in applying the rule for transformation by substitution. The following example affords a good illustration of this.

Let
$$J=\int_1^7 (x^2-6x+13)\,dx.$$

By direct integration we find that $J=48$. Now let us apply the substitution

$$y=x^2-6x+13,\quad x=3\pm\sqrt{(y-4)}.$$

Since $y=8$ when $x=1$ and $y=20$ when $x=7$, we appear to be led to the result

$$J=\int_8^{20} y\frac{dx}{dy}\,dy=\pm\tfrac{1}{2}\int_8^{20}\frac{y\,dy}{\sqrt{(y-4)}}=\pm\left[\tfrac{1}{3}(y-4)^{3/2}+4(y-4)^{1/2}\right]_8^{20}=\pm\tfrac{80}{3},$$

which is certainly untrue whichever sign we choose.

The explanation is to be found in a closer consideration of the relation between x and y. The function $x^2-6x+13$ has a minimum for $x=3$, when $y=4$. As x increases from 1 to 3, y decreases from 8 to 4, and dx/dy is negative, so that

$$\frac{dx}{dy}=-\frac{1}{2\sqrt{(y-4)}}.$$

As x increases from 3 to 7, y increases from 4 to 20, and the other sign must be chosen. Thus

$$\begin{aligned}J=\int_1^7 y\,dx&=\int_8^4\left\{-\frac{y}{2\sqrt{(y-4)}}\right\}dy+\int_4^{20}\frac{y}{2\sqrt{(y-4)}}\,dy\\&=\left[\tfrac{1}{3}(y-4)^{3/2}+4(y-4)^{1/2}\right]_4^8+\left[\tfrac{1}{3}(y-4)^{3/2}+4(y-4)^{1/2}\right]_4^{20}\\&=\tfrac{144}{3}=48.\end{aligned}$$

Similarly, if we wish to transform the integral $\int_0^\pi dx = \pi$ by the substitution $x = \arcsin y$, we must observe that $dx/dy = 1/\sqrt{(1-y^2)}$ or $-1/\sqrt{(1-y^2)}$ according as $0 \leqq x < \frac{1}{2}\pi$ or $\frac{1}{2}\pi < x \leqq \pi$.

Example. Verify the results of transforming the integrals

$$\int_0^1 (4x^2 - x + \tfrac{1}{16})\,dx, \quad \int_0^\pi \cos^2 x\,dx$$

by the substitutions $4x^2 - x + \frac{1}{16} = y$, $x = \arcsin y$ respectively.

165. Series of positive and negative terms. Our definitions of the sum of an infinite series, and the value of an infinite integral, whether of the first or the second kind, apply to series of terms or integrals of functions whose values may be either positive or negative. But the special tests for convergence or divergence which we have established in this chapter, and the examples by which we have illustrated them, have had reference almost entirely to the case in which all these values are positive. Of course the case in which they are all negative is not essentially different, as it can be reduced to the former by merely changing u_n into $-u_n$ or $\phi(x)$ into $-\phi(x)$.

In the case of a series it has always been explicitly or tacitly assumed that any conditions imposed upon u_n may be violated for a finite number of terms: all that is necessary is that such a condition (e.g. that all the terms are positive) should be satisfied *from some definite term onwards.* Similarly in the case of an infinite integral the conditions have been supposed to be satisfied for *all values of x greater than some definite value* or for all values of x within some definite interval $(a, a+\delta)$ which includes the value a near which the subject of integration tends to $+\infty$. Thus our tests apply to such a series as $\Sigma(n^2-10)/n^4$, since $n^2 - 10 > 0$ if $n \geqq 4$, and to such integrals as

$$\int_1^\infty \frac{3x-7}{(x+1)^3}\,dx, \quad \int_0^1 \frac{1-2x}{\sqrt{x}}\,dx,$$

since $3x - 7 > 0$ if $x > \frac{7}{3}$, and $1 - 2x > 0$ if $0 < x < \frac{1}{2}$.

But when the changes of sign of u_n *persist throughout the series,* i.e. when the number of both positive and negative terms is infinite (as in the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$)—or when $\phi(x)$ continually changes sign as $x \to +\infty$, as in the integral

$$\int_1^\infty \frac{\sin x}{x^s}\,dx,$$

or as $x \rightarrow a$, where a is a point of discontinuity of $\phi(x)$, as in the integral

$$\int_a^A \sin\left(\frac{1}{x-a}\right)\frac{dx}{x-a},$$

—the problem of discussing convergence or divergence becomes more difficult. For now we have to consider the possibility of oscillation as well as of convergence or divergence.

We shall not, in this volume, have to consider the more general problem for integrals. But we shall, in the ensuing chapters, have to consider certain simple examples of series containing an infinite number of both positive and negative terms.

166. Absolutely Convergent Series. Let us then consider a series Σu_n in which any term may be either positive or negative. Let

$$|u_n| = \alpha_n,$$

so that $\alpha_n = u_n$ if u_n is positive and $\alpha_n = -u_n$ if u_n is negative. Further let $v_n = u_n$ or 0, according as u_n is positive or negative, and $w_n = -u_n$ or 0, according as u_n is negative or positive; or, what is the same thing, let v_n or w_n be equal to α_n according as u_n is positive or negative, the other being in either case equal to zero. Then it is evident that v_n and w_n are always positive, and

$$u_n = v_n - w_n, \quad \alpha_n = v_n + w_n.$$

If, for example, our series is $1-(1/2^2)+(1/3^2)-\ldots$, $u_n=(-1)^n/(n+1)^2$ and $\alpha_n=1/(n+1)^2$, while $v_n=1/(n+1)^2$ or 0 according as n is even or odd, and $w_n=1/(n+1)^2$ or 0 according as n is odd or even.

We can now distinguish two cases.

A. Suppose that the series $\Sigma\alpha_n$ is convergent. This is the case, for instance, in the example above, where $\Sigma\alpha_n$ is

$$1 + (1/2^2) + (1/3^2) + \ldots.$$

Then both Σv_n and Σw_n are convergent: for (Ex. XXXII. 18) any series selected from the terms of a convergent series of positive terms is convergent. And hence (§ 70) Σu_n or $\Sigma(v_n - w_n)$ is convergent and equal to $\Sigma v_n - \Sigma w_n$.

We are thus led to formulate the following definition.

DEFINITION. *When $\Sigma\alpha_n$ or $\Sigma|u_n|$ is convergent the series Σu_n is said to be* **absolutely convergent.**

And what we have proved above amounts to this: *if Σu_n is absolutely convergent it is convergent: so are the series formed* *by its positive and negative terms taken separately: and the sum of the series is equal to the sum of the positive terms plus the sum of the negative terms.*

The reader should carefully guard himself against supposing that the statement '*an absolutely convergent series is convergent*' is a mere tautology. When we say that Σu_n is 'absolutely convergent' we do *not* directly assert the convergence of Σu_n: we assert the convergence of *another* series $\Sigma|u_n|$, and it is by no means evident *a priori* that convergence of $\Sigma|u_n|$ and oscillation of Σu_n might not go together.

167. Extension of Dirichlet's Theorem to absolutely convergent series. Dirichlet's Theorem (§ 152) shows that the terms of a series of positive terms may be rearranged in any way without affecting its sum. It is now easy to see that any absolutely convergent series has the same property. For let Σu_n be so rearranged as to become $\Sigma u_n'$, and let α_n', v_n', w_n' be formed from u_n' as α_n, v_n, w_n were formed from u_n. Then $\Sigma\alpha_n'$ is convergent, as it is a rearrangement of $\Sigma\alpha_n$, and so are $\Sigma v_n'$, $\Sigma w_n'$, which are rearrangements of Σv_n, Σw_n. Also, by Dirichlet's Theorem, $\Sigma v_n' = \Sigma v_n$ and $\Sigma w_n' = \Sigma w_n$, and so

$$\Sigma u_n' = \Sigma v_n' - \Sigma w_n' = \Sigma v_n - \Sigma w_n = \Sigma u_n.$$

Examples LXXIX. 1. Verify that if $0 < r < 1$ then

$$1 - r + r^2 - r^3 + \dots = 1 + r^2 - r + r^4 + r^6 - r^3 + r^8 + r^{10} - r^5 + \dots$$
$$= 1 + r^2 + r^4 + \dots - r - r^3 - \dots.$$

2. If Σa_n is a convergent series of positive terms, and $|b_n| \leqq K a_n$, then Σb_n is absolutely convergent.

3. If Σa_n is a convergent series of positive terms, the series $\Sigma a_n r^n$ is absolutely convergent for $-1 \leqq r \leqq 1$.

4. If Σa_n is a convergent series of positive terms, the series $\Sigma a_n \cos n\theta$, $\Sigma a_n \sin n\theta$ are absolutely convergent for all values of θ. [Examples are afforded by the series $\Sigma r^n \cos n\theta$, $\Sigma r^n \sin n\theta$ of Ch. IV, § 76.]

5. Any series selected from the terms of an absolutely convergent series is absolutely convergent. [For the series of the moduli of its terms is a selection from the series of the moduli of the terms of the original series.]

6. Prove that if $\Sigma|u_n|$ is convergent then

$$|\Sigma u_n| \leqq \Sigma|u_n|,$$

and that the only case to which the sign of equality can apply is that in which every term is positive.

168. Conditionally convergent series. B. We have now to consider the second case indicated above, viz. that in which the series of moduli $\Sigma\alpha_n$ diverges to $+\infty$ (this being of course the only alternative to its convergence, as α_n is positive).

DEFINITION. *If* Σu_n *is convergent, but* $\Sigma |u_n|$ *divergent, the original series is said to be* **conditionally convergent.**

In the first place we note that, if Σu_n is conditionally convergent, the series Σv_n, Σw_n of § 166 must both diverge to $+\infty$. For they obviously cannot both converge, as this would involve the convergence of $\Sigma(v_n + w_n)$ or $\Sigma \alpha_n$. But if one of them, say Σw_n were convergent, and Σv_n divergent, then since

$$\sum_0^N u_n = \sum_0^N v_n - \sum_0^N w_n \quad \text{....................(1)},$$

the left-hand side would tend to $+\infty$ with N, which is contrary to the hypothesis that Σu_n is convergent.

Hence Σv_n, Σw_n are both divergent. It is clear from equation (1) above that the sum of a conditionally convergent series is the limit of the difference of two functions each of which tends to $+\infty$ with n. It is obvious too that Σu_n no longer possesses the property of convergent series of positive terms (Ex. XXXII. 18) and all absolutely convergent series (Ex. LXXIX. 5) that any selection from the terms itself forms a convergent series. And it seems more than likely that the property prescribed by Dirichlet's Theorem will not be possessed by conditionally convergent series; at any rate the proof of § 167 fails completely, as it depended essentially on the convergence of Σv_n and Σw_n separately. We shall see in a moment that this conjecture is well founded, and that the theorem is not true for series such as we are now considering.

169. Tests of convergence for conditionally convergent series. It is not to be expected that we should be able to find tests for conditional convergence as simple and general as those of §§ 150 *et seq.* It is naturally a much more difficult matter to formulate tests of convergence for series whose convergence, as is shown by equation (1) above, depends essentially on the cancelling of the positive by the negative terms. In the first instance *there are no comparison tests for convergence of conditionally convergent series.*

For suppose we wished to infer the convergence of Σv_n from that of Σu_n. We have to compare

$$v_0 + v_1 + \dots + v_n, \quad u_0 + u_1 + \dots + u_n.$$

If every u and every v were positive and every v less than the corresponding u, we could at once infer that

$$v_0 + v_1 + \dots + v_n < u_0 + \dots + u_n,$$

and so the convergence of Σv_n. If the u's only were positive and every v *numerically* less than the corresponding u, we could infer that

$$|v_0| + |v_1| + \dots + |v_n| < u_0 + \dots + u_n,$$

and so the absolute convergence of Σv_n. But in the general case, when the u's and v's are both unrestricted as to sign, all that we can infer is that

$$|v_0| + |v_1| + \dots + |v_n| < |u_0| + \dots + |u_n|.$$

This would enable us to infer the absolute convergence of Σv_n from the absolute convergence of Σu_n: but if Σu_n is only conditionally convergent we can draw no inference at all.

Example. We shall see shortly that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is convergent. But the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$ is divergent, although each of its terms is numerically less than the corresponding term of the former series.

It is therefore only natural that such tests as we can obtain should be of a much more special character than those given in the early part of this chapter.

170. Alternating Series. The simplest and most common conditionally convergent series are what is known as *alternating series*, series whose terms are alternately positive and negative. The convergence of the most important series of this type is established by the following theorem.

If $\phi(n)$ is a positive function of n which tends **steadily** *to zero as $n \to \infty$, the series*

$$\phi(0) - \phi(1) + \phi(2) - \dots$$

is convergent, and its sum lies between $\phi(0)$ and $\phi(0) - \phi(1)$.

Let us write $\phi_0, \phi_1, \dots$ for $\phi(0), \phi(1); \dots$ and let

$$s_n = \phi_0 - \phi_1 + \phi_2 - \dots + (-1)^n \phi_n.$$

Then

$$s_{2n+1} - s_{2n-1} = \phi_{2n} - \phi_{2n+1} \geqq 0, \quad s_{2n} - s_{2n-2} = -(\phi_{2n-1} - \phi_{2n}) \leqq 0.$$

Hence $s_0, s_2, s_4, \dots, s_{2n}, \dots$ is a decreasing sequence, and therefore tends to a limit or to $-\infty$, and $s_1, s_3, s_5, \dots, s_{2n+1}, \dots$ is an increasing sequence, and therefore tends to a limit or to $+\infty$. But since $\lim (s_{2n+1} - s_{2n}) = \lim (-1)^{2n+1}\phi_{2n+1} = 0$, it is obvious that both sequences must tend to limits, and that the two limits must be the same. That is to say, the sequence $s_0, s_1, \dots, s_n, \dots$ tends to a limit. Since $s_0 = \phi_0$, $s_1 = \phi_0 - \phi_1$, it is clear that this limit lies between ϕ_0 and $\phi_0 - \phi_1$.

Examples LXXX. 1. The series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots, \quad 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots,$$

$\Sigma(-1)^n/(n+a)$, $\Sigma(-1)^n/\sqrt{(n+a)}$, $\Sigma(-1)^n/(\sqrt{n}+\sqrt{a})$, $\Sigma(-1)^n/(\sqrt{n}+\sqrt{a})^2$, where $a > 0$, are conditionally convergent.

2. The series $\Sigma(-1)^n/(n+a)^s$ $(a > 0)$ is absolutely convergent if $s > 1$, conditionally convergent if $0 < s \leqq 1$, oscillatory if $s \leqq 0$.

3. The sum of the series of § 170 lies between s_n and s_{n+1} for all values of n; and the error committed by taking the sum of the first n terms instead of the sum of the whole series is numerically not greater than the modulus of the $(n+1)$th term.

4. Consider the series $\Sigma(-1)^n/\{\sqrt{n}+(-1)^n\}$, which we suppose to begin with the term for which $n=2$, to avoid any difficulty as to the definitions of the first few terms. This series may be written in the form

$$\Sigma\left[\left\{\frac{(-1)^n}{\sqrt{n}+(-1)^n} - \frac{(-1)^n}{\sqrt{n}}\right\} + \frac{(-1)^n}{\sqrt{n}}\right]$$

or

$$\Sigma\left\{\frac{(-1)^n}{\sqrt{n}} - \frac{1}{n+(-1)^n\sqrt{n}}\right\}.$$

The series $\Sigma(-1)^n/\sqrt{n}$ is convergent; but $\Sigma 1/\{n+(-1)^n\sqrt{n}\}$ is divergent, as all its terms are positive, and $\lim n/\{n+(-1)^n\sqrt{n}\}=1$. Hence the original series is divergent, although it is of the form $\phi(2)-\phi(3)+\phi(4)-\dots$ where $\phi(n) \to 0$. This example shows that the condition that $\phi(n)$ should tend *steadily* to zero is essential to the theorem. The reader will easily verify that $\sqrt{(2n+1)}-1 < \sqrt{(2n)}+1$, so that this condition is not satisfied.

5. If the conditions of § 170 are satisfied except that $\phi(n)$ tends steadily to a positive limit l, then the series $\Sigma(-1)^n\phi(n)$ oscillates finitely.

6. **Alteration of the sum of a conditionally convergent series by rearrangement of the terms.** Let s be the sum of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$, and s_{2n} the sum of its first $2n$ terms, so that $\lim s_{2n} = s$.

Now consider the series

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \dots \quad \dots\dots\dots\dots(1)$$

in which two positive terms are followed by one negative term. And let t_{3n} denote the sum of the first $3n$ terms. Then

$$t_{3n} = 1 + \frac{1}{3} + \dots + \frac{1}{4n-1} - \frac{1}{2} - \frac{1}{4} - \dots - \frac{1}{2n}$$

$$= s_{2n} + \frac{1}{2n+1} + \frac{1}{2n+3} + \dots + \frac{1}{4n-1}.$$

Now $$\lim\left[\frac{1}{2n+1}-\frac{1}{2n+2}+\frac{1}{2n+3}-\dots+\frac{1}{4n-1}-\frac{1}{4n}\right]=0,$$
since the sum of the terms inside the bracket is clearly less than $n/(2n+1)(2n+2)$; and
$$\lim\left(\frac{1}{2n+2}+\frac{1}{2n+4}+\dots+\frac{1}{4n}\right)=\tfrac{1}{2}\lim\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+(r/n)}=\tfrac{1}{2}\int_1^2\frac{dx}{x},$$
by § 138. Hence $\lim t_{3n}=s+\frac{1}{2}\int_1^2\frac{dx}{x}$, and it follows that the sum of the series (1) is not s, but the right-hand side of the last equation. Later on (§ 195) we shall give the actual values of the sums of the two series.

It can indeed be proved that a conditionally convergent series can always be so rearranged as to converge to *any sum whatever*, or to diverge to $+\infty$ or to $-\infty$. For a proof we may refer to Bromwich's *Infinite Series*, p. 68.

7. The series $1+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{4}}+\dots$ diverges to $+\infty$. [Here
$$t_{3n}=s_{2n}+\frac{1}{\sqrt{(2n+1)}}+\frac{1}{\sqrt{(2n+3)}}+\dots+\frac{1}{\sqrt{(4n-1)}}>s_{2n}+\frac{n}{\sqrt{(4n-1)}},$$
where $s_{2n}=1-\frac{1}{\sqrt{2}}+\dots-\frac{1}{\sqrt{2n}}$, which tends to a limit as $n\to\infty$.]

171. Abel's and Dirichlet's Tests of Convergence. A more general test, which includes the test of § 170 as a particular test case, is the following.

Dirichlet's Test. *If* ϕ_n *satisfies the same conditions as in* § 170, *and* Σa_n *is a series which* **converges or oscillates finitely**, *then the series*
$$a_0\phi_0+a_1\phi_1+a_2\phi_2+\dots$$
is convergent.

The reader will easily verify the identity
$$a_0\phi_0+a_1\phi_1+\dots+a_n\phi_n=s_0(\phi_0-\phi_1)+s_1(\phi_1-\phi_2)+\dots+s_{n-1}(\phi_{n-1}-\phi_n)+s_n\phi_n,$$
where $s_n=a_0+a_1+\dots+a_n$. Now the series $(\phi_0-\phi_1)+(\phi_1-\phi_2)+\dots$ is convergent, since the sum to n terms is $\phi_0-\phi_n$ and $\lim\phi_n=0$; and all its terms are positive. Also since Σa_n, if not actually convergent, at any rate oscillates finitely, we can determine a constant K so that $|s_\nu|<K$ for all values of ν. Hence the series
$$\Sigma s_\nu(\phi_\nu-\phi_{\nu+1})$$
is absolutely convergent, and so
$$s_0(\phi_0-\phi_1)+s_1(\phi_1-\phi_2)+\dots+s_{n-1}(\phi_{n-1}-\phi_n)$$
tends to a limit as $n\to\infty$. Also ϕ_n, and therefore $s_n\phi_n$, tends to zero. And therefore
$$a_0\phi_0+a_1\phi_1+\dots+a_n\phi_n$$
tends to a limit, i.e. the series $\Sigma a_\nu\phi_\nu$ is convergent.

The most important case of this theorem is obtained by supposing $a_n=\cos n\theta$ or $\sin n\theta$, in which case Σa_n oscillates finitely (Exs. XXXIV. 1, 2) except when $\theta=0$ or a multiple of 2π. It follows that *if $\phi(n)$ is a positive function of n which tends steadily to zero as $n\to\infty$, the series*

$$\Sigma\,\phi(n)\cos n\theta, \quad \Sigma\,\phi(n)\sin n\theta$$

are convergent, except perhaps the first series when $\theta=0$ or a multiple of 2π. In this case the first series reduces to $\Sigma\,\phi(n)$, which may or may not be convergent: the second series vanishes identically. If $\Sigma\,\phi(n)$ is convergent, both series are absolutely convergent (Ex. LXXIX. 4) for all values of θ, and the whole interest of Dirichlet's Test lies in its application to the case in which $\Sigma\,\phi(n)$ is divergent. And in this case the series above written are conditionally and *not* absolutely convergent, as will be proved in Ex. LXXXI. 6 below.

If we put $\theta=\pi$ in the cosine series we are led back to the result of § 170, since $\cos n\pi=(-1)^n$.

Abel's Test. There is another test, due to Abel, which, though of less frequent application than Dirichlet's, is sometimes useful.

Suppose that ϕ_n, as in Dirichlet's Test, is a positive and decreasing function of n, but that its limit as $n\to\infty$ is not necessarily zero. Thus we postulate less about ϕ_n, but to make up for this we postulate more about $\Sigma\,a_n$, viz. that it is *convergent*. Then we have the theorem: *if ϕ_n is a positive and decreasing function of n, and $\Sigma\,a_n$ is convergent, then $\Sigma\,a_n\phi_n$ is convergent.*

For ϕ_n has a limit as $n\to\infty$, say l: and $\lim(\phi_n-l)=0$. Hence, by Dirichlet's Test, $\Sigma\,a_n(\phi_n-l)$ is convergent; and as $\Sigma\,a_n$ is convergent it follows that $\Sigma\,a_n\phi_n$ is convergent.

This theorem may be stated as follows: *a convergent series remains convergent if we multiply its terms by any sequence of positive and decreasing factors.*

Examples LXXXI. 1. Show that a convergent series remains convergent if we multiply the nth term by ϕ_n, where ϕ_n is an *increasing* function of n which tends to a limit l as $n\to\infty$. [Write $\psi_n=l-\phi_n$ and consider the series $\Sigma\,a_n\psi_n$.]

2. Deduce Abel's Test directly from the identity of p. 328, in a manner similar to that in which we deduced Dirichlet's Test.

3. The series $\Sigma(\cos n\theta)/n^s$, $\Sigma(\sin n\theta)/n^s$ are convergent if $s>0$, unless (in the case of the first series) $\theta=0$ or a multiple of 2π.

4. The series of Ex. 3 are in general absolutely convergent if $s>1$, conditionally convergent if $0<s\leqq 1$, and oscillatory if $s\leqq 0$ (finitely if $s=0$ and infinitely if $s<0$). Mention any exceptional cases.

5. If $\Sigma\,a_n n^{-s}$ is convergent, $\Sigma\,a_n n^{-t}$ is convergent for $t>s$.

6. If ϕ_n is a positive function of n which tends steadily to 0 as $n\to\infty$, and $\Sigma\,\phi_n$ is divergent, the series $\Sigma\,\phi_n\cos n\theta$, $\Sigma\,\phi_n\sin n\theta$ are *not* absolutely convergent, except the sine-series when $\theta=0$ or a multiple of π. [For suppose,

e.g., that $\Sigma\phi_n|\cos n\theta|$ were convergent. Since $\cos^2 n\theta \leqq |\cos n\theta|$ it follows that $\Sigma\phi_n \cos^2 n\theta$ or

$$\tfrac{1}{2}\Sigma\phi_n(1+\cos 2n\theta)$$

would be convergent. But this is impossible, since $\Sigma\phi_n$ is divergent and $\Sigma\phi_n \cos 2n\theta$, by Dirichlet's Test, convergent, unless θ is a multiple of π. And in this case it is obvious that $\Sigma\phi_n|\cos n\theta|$ is divergent. The reader should write out the corresponding argument for the sine-series, noting where it fails if θ is a multiple of π.]

172. Series of complex terms. So far we have confined ourselves to series all of whose terms are real. We shall now consider the series

$$\Sigma u_n \equiv \Sigma\,(v_n + iw_n),$$

where v_n and w_n are real. The consideration of such series does not, of course, introduce anything really novel. The series is convergent if, and only if, the series

$$\Sigma v_n,\ \Sigma w_n$$

are separately convergent. There is however one class of such series so important as to require special treatment. Accordingly we give the following definition. This definition is an obvious extension of that of § 166.

DEFINITION. *The series* Σu_n, *where* $u_n = v_n + iw_n$, *is said to be* **absolutely convergent** *if the series* Σv_n *and* Σw_n *are absolutely convergent* (§ 166).

THEOREM. *The necessary and sufficient condition for the absolute convergence of* Σu_n *is the convergence of* $\Sigma|u_n|$ *or* $\Sigma\sqrt{(v_n^2 + w_n^2)}$.

For if Σu_n is absolutely convergent, both of the series $\Sigma|v_n|$, $\Sigma|w_n|$ are convergent, and so $\Sigma\{|v_n| + |w_n|\}$ is convergent: but

$$\sqrt{(v_n^2 + w_n^2)} \leqq |v_n| + |w_n|$$

and therefore $\Sigma|u_n|$ is convergent. And conversely, since

$$|v_n| \leqq \sqrt{(v_n^2 + w_n^2)}, \quad |w_n| \leqq \sqrt{(v_n^2 + w_n^2)},$$

$\Sigma|v_n|$ and $\Sigma|w_n|$ are convergent if $\Sigma|u_n|$ is convergent.

It is obvious that *an absolutely convergent series is convergent*, since its real and imaginary parts converge separately. And Dirichlet's Theorem (§§ 152, 167) may obviously be extended to absolutely convergent complex series, since it is true of the separate series Σv_n and Σw_n.

173. Power Series. One of the most important parts of the theory of the ordinary functions which occur in elementary analysis (such as the sine and cosine, and the logarithm and exponential, which will be discussed in the next two chapters) is that which is concerned with their expansion in series of the form $\Sigma a_n x^n$. Such a series is called a **power series.** We have already, in Ch. VII, in connection with Taylor's and Maclaurin's series, come across some cases of expansion in series of this kind. There, however, we were concerned only with *real* values of x. We shall now consider a few general properties of power series in x, where x is not restricted to be real, which will be useful in Chapters IX and X.

A. *A power series may be convergent for all values of x, for a certain region of values, or for no values except $x = 0$.*

It is sufficient to give an example of each possibility.

1. *The series $\Sigma(x^n/n!)$ is convergent for all values of x.* For if $u_n = x^n/n!$, then

$$|u_{n+1}|/|u_n| = |x|/(n+1) \to 0$$

as $n \to \infty$, whatever value x may have. Hence, by d'Alembert's test, $\Sigma|u_n|$ is convergent for all values of x, and the original series is absolutely convergent for all values of x. We shall see later on that a power series, when convergent, is *generally* absolutely convergent.

2. *The series $\Sigma n!\,x^n$ is not convergent for any value of x except $x=0$.* For if $u_n = n!\,x^n$, $|u_{n+1}|/|u_n| = (n+1)|x|$, which tends to $+\infty$ with n, unless $x=0$. Hence (Exs. XXX. 1, 2) the modulus of the nth term tends to $+\infty$ with n; and so the series cannot converge, except when $x=0$. It is obvious that any power series converges for $x=0$.

3. *The series Σx^n converges if $|x| < 1$, and not if $|x| \geqq 1$,* by § 76. Thus we have an actual example of each of the three possibilities.

174. B. *If a power series $\Sigma a_n x^n$ is convergent for a particular value of x, say $x_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, it is absolutely convergent for all values of x such that $|x| < r_1$.*

For since $\Sigma a_n x_1^n$ is convergent, $\lim a_n x_1^n = 0$, and we can certainly find a constant K such that $|a_n x_1^n| < K$ for all values of n. But, if $|x| = r < r_1$,

$$|a_n x^n| = |a_n x_1^n| \left(\frac{r}{r_1}\right)^n < K\left(\frac{r}{r_1}\right)^n;$$

and the result follows at once by comparison with the convergent geometrical series $\Sigma(r/r_1)^n$.

In other words, if the series converges at P, *it converges absolutely at all points nearer to the origin than* P.

Example. Show that the result is true even if the series oscillates finitely for $x=x_1$. [If $s_n=a_0+a_1x_1+\dots+a_nx_1^n$, we can find K so that $|s_n|<K$ for all values of n. But $|a_nx_1^n|=|s_n-s_{n-1}|\leqq|s_{n-1}|+|s_n|<2K$, and the argument can be completed as before.]

175. The region of convergence of a power series. The circle of convergence. Let $x=r_1$ be any point on the positive real axis. If the power series converges when $x=r_1$ it converges absolutely for all points inside the circle $|x|=r_1$. In particular it converges for all real values of x less than r_1.

Now let us divide the points r_1 of the positive real axis into two classes, the class for which the series converges and the class for which it does not. The first class must contain at least the one point $x=0$. The second class, on the other hand, need not exist, as the series may converge for all values of x. Suppose however that it does exist: and that the first class of points does include points besides $x=0$. Then it is clear that every point of the first class lies to the left of every point of the second class. Hence there is a point A ($x=R$, say) which divides the two classes, and may itself belong to either one or the other. *Then the series is absolutely convergent at all points inside the circle* $|x|=R$.

For let P be any such point. We can draw a circle, whose centre is O and whose radius is less than R, so as to include P inside it. Let this circle cut OA in Q. Then the series is convergent at Q, and therefore, by Theorem B, absolutely convergent at P.

On the other hand the series cannot converge at any point P' *outside* the circle. For if it converged at P' it would converge absolutely at all points nearer to O than P: and this is absurd, as it does not converge at any point between A and Q' (Fig. 66).

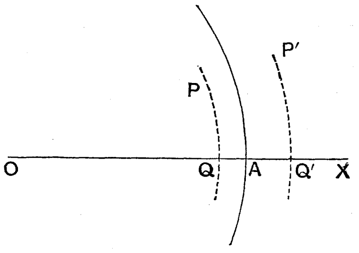

Fig. 66.

So far we have excepted the cases in which the power series (1) does not converge at any point on the positive real axis except $x=0$, or (2) converges for all points on the positive real axis. It is clear that in case (1) the power series converges nowhere except for $x=0$, and in case (2) it is everywhere absolutely convergent. Thus we obtain the following result: *a power series either*

(1) *converges for $x=0$ and for no other value of x; or*

(2) *converges absolutely for all values of x; or*

(3) *converges absolutely for all values of x within a certain circle of radius R, and does not converge for any value of x outside this circle.*

In case (3) the circle is called the **circle of convergence** and its radius the **radius of convergence** of the power series.

It should be observed that this general result gives absolutely no information about the behaviour of the series *on* the circle of convergence. The examples which follow show that as a matter of fact there are very diverse possibilities as to this.

Examples LXXXII. 1. The series $1+ax+a^2x^2+\dots$ $(a>0)$ has a radius of convergence equal to $1/a$. It does not converge anywhere on its circle of convergence, diverging for $x=1/a$ and oscillating for all other points on the circle.

2. The series $\frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\dots$ has its radius of convergence equal to 1; it converges absolutely at all points on its circle of convergence.

3. Generally if $|a_{n+1}|/|a_n|\rightarrow\lambda$ or $|a_n|^{1/n}\rightarrow\lambda$ as $n\rightarrow\infty$, the series $a_0+a_1x+a_2x^2+\dots$ has $1/\lambda$ as its radius of convergence. For in the first case

$$\lim|a_{n+1}x^{n+1}|/|a_nx^n|=\lambda x$$

which is less or greater than unity according as $|x|$ is less or greater than $1/\lambda$, so that we can use D'Alembert's Test (§ 151. 3). In the second case we can use Cauchy's Test (§ 151. 1) similarly.

4. The series $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\dots$ has its radius of convergence equal to unity. It diverges for $x=-1$, but is convergent (though not absolutely) for all other points on the circle of convergence, since its real and imaginary parts are $\cos\theta-\frac{1}{2}\cos 2\theta+\dots$, $\sin\theta-\frac{1}{2}\sin 2\theta+\dots$.

176. Uniqueness of a power series. If Σa_nx^n is a power series which is convergent for some values of x at any rate besides $x=0$, and $f(x)$ is its sum, it is easy to see that $f(x)$ can be expressed in the form

$$a_0+a_1x+a_2x^2+\dots+(a_n+\epsilon_x)\,x^n,$$

where $\epsilon_x\rightarrow 0$ with x. For if μ is any number less than the radius of convergence of the series, and $|x|<\mu$, we have

$$|f(x)-\sum_0^n a_\nu x^\nu|<|x|^{\nu+1}(|a_{\nu+1}|+|a_{\nu+2}|\mu+|a_{\nu+3}|\mu^2+\dots)<K|x|^{\nu+1},$$

where K is a number independent of x. It follows (cf. Ex. LVII. 15) that if $\Sigma a_n x^n = \Sigma b_n x^n$, for all real values of x whose modulus is less than some fixed number, then $a_n = b_n$ for all values of n. This result is capable of considerable generalisations into which we cannot enter now. It shows that *the same function $f(x)$ cannot be represented by two different power series.*

177. Multiplication of Series. We saw in § 153 that if Σu_n and Σv_n are two convergent series of positive terms, then $\Sigma u_n \times \Sigma v_n = \Sigma w_n$, where

$$w_n = u_0 v_n + u_1 v_{n-1} + \dots + u_n v_0.$$

We can now extend this result to all cases in which Σu_n and Σv_n are *absolutely* convergent; for our proof was merely a simple application of Dirichlet's theorem, which we have already extended to all absolutely convergent series.

Examples LXXXIII. 1. If $|x|$ is less than the radius of convergence of either of the series $\Sigma a_n x^n$, $\Sigma b_n x^n$, then the product of the two series is $\Sigma c_n x^n$, where $c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$.

2. If the radius of convergence of $\Sigma a_n x^n$ is r, and $f(x)$ is the sum of the series when $|x| < r$, and $|x|$ is less than either r or unity, then $f(x)/(1-x) = \Sigma s_n x^n$, where $s_n = a_0 + a_1 + \dots + a_n$.

3. Prove by squaring the series for $1/(1-x)$ that $1/(1-x)^2 = 1 + 2x + 3x^2 + \dots$ if $|x| < 1$.

4. Prove similarly that $1/(1-x)^3 = 1 + 3x + 6x^2 + \dots$, the general term being $\frac{1}{2}(n+1)(n+2)x^n$.

5. **The Binomial Theorem for a negative integral exponent.** If $|x| < 1$ and n is a positive integer, then

$$\frac{1}{(1-x)^n} = 1 + nx + \frac{n(n+1)}{1 \, . \, 2} n^2 + \dots + \frac{n(n+1)\dots(n+r-1)}{1 \, . \, 2 \dots r} x^r + \dots$$

[Assume the truth of the theorem for all indices up to n. Then, by Ex. 2, $1/(1-x)^{n+1} = \Sigma s_r x^r$, where

$$s_r = 1 + n + \frac{n(n+1)}{1 \, . \, 2} + \dots + \frac{n(n+1)\dots(n+r-1)}{1 \, . \, 2 \dots r} = \frac{(n+1)(n+2)\dots(n+r)}{1 \, . \, 2 \dots r}.$$

The last identity is easily proved by induction. We leave it to the reader to supply the details.]

6. If $f(m, x) = 1 + \binom{m}{1} x + \binom{m}{2} x^2 + \dots$, prove by multiplication of series that, when $|x| < 1$, $f(m, x) f(n, x) = f(m+n, x)$. [This equation forms the basis of Euler's proof of the Binomial Theorem. The coefficient of x^k in the product series is

$$\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \binom{m}{2}\binom{n}{k-2} + \dots + \binom{m}{k-1}\binom{n}{1} + \binom{m}{k}.$$

This is a polynomial in m and n: but when m and n are positive integers this polynomial must reduce to $\binom{m+n}{k}$, in virtue of the Binomial Theorem for a positive integral exponent. And it is easy to see that if two such polynomials are equal for all positive integral values of m and n they must be identically equal.]

7. If $f(x)=1+x+\dfrac{x^2}{2!}+\dots$ then $f(x)f(y)=f(x+y)$. [For the series for $f(x)$ is absolutely convergent for all values of x: and if $u_n=x^n/n!$, $v_n=y^n/n!$ it is easy to see that $w_n=(x+y)^n/n!$.]

8. If $$C(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots,\quad S(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots,$$
then $C(x+y)=C(x)C(y)-S(x)S(y)$, $S(x+y)=S(x)C(y)+C(x)S(y)$,
and $$\{C(x)\}^2+\{S(x)\}^2=1.$$

9. **Failure of the Multiplication Theorem.** That the theorem is not always true when Σu_n and Σv_n are not *absolutely* convergent may be seen by considering the case in which $u_n=v_n=(-1)^n/\sqrt{(n+1)}$. Then
$$w_n=(-1)^n\sum_{r=0}^{n}\frac{1}{\sqrt{\{(r+1)(n+1-r)\}}}.$$
But $\sqrt{\{(r+1)(n+1-r)\}}\leq\frac{1}{2}(n+2)$, and so $|w_n|>(2n+2)/(n+2)$, which shows that Σw_n is not convergent.

MISCELLANEOUS EXAMPLES ON CHAPTER VIII.

1. Discuss the convergence of the series $\Sigma n^k\{\sqrt{(n+1)}-2\sqrt{n}+\sqrt{(n-1)}\}$, where k is real. (*Math. Trip.* 1890.)

2. Show that the series
$$1-\frac{1}{1+x}+\frac{1}{2}-\frac{1}{2+x}+\frac{1}{3}-\frac{1}{3+x}+\dots$$
is convergent provided x is not a negative integer.

3. Investigate the convergence or divergence of the series
$$\Sigma\sin\frac{a}{n},\quad \Sigma\frac{1}{n}\sin\frac{a}{n},\quad \Sigma(-1)^n\sin\frac{a}{n},\quad \Sigma\left(1-\cos\frac{a}{n}\right),\quad \Sigma(-1)^n n\left(1-\cos\frac{a}{n}\right),$$
where a is real.

4. Discuss the convergence of the series
$$\sum_{1}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)\frac{\sin(n\theta+a)}{n},$$
where θ and a are real. (*Math. Trip.* 1899.)

5. Show that, if $R(n)$ is any rational function of n, we can determine a polynomial $P(n)$ and a constant A such that $\Sigma\{R(n)-P(n)-(A/n)\}$ is convergent. Consider in particular the cases in which $R(n)$ is one of the functions $1/(an+b)$, $(an^2+2bn+c)/(\alpha n^2+2\beta n+\gamma)$.

6. If u_1, u_2, u_3, ... is a decreasing sequence of positive numbers whose limit is zero, the series

$$u_1 - \tfrac{1}{2}(u_1+u_2) + \tfrac{1}{3}(u_1+u_2+u_3) - \dots, \quad u_1 - \tfrac{1}{3}(u_1+u_3) + \tfrac{1}{5}(u_1+u_3+u_5) - \dots$$

are convergent. [For if $(u_1+u_2+\dots+u_n)/n = v_n$, then v_1, v_2, v_3, ... is also a decreasing sequence whose limit is zero (see Ch. IV, Misc. Ex. 9). This shows that the first series is convergent; the second we leave to the reader. In particular the series

$$1 - \tfrac{1}{2}(1+\tfrac{1}{2}) + \tfrac{1}{3}(1+\tfrac{1}{2}+\tfrac{1}{3}) - \dots, \quad 1 - \tfrac{1}{3}(1+\tfrac{1}{5}) + \tfrac{1}{5}(1+\tfrac{1}{3}+\tfrac{1}{5}) - \dots$$

are convergent.]

7. The series $\Sigma \cos n\theta$, $\Sigma \sin n\theta$ oscillate finitely, the sum of the first n terms of either series always lying between $-\operatorname{cosec} \frac{1}{2}\theta$ and $\operatorname{cosec} \frac{1}{2}\theta$, while we can find values of n, as large as we please, and for which the sum lies as near as we please to either of these limits, or to any number which lies between them.

If a_n increases or decreases steadily to a limit which is not equal to zero, the series $\Sigma a_n \cos n\theta$, $\Sigma a_n \sin n\theta$ also oscillate finitely.

8. Find the sums of the series

$$\frac{x}{1+x} + \frac{2x^2}{1+x^2} + \frac{4x^4}{1+x^4} + \dots, \quad \frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \dots$$

(in which all the indices are powers of 2), whenever they are convergent. [The first series converges only if $|x| < 1$, its sum then being $x/(1-x)$; the second series converges to $x/(1-x)$ if $|x| < 1$ and to $1/(1-x)$ if $|x| > 1$.]

9. Find the sum of the series $\sum\limits_{1}^{\infty} u_n$, where

$$u_n = \frac{x^n - x^{-n-1}}{(x^n + x^{-n})(x^{n+1} + x^{-n-1})} = \frac{1}{x-1}\left(\frac{1}{x^n + x^{-n}} - \frac{1}{x^{n+1} + x^{-n-1}}\right),$$

for all real values of x for which the series is convergent. (*Math. Trip.* 1901.)

[If $|x|$ is not equal to unity, the series has the sum $x/\{(x-1)(x^2+1)\}$. If $x=1$, then $u_n=0$ and the sum is 0. If $x=-1$, then $u_n = \frac{1}{2}(-1)^{n+1}$ and the series oscillates finitely.]

10. If $u_0 + u_1 + u_2 + \dots$ is a divergent series of positive and decreasing terms, then

$$(u_0 + u_2 + \dots + u_{2n})/(u_1 + u_3 + \dots + u_{2n+1}) \rightarrow 1.$$

11. If $|a_n| \leqq 1$ for all values of n, and $1 + a_1 x + a_2 x^2 + \dots$ is a finite series, or an infinite series convergent for $|x| < 1$, the equation $0 = 1 + a_1 x + a_2 x^2 + \dots$ cannot have a root whose modulus is less than $\frac{1}{2}$, and the only case in which it can have a root whose modulus is equal to $\frac{1}{2}$ is that in which $a_n = -\operatorname{Cis}(n\theta)$, when $x = \frac{1}{2}\operatorname{Cis}(-\theta)$ is a root.

12. Prove that, if $a > 0$, then $\lim\limits_{p \to \infty} \sum\limits_{n=0}^{\infty} \dfrac{1}{(p+n)^{1+a}} = 0$.

13. Prove that $\lim\limits_{a \to +0} a \sum\limits_{1}^{\infty} \dfrac{1}{n^{1+a}} = 1$. [Use the inequalities of §§ 157–8 to show that the sum of the series lies between $1/a$ and $(1+a)/a$.

14. **Recurring Series.** A power series $\Sigma a_n x^n$ is said to be a *recurring series* if its coefficients satisfy a relation of the type

$$a_n + p_1 a_{n-1} + p_2 a_{n-2} + \dots + p_k a_{n-k} = 0 \quad \text{.....................(1)},$$

where $n \geqq k$ and $p_1, p_2, \dots, p_k$ are independent of n. Any recurring series is the expansion of a rational function of x. To prove this we observe in the first place that the series is certainly convergent for sufficiently small values of x. For let G be the greater of the two numbers

$$1, \ |p_1| + |p_2| + \dots + |p_k|.$$

Then it follows from the equation (1) that $|a_n| \leqq G\alpha_n$, where α_n is the modulus of the numerically greatest of the preceding coefficients; and from this that $|a_n| < KG^n$, where K is independent of n. Thus the recurring series is certainly convergent for values of x numerically less than $1/G$.

But if we multiply the series $f(x) = \Sigma a_n x^n$ by $p_1 x, p_2 x^2, \dots p_k x^k$, and add the results, we obtain a new series in which all the coefficients after the $(k-1)$th vanish in virtue of the relation (1), so that

$$(1 + p_1 x + p_2 x^2 + \dots + p_k x^k) f(x) = P_0 + P_1 x + \dots + P_{k-1} x^{k-1},$$

where $P_0, P_1, \dots, P_{k-1}$ are constants. The polynomial $1 + p_1 x + p_2 x^2 + \dots + p_k x^k$ is called the *scale of relation* of the series.

Conversely, it follows from the known results as to the expression of any rational function as the sum of a polynomial and certain partial fractions of the type $A/(x-\alpha)^p$, and from the Binomial Theorem for a negative integral exponent, that any rational function whose denominator is not divisible by x can be expanded in a power series convergent for sufficiently small values of x, in fact for $|x| < \rho$, where ρ is the least of the moduli of the roots of the denominator (cf. Ch. IV, Misc. Exs. 19 *et seq.*). And it is easy to see, by reversing the argument above, that the series is a recurring series. Thus *the necessary and sufficient condition that a power series should be a recurring series is that it should be the expansion of such a rational function of x.*

15. **Solution of Difference-Equations.** A relation of the type of (1) in Ex. 14 is called a *linear difference-equation in a_n with constant coefficients.* Such equations may be solved by a method which will be sufficiently explained by an example. Let the equation be

$$a_n - a_{n-1} - 8a_{n-2} + 12a_{n-3} = 0.$$

Consider the recurring power series $\Sigma a_n x^n$. We find, as in Ex. 14, that its sum is

$$\frac{a_0 + (a_1 - a_0)\,x + (a_2 - a_1 - 8a_0)\,x^2}{1 - x - 8x^2 + 12x^3} = \frac{A_1}{1-2x} + \frac{A_2}{(1-2x)^2} + \frac{B}{1+3x},$$

where A_1, A_2, B are numbers easily expressible in terms of a_0, a_1, and a_2. Expanding each fraction separately we see that the coefficient of x^n is

$$a_n = 2^n \{A_1 + (n+1)\,A_2\} + (-3)^n B.$$

The values of A_1, A_2, B depend upon the first three coefficients a_0, a_1, a_2, which may of course be chosen arbitrarily.

16. The solution of the difference-equation $u_n - 2\cos\theta\, u_{n-1} + u_{n-2} = 0$ is $u_n = A\cos n\theta + B\sin n\theta$, where A and B are arbitrary constants.

17. If u_n is a polynomial in n of degree r, $\Sigma\, u_n x^n$ is a recurring series whose scale of relation is $(1-x)^{r+1}$. (*Math. Trip.* 1904.)

18. If $f(n)$ is the coefficient of x^n in the expansion of $x/(1+x+x^2)$ in powers of x, prove that

$$(1)\ f(n) + f(n-1) + f(n-2) = 0, \quad (2)\ f(n) = (\omega_3{}^n - \omega_3{}^{2n})/(\omega_3 - \omega_3{}^2),$$

where ω_3 (Ch. III, § 41) is a complex cube root of unity. Deduce that $f(n) = 0$ or 1 or -1 according as n is of the form $3k$ or $3k+1$ or $3k+2$, and verify this by means of the identity $x/(1+x+x^2) = x(1-x)/(1-x^3)$.

19. A player tossing a coin is to score one point for every head he turns up and two for every tail, and is to play on until his score reaches or passes a total n. Show that his chance of making exactly the total n is $\frac{1}{3}\{2 + (-\frac{1}{2})^n\}$. (*Math. Trip.* 1898.)

[If p_n is the probability, $p_n = \frac{1}{2}(p_{n-1} + p_{n-2})$: also $p_0 = 1$, $p_1 = \frac{1}{2}$.]

20. If $v_n = (-1)^{n-1}(u_1 + u_2 + \dots + u_n)$ and $s_n = v_1 + v_2 + \dots + v_n$, then

$$\lim (s_1 + s_2 + \dots + s_n)/n = \tfrac{1}{2}(u_1 - u_2 + u_3 - \dots),$$

whenever the series is convergent.

[It is easy to verify that

$$s_{2n-1} = u_1 + u_3 + \dots + u_{2n-1}, \quad s_{2n} = -u_2 - u_4 - \dots - u_{2n},$$
$$s_{2\nu-1} + s_{2\nu} = u_1 - u_2 + u_3 - u_4 + \dots + u_{2\nu-1} - u_{2\nu},$$

so that $s_{2\nu-1} + s_{2\nu} \to u$, where u is the sum of the series on the right-hand side when continued to infinity. Hence (Ch. IV, Misc. Ex. 28)

$$\lim \{(s_1 + s_2) + (s_3 + s_4) + \dots + (s_{2\nu-1} + s_{2\nu})\}/\nu = u,$$

which proves the result when $n \to \infty$ through even values. Also, since $u_{2n} \to 0$, $s_{2n}/n \to 0$, and so the result is true when $n \to \infty$ in any way.]

21. Prove that

$$\frac{1}{a+1} + \frac{1}{a+2} + \dots + \frac{1}{a+n} = \binom{n}{1}\frac{1}{a+1} - \binom{n}{2}\frac{1!}{(a+1)(a+2)} + \dots$$

if n is a positive integer and a is not one of the numbers $-1, -2, \dots, -n$. [This follows from splitting up each term on the right-hand side into partial fractions. If $a > -1$ it may be proved very simply from the equation

$$\int_0^1 x^a \frac{1-x^n}{1-x}\, dx = \int_0^1 (1-x)^a \{1 - (1-x)^n\} \frac{dx}{x}$$

by expanding $(1-x^n)/(1-x)$ and $1-(1-x)^n$ in powers of x and integrating each term separately. The result, being merely an algebraical identity, must be true for all values of a save $-1, -2, \dots, -n$.]

22. Prove by multiplication of series that

$$\sum_0^\infty \frac{x^n}{n!} \sum_1^\infty \frac{(-1)^{n-1}x^n}{n \cdot n!} = \sum_1^\infty \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)\frac{x^n}{n!}.$$

[The coefficient of x^n will be found to be

$$\frac{1}{n!}\left\{\binom{n}{1}-\tfrac{1}{2}\binom{n}{2}+\tfrac{1}{3}\binom{n}{3}-\dots\right\}.$$

Now use Ex. 21, putting $a=0$.]

23. If $A_n\rightarrow A$ and $B_n\rightarrow B$ as $n\rightarrow\infty$, then

$$(A_1B_n+A_2B_{n-1}+\dots+A_nB_1)/n\rightarrow AB.$$

[Let $A_n=A+\epsilon_n$. Then the expression given is equal to

$$A\,\frac{B_1+B_2+\dots+B_n}{n}+\frac{\epsilon_1B_n+\epsilon_2B_{n-1}+\dots+\epsilon_nB_1}{n}.$$

The first term tends to AB (Ch. IV, Misc. Ex. 28). The modulus of the second is less than $\beta\{|\epsilon_1|+|\epsilon_2|+\dots+|\epsilon_n|\}/n$, where β is any number greater than the greatest value of $|B_\nu|$: and the last expression tends to zero.]

24. If $c_n=a_1b_n+a_2b_{n-1}+\dots+a_nb_1$ and

$$A_n=a_1+a_2+\dots+a_n,\quad B_n=b_1+b_2+\dots+b_n,\quad C_n=c_1+c_2+\dots+c_n,$$

prove that

$$C_n=a_1B_n+a_2B_{n-1}+\dots+a_nB_1=b_1A_n+b_2A_{n-1}+\dots+b_nA_1$$

and

$$C_1+C_2+\dots+C_n=A_1B_n+A_2B_{n-1}+\dots+A_nB_1.$$

Hence prove that if the series Σa_n, Σb_n are convergent and have the sums A, B, so that $A_n\rightarrow A$, $B_n\rightarrow B$, then

$$(C_1+C_2+\dots+C_n)/n\rightarrow AB.$$

Deduce that *if* Σc_n is convergent, its sum is AB. This result is known as **Abel's Theorem on the multiplication of Series.** We have already seen that we can multiply the series Σa_n, Σb_n in this way if both series are *absolutely* convergent: Abel's Theorem shows that we can do so even if one or both are not absolutely convergent, *provided only the product series is convergent.*

25. Prove that

$$\tfrac{1}{2}(1-\tfrac{1}{2}+\tfrac{1}{3}-\dots)^2=\tfrac{1}{2}-\tfrac{1}{3}(1+\tfrac{1}{2})+\tfrac{1}{4}(1+\tfrac{1}{2}+\tfrac{1}{3})-\dots$$

$$\tfrac{1}{2}(1-\tfrac{1}{3}+\tfrac{1}{5}-\dots)^2=\tfrac{1}{2}-\tfrac{1}{4}(1+\tfrac{1}{3})+\tfrac{1}{6}(1+\tfrac{1}{3}+\tfrac{1}{5})-\dots.$$

[Use Ex. 6 to establish the convergence of the series.]

26. For what values of m and n has the integral $\int_0^\pi(\sin x)^m(1-\cos x)^n\,dx$ a meaning? [It has a meaning if $m+1$ and $m+2n+1$ are positive.]

27. Prove that, if $a>1$, then

$$\int_0^1\frac{dx}{(a-x)\sqrt{(1-x^2)}}=\frac{\pi}{\sqrt{(a^2-1)}}.$$

28. Establish the formulae

$$\int_0^\infty F\{\surd(x^2+1)+x\}\,dx=\tfrac{1}{2}\int_1^\infty\left(1+\frac{1}{y^2}\right)F(y)\,dy,$$

$$\int_0^\infty F\{\surd(x^2+1)-x\}\,dx=\tfrac{1}{2}\int_0^1\left(1+\frac{1}{y^2}\right)F(y)\,dy,$$

where F is such a function that the integrals have a meaning. In particular, prove that if $n>1$ then

$$\int_0^\infty \frac{dx}{\{\surd(x^2+1)+x\}^n}=\int_0^\infty \{\surd(x^2+1)-x\}^n\,dx=\frac{n}{n^2-1}.$$

[First put $x=\sinh u$ and then $e^u=y$.]

29. Show that if $2y=ax-(b/x)$, where a and b are positive, y increases steadily from $-\infty$ to $+\infty$ as x increases from 0 to $+\infty$. Hence show that

$$\int_0^\infty f\left\{\tfrac{1}{2}\left(ax+\frac{b}{x}\right)\right\}dx=\frac{1}{a}\int_{-\infty}^\infty f\{\surd(y^2+ab)\}\left\{1+\frac{y}{\surd(y^2+ab)}\right\}dy$$

$$=\frac{2}{a}\int_0^\infty f\{\surd(y^2+ab)\}\,dy.$$

30. Show that if $2y=ax+(b/x)$, where a and b are positive, two values of x correspond to any value of y greater than $\surd(ab)$. Denoting the greater of these by x_1 and the less by x_2, show that, as y increases from $\surd(ab)$ towards ∞, x_1 increases from $\surd(b/a)$ towards ∞, and x_2 decreases from $\surd(b/a)$ to 0. Hence show that

$$\int_{\surd(b/a)}^\infty f(y)\,dx_1=\frac{1}{a}\int_{\surd(ab)}^\infty f(y)\left\{\frac{y}{\surd(y^2-ab)}+1\right\}dy,$$

$$\int_0^{\surd(b/a)} f(y)\,dx_2=\frac{1}{a}\int_{\surd(ab)}^\infty f(y)\left\{\frac{y}{\surd(y^2-ab)}-1\right\}dy,$$

and that

$$\int_0^\infty f\left\{\tfrac{1}{2}\left(ax+\frac{b}{x}\right)\right\}dx=\frac{2}{a}\int_{\surd(ab)}^\infty \frac{yf(y)}{\surd(y^2-ab)}\,dy=\frac{2}{a}\int_0^\infty f\{\surd(z^2+ab)\}\,dz,$$

f denoting any function such that these integrals have a meaning in accordance with the definitions of §§ 160 *et seq.*

31. Prove the formula

$$\int_0^\pi f(\sec\tfrac{1}{2}x+\tan\tfrac{1}{2}x)\frac{dx}{\surd(\sin x)}=2\int_0^\pi f(\operatorname{cosec} x)\frac{dx}{\surd(\sin x)}.$$

32. If a and b are positive, then

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}=\frac{\pi}{2ab(a+b)},\quad \int_0^\infty \frac{x^2dx}{(x^2+a^2)(x^2+b^2)}=\frac{\pi}{2(a+b)}.$$

Deduce that if α, β, and γ are positive, and $\beta^2\geqq\alpha\gamma$, then

$$\int_0^\infty \frac{dx}{\alpha x^4+2\beta x^2+\gamma}=\frac{\pi}{2\surd(2\gamma A)},\quad \int_0^\infty \frac{x^2dx}{\alpha x^4+2\beta x^2+\gamma}=\frac{\pi}{2\surd(2\alpha A)},$$

where $A=\beta+\surd(\alpha\gamma)$. Also deduce the last result from Ex. 29, by putting $f(y)=1/(c^2+y^2)$. The last two results remain true when $\beta^2<\alpha\gamma$, but their proof is then not quite so simple.

CHAPTER IX.

THE LOGARITHMIC AND EXPONENTIAL FUNCTIONS OF A REAL VARIABLE.

178. THE number of essentially different types of functions with which we have been concerned in the foregoing chapters is not very large. Among those which have occurred the most important for ordinary purposes are polynomials, rational functions, algebraical functions, explicit or implicit, and trigonometrical functions, direct or inverse.

We are however far from having exhausted the list of functions which are important in mathematics. As the range of mathematical knowledge has widened so have new classes of functions, one after another, been introduced and defined, and their properties investigated. These new functions have generally been introduced because it appeared that some problem which was occupying the attention of mathematicians was incapable of solution by means of the functions already known. The process may fairly be compared with that by which the irrational and complex numbers were first introduced, when it was found that certain algebraical equations could not be solved by means of the numbers already recognised. One of the most fruitful sources of new functions has been the problem of *integration*. Attempts have been made to integrate some function $f(x)$ in terms of functions already known. These attempts have failed; and after a certain number of failures it has begun to appear probable that the problem is insoluble. Sometimes it has been *proved* that this is so; but as a rule such a strict proof has not been forthcoming until later on. Generally it has happened that mathematicians have taken the impossibility for granted as soon as they have become reasonably convinced of it, and have introduced a new function $F(x)$ *defined* by its possessing the required property, viz. that $F'(x) = f(x)$. The properties of $F(x)$ have been investigated by starting from this definition, and it has then appeared that $F(x)$ has properties which no finite combination of the functions previously known could possibly have; and thus the correctness of the assumption that the original problem could not possibly be solved has been established. One such case has occurred in the preceding pages, when in Ch. VI we defined the function $\log x$ by means of the equation

$$\log x = \int \frac{dx}{x}.$$

Let us consider what grounds we have for supposing $\log x$ to be a really new function. We have already seen (Ex. XLIV. 5) that it cannot be a rational function, since the derivative of a rational function is itself a rational function, whose denominator contains only repeated factors. The question whether it can be an algebraical or trigonometrical function is more difficult. But it is very easy to become convinced by a few experiments that differentiation will never get rid of *algebraical irrationalities*. For example if we differentiate $\sqrt{(1+x)}$ any number of times the derivative is always the product of $\sqrt{(1+x)}$ by a rational function. And so generally—the reader should test the correctness of the statement by experimenting with a number of examples, such as $\sqrt{(x+\sqrt{x})}$, $(1-\sqrt[3]{x})/(1+\sqrt[3]{x})$. Similarly with the trigonometrical functions: if we differentiate a function which involves $\sin x$, or $\cos x$, one or other of these functions persists in the result; while differentiation of a function which involves $\arcsin x$ or $\arctan x$ always leads to a function involving $\sqrt{(1-x^2)}$ or $1+x^2$ in a form impossible to eliminate.

We have therefore, not indeed a strict proof that $\log x$ is a new function—that we do not profess to give*—but a reasonable presumption that it is. We shall therefore treat it as such, and we shall find on examination that its properties are quite unlike those of any function which we have come across hitherto.

179. Definition of $\log x$. We define $\log x$, the logarithm of x, by the equation

$$\log x = \int_1^x \frac{dt}{t}.$$

Here x is *positive*: if x is negative the integral has no meaning (Ex. LXXVIII. 2). We might have chosen a lower limit other than 1; but 1 proves to be the most convenient. With this definition $\log 1 = 0$.

We shall now consider how $\log x$ behaves as x varies from 0

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[*] For such a proof see p. 35 of the author's tract quoted on p. 225.

towards ∞. It follows at once from the definition that $\log x$ is a continuous function of x which increases steadily with x and has a derivative

$$D_x \log x = 1/x.$$

Moreover (§ 158) $\log x$ tends to $+\infty$ with x.

If x is positive but less than 1, $\log x$ is negative. For

$$\log x = \int_1^x \frac{dt}{t} = -\int_x^1 \frac{dt}{t} < 0.$$

Moreover, if we make the substitution $t = 1/u$ in the integral, we obtain

$$\log x = \int_1^x \frac{dt}{t} = -\int_1^{1/x} \frac{du}{u} = -\log(1/x).$$

Thus as x varies from 1 downwards towards 0, $\log x$ tends steadily to $-\infty$.

The general form of the graph of the logarithmic function is shown in Fig. 67. Since the derivative of $\log x$ is $1/x$, the slope of the curve is very gentle when x is very large, and very steep when x is very small.

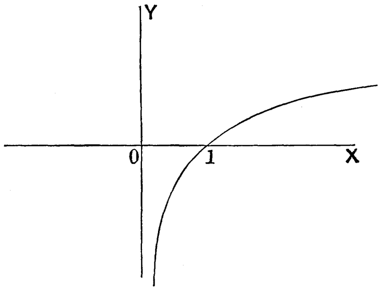

Fig. 67.

Examples LXXXIV. 1. Prove from the definition that if $u > 0$ then

$$u/(1+u) < \log(1+u) < u.$$

[For $\log(1+u) = \int_0^u \frac{dt}{1+t}$, and the subject of integration lies between 1 and $1/(1+u)$.]

2. Prove that, if $u > 0$, $\log(1+u)$ lies between $u - \frac{1}{2}u^2$ and $u - \{\frac{1}{2}u^2/(1+u)\}$. [Use the fact that $\log(1+u) = u - \int_0^u \frac{t\,dt}{1+t}$.]

3. If $0 < u < 1$ then $u < -\log(1-u) < u/(1-u)$.

4. Prove that

$$\lim_{x\to 1} \frac{\log x}{x-1} = \lim_{t\to 0} \frac{\log(1+t)}{t} = 1.$$

[Use Ex. 1.]

180. The functional equation satisfied by $\log x$. *The function* $\log x$ *is a solution of the functional equation*

$$f(xy) = f(x) + f(y) \quad \text{.....................(1)}.$$

For, making the substitution $t = yu$, we see that

$$\log(xy) = \int_1^{xy} \frac{dt}{t} = \int_{1/y}^{x} \frac{du}{u} = \int_1^x \frac{du}{u} - \int_1^{1/y} \frac{du}{u}$$

$$= \log x - \log\left(\frac{1}{y}\right) = \log x + \log y,$$

which proves the theorem.

Examples LXXXV. 1. It can be shown that there is no solution of the equation (1) fundamentally distinct from $\log x$. For, if we suppose $f(x)$ to be a function which has a differential coefficient (and other functions may be neglected as of no practical importance) we obtain by differentiating the functional equation, first with respect to x, and then with respect to y, the two equations

$$yf'(xy) = f'(x), \quad xf'(xy) = f'(y);$$

and so, eliminating $f'(xy)$, $xf'(x) = yf'(y)$. But if this if true for every pair of values of x and y, we must have $xf'(x) = C$, or $f'(x) = C/x$, where C is a constant. Hence

$$f(x) = \int \frac{C}{x}\,dx + C' = C\log x + C'.$$

Thus there is no solution fundamentally distinct from $\log x$—except the trivial solution $f(x) = 0$, given by taking $C = 0$.

2. Show in the same way that there is no solution of the equation

$$f(x) + f(y) = f\{(x+y)/(1-xy)\}$$

fundamentally distinct from $\arctan x$.

181. The manner in which $\log x$ **tends to** $+\infty$ **with** x. It will be remembered that in Ch. V we defined functions of x which tend to $+\infty$ with x in certain different ways, distinguishing between those which, when x is large, are of the first, second, third, ... orders of greatness. A function $f(x)$ was said to be of the kth order of greatness when $f(x)/x^k$ tends to a limit different from zero as x tends to $+\infty$.

It is easy to define a whole series of functions which tend to $+\infty$ with x, but whose order of greatness is smaller than the first.

Thus $\sqrt{x}$, $\sqrt[3]{x}$, $\sqrt[4]{x}$, ... are such functions. We may say generally that x^α, where α is any positive rational number, is of the αth order of greatness when x is large. We may suppose α as small as we please, e.g. less than ·0000001. And it might be thought that by giving α all possible values we should exhaust the possible 'orders of infinity' of $f(x)$. At any rate it might be supposed that if $f(x)$ tends to $+\infty$ with x, however slowly, we could always find a value of α so small that x^α would tend to $+\infty$ more slowly still: and conversely that if $f(x)$ tends to $+\infty$ with x, however rapidly, we could always find a value of α so great that x^α would tend to $+\infty$ more rapidly still.

Perhaps the most interesting feature of the function $\log x$ is its behaviour as x tends to $+\infty$. It shows that the presupposition stated above, which seems so natural, is unfounded. *The logarithm of x tends to $+\infty$ with x, but more slowly than* **any** *positive power of x, integral or fractional.* In other words $\log x \rightarrow +\infty$ but $(\log x)/x^\alpha \rightarrow 0$ for *all* positive values of α.

This fact is sometimes loosely expressed by saying that 'the order of infinity of $\log x$ is infinitely small'—but to say that anything is 'infinitely small' is, strictly speaking, just as meaningless as to say that anything is 'infinitely great,' or that $x = \infty$, and we therefore advise the reader to avoid such modes of expression.

182. Proof that $(\log x)/x^\alpha \rightarrow 0$ **as** $x \rightarrow +\infty$. Let β be any positive number. Then if $t > 1$, $1/t < 1/t^{1-\beta}$, and so

$$\log x = \int_1^x \frac{dt}{t} < \int_1^x \frac{dt}{t^{1-\beta}},$$

or

$$\log x < (x^\beta - 1)/\beta < x^\beta/\beta.$$

Now if α is any positive number (e.g. ·01) we can choose a smaller positive value of β (e.g. ·001). And then

$$0 < (\log x)/x^\alpha < x^{\beta-\alpha}/\beta \quad (x > 1).$$

But since $\alpha > \beta$, $x^{\beta-\alpha}/\beta \rightarrow 0$ as $x \rightarrow +\infty$, and therefore

$$(\log x)/x^\alpha \rightarrow 0.$$

183. Since $(\log x)/x^\alpha = -y^\alpha \log y$ if $x = 1/y$, it follows from the theorem proved above that

$$\lim_{y \rightarrow +0} y^\alpha \log y = -\lim_{x \rightarrow +\infty} (\log x)/x^\alpha = 0.$$

Thus $\log x$ tends to $-\infty$ and $\log(1/x) = -\log x$ to $+\infty$ as x tends to zero by positive values, but $\log(1/x)$ tends to $+\infty$ more slowly than any positive power of $1/x$, integral or fractional.

184. Scales of infinity. The logarithmic scale. Let us consider once more the series of functions

$$x,\quad \sqrt{x},\quad \sqrt[3]{x},\ \dots,\quad \sqrt[n]{x},\ \dots,$$

which possesses the property that, if $f(x)$ and $\phi(x)$ are any two of the functions contained in it, $f(x)$ and $\phi(x)$ both tend to $+\infty$ with x, while $f(x)/\phi(x)$ tends to 0 or to $+\infty$ according as $f(x)$ occurs to the right or the left of $\phi(x)$ in the series. We can now continue this series by the insertion of new terms to the right of all those already written down. We can begin with $\log x$, which tends to infinity more slowly than any of the old terms. Then $\sqrt{(\log x)}$ tends to $+\infty$ more slowly than $\log x$, $\sqrt[3]{(\log x)}$ than $\sqrt{(\log x)}$, and so on. Thus we obtain a series

$$x,\quad \sqrt{x},\quad \sqrt[3]{x},\ \dots,\quad \sqrt[n]{x},\ \dots\qquad \log x,\quad \sqrt{(\log x)},\quad \sqrt[3]{(\log x)},\ \dots\dots\quad \sqrt[n]{(\log x)},\ \dots$$

formed of two simply infinite series arranged one after the other. But this is not all. Consider the function $\log\log x$, the logarithm of $\log x$. Since $(\log x)/x^{\alpha} \rightarrow 0$, for all positive values of x, it follows on putting $x = \log y$ that

$$(\log\log y)/(\log y)^{\alpha} = (\log x)/x^{\alpha} \rightarrow 0.$$

Thus $\log\log y$ tends to $+\infty$ with y, but more slowly than any power of $\log y$. Hence we may continue our series in the form

$$x, \sqrt{x}, \sqrt[3]{x}, \dots \log x, \sqrt{(\log x)}, \sqrt[3]{(\log x)}, \dots \log\log x, \sqrt{(\log\log x)}, \dots \sqrt[n]{(\log\log x)}, \dots;$$

and it will by now be obvious that by introducing the functions $\log\log\log x$, $\log\log\log\log x$, ... we can prolong the series to any extent we like. By putting $x = 1/y$ we obtain a similar scale of infinity for functions of y which tend to $+\infty$ as y tends to 0 by positive values.

Examples LXXXVI. 1. Between any two terms $f(x)$, $F(x)$ of the series we can insert a new term $\phi(x)$ such that $\phi(x)$ tends to $+\infty$ more slowly than $f(x)$ and more rapidly than $F(x)$. [Thus between $\sqrt{x}$ and $\sqrt[3]{x}$ we could insert $x^{5/12}$: between $\sqrt{(\log x)}$ and $\sqrt[3]{(\log x)}$ we could insert $(\log x)^{5/12}$. And, generally, $\phi(x) = \sqrt{\{f(x)\,F(x)\}}$ satisfies the conditions stated.]

2. Find a function which tends to $+\infty$ more slowly than $\sqrt{x}$, but more rapidly than x^{α}, where α is any rational number less than $1/2$. [$\sqrt{x}/(\log x)$ is such a function; or $\sqrt{x}/(\log x)^{\beta}$, where β is any positive rational number.]

3. Find a function which tends to $+\infty$ more slowly than $\sqrt{x}$, but more rapidly than $\sqrt{x}/(\log x)^{\alpha}$, where α is any rational number. [Such a function is $\sqrt{x}/(\log\log x)$. It will be gathered from these examples that *incompleteness* is an inherent characteristic of the logarithmic scale of infinity.]

4. How does the function

$$f(x) = \{x^{\alpha}(\log x)^{\alpha'}(\log\log x)^{\alpha''}\}/\{x^{\beta}(\log x)^{\beta'}(\log\log x)^{\beta''}\}$$

behave as x tends to $+\infty$? [If $\alpha \neq \beta$ the behaviour of

$$f(x) = x^{\alpha-\beta} (\log x)^{\alpha'-\beta'} (\log\log x)^{\alpha''-\beta''}$$

is dominated by that of $x^{\alpha-\beta}$. If $\alpha = \beta$ the power of x disappears and the behaviour of $f(x)$ is dominated by that of $(\log x)^{\alpha'-\beta'}$, unless $\alpha' = \beta'$, when it is dominated by that of $(\log\log x)^{\alpha''-\beta''}$. Thus $f(x) \to +\infty$ if $\alpha > \beta$, or $\alpha = \beta$, $\alpha' > \beta'$, or $\alpha = \beta$, $\alpha' = \beta'$, $\alpha'' > \beta''$, and $f(x) \to 0$ if $\alpha < \beta$, or $\alpha = \beta$, $\alpha' < \beta'$, or $\alpha = \beta$, $\alpha' = \beta'$, $\alpha'' < \beta''$.]

5. Arrange the functions $x/\sqrt{(\log x)}$, $x\sqrt{(\log x)}/\log\log x$, $x \log\log x/\sqrt{(\log x)}$, $(x \log\log\log x)/\sqrt{(\log\log x)}$ according to the rapidity with which they tend to $+\infty$ with x.

6. Arrange

$$\log\log x/(x \log x), \quad (\log x)/x, \quad x \log\log x/\sqrt{(x^2+1)}, \quad \{\sqrt{(x+1)}\}/x(\log x)^2$$

according to the rapidity with which they tend to zero as x tends to $+\infty$.

7. Arrange

$$x \log\log(1/x), \quad \sqrt{[x/\{\log(1/x)\}^3]}, \quad \sqrt{\{x \sin x \log(1/x)\}}, \quad (1-\cos x)\log(1/x)$$

according to the rapidity with which they tend to zero as $x \to +0$.

8. Show that

$$D_x \log\log x = 1/(x \log x), \quad D_x \log\log\log x = 1/(x \log x \log\log x),$$

and so on.

9. Show that

$$D_x (\log x)^\alpha = \alpha/\{x(\log x)^{1-\alpha}\}, \quad D_x (\log\log x)^\alpha = \alpha/\{x \log x (\log\log x)^{1-\alpha}\},$$

and so on.

185. The number e. We shall now introduce a number, usually denoted by e, which is of immense importance in higher mathematics. It is, like π, one of the fundamental constants which perpetually occur in analysis.

We define e as *the number whose logarithm is* 1. In other words e is defined by the equation

$$1 = \int_1^e \frac{dt}{t}.$$

Since $\log x$ continually increases with x, it can only pass *once* through the value 1. Hence our definition does in fact define one definite number.

Example. Prove that $2 < e < 3$. [In the first place it is evident that

$$\int_1^2 \frac{dt}{t} < 1,$$

and so $2 < e$. Also

$$\int_1^3 \frac{dt}{t} = \left(\int_1^{5/4} + \int_{5/4}^{3/2} + \int_{3/2}^{7/4} + \int_{7/4}^{2} + \dots + \int_{11/4}^{3}\right)\frac{dt}{t}$$

$$> \tfrac{1}{4}\{\tfrac{4}{5} + \tfrac{2}{3} + \tfrac{4}{7} + \tfrac{1}{2} + \tfrac{4}{9} + \tfrac{2}{5} + \tfrac{4}{11} + \tfrac{1}{3}\} > 1,$$

so that $e < 3$.]

Now $\log(xy) = \log x + \log y$ and in particular

$$\log x^2 = 2\log x,\ \log x^3 = 3\log x,\ \dots,\ \log x^n = n\log x,$$

where n is any positive integer. Hence

$$\log e^n = n\log e = n.$$

Again, if p and q are any positive integers, and $e^{p/q}$ denotes the positive qth-root of e^p, we have

$$p = \log e^p = \log(e^{p/q})^q = q\log e^{p/q},$$

so that $\log(e^{p/q}) = p/q$. Thus if y has any positive rational value and e^y denotes the positive yth-power of e, we have

$$\log e^y = y \quad \dots\dots\dots\dots\dots\dots(1),$$

and $\log e^{-y} = -\log e^y = -y$. Hence the equation (1) is true for all rational values of y, positive or negative. In other words the equations

$$y = \log x, \quad x = e^y \dots\dots\dots\dots\dots\dots(2)$$

are consequences of one another so long as y is rational and e^y has its positive value. At present we have not given any definition of a power such as e^y in which the index is irrational, and the function e^y is defined for rational values of y only.

186. The exponential function e^y. We now define the *exponential function* e^y for all real values of y as the inverse of the logarithmic function. In other words, if $y = \log x$, we write

$$x = e^y.$$

We saw that, as x varies from 0 towards $+\infty$, y increases steadily (in the stricter sense) from $-\infty$ towards $+\infty$. To one value of x corresponds one value of y and conversely. Also y is a continuous function of x, and it follows from § 88 that x is likewise a continuous function of y.

A direct proof of the continuity of the exponential function is easily given. For if $x = e^y$ and $x + \xi = e^{y+\eta}$, it is clear that

$$\eta = \int_x^{x+\xi} \frac{dt}{t}.$$

Thus $|\eta|$ is greater than $\xi/(x+\xi)$ if $\xi>0$, and than $|\xi|/x$ if $\xi<0$; and if η is very small ξ must also be very small.

Thus e^y is a continuous function of y which increases steadily from 0 towards $+\infty$ as y increases from $-\infty$ towards $+\infty$. Moreover, by the results of § 185, e^y is the positive yth-power of the number e, according to the elementary definitions, whenever y is a rational number. In particular $e^y=1$ when $y=0$. The general form of the graph of e^y is therefore as shown in Fig. 68. It is to be observed that e^y is positive for all values of y.

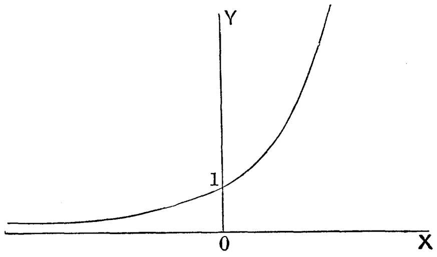

FIG. 68.

187. The principal properties of e^y. (1) If $x=e^y$, so that $y=\log x$, we have $dy/dx=1/x$ and

$$dx/dy=x=e^y.$$

Thus *the derivative of the exponential function is equal to the function itself.* In other words the exponential function is a function whose rate of increase is always equal to its own value. More generally, if $y=e^{ax}$ then $dy/dx=ae^{ax}$.

(2) *The exponential function satisfies the functional equation*

$$f(x+y)=f(x)f(y).$$

This is evident if x and y are rational, by the ordinary rules of indices. If x or y, or both, are irrational we can choose two series of rational numbers $x_1, x_2, \dots, x_n, \dots; y_1, y_2, \dots, y_n, \dots,$ such that $\lim x_n=x$, $\lim y_n=y$. Then, since the exponential function is continuous,

$$e^x\times e^y=\lim e^{x_n}\times\lim e^{y_n}=\lim e^{x_n+y_n}=e^{x+y}.$$

In particular $e^x\times e^{-x}=e^0=1$, or $e^{-x}=1/e^x$.

Or we may deduce the functional equation satisfied by e^y from that satisfied by $\log x$. For if $y_1 = \log x_1$, $y_2 = \log x_2$, so that $x_1 = e^{y_1}$, $x_2 = e^{y_2}$, we have $y_1 + y_2 = \log x_1 + \log x_2 = \log (x_1 x_2)$ and

$$e^{y_1+y_2} = e^{\log (x_1 x_2)} = x_1 x_2 = e^{y_1} \times e^{y_2}.$$

Examples LXXXVII. 1. If $dx/dy = x$ then $x = Ke^y$, where K is a constant.

2. There is no solution of the equation $f(x+y) = f(x) f(y)$ fundamentally distinct from $f(y) = e^y$. [For, differentiating the equation with respect to x and y in turn, we obtain

$$f'(x+y) = f'(x) f(y), \quad f'(x+y) = f(x) f'(y)$$

and so $f'(x)/f(x) = f'(y)/f(y)$, and therefore each is constant. Thus if $z = f(y)$, $dz/dy = z/A$, or

$$y = A \int \frac{dz}{z} = A \log z + B,$$

A and B being constants; so that $z = e^{(y-B)/A}$.]

3. Prove that $(e^{ax} - 1)/x \rightarrow a$ as $x \rightarrow 0$. [Applying the Mean Value Theorem, we obtain $e^{ax} - 1 = axe^{a\xi}$ where $0 < \xi < x$.]

188. (3) *The function e^y tends to $+\infty$ with y more rapidly than any power of y, or*

$$\lim y^a/e^y = \lim e^{-y} y^a = 0$$

as $y \rightarrow +\infty$, *for all values of α, however great.*

We saw that $(\log x)/x^\beta \rightarrow 0$ as $x \rightarrow +\infty$, for any positive value of β, however small. Hence, if $\alpha = 1/\beta$, $(\log x)^\alpha/x \rightarrow 0$ for any value of α, however large. The result follows by putting $x = e^y$.

From this result it follows that we can construct a 'scale of infinity' similar to that constructed in § 184, but extended in the opposite direction—i.e. a scale of functions which tend to $+\infty$ with x more and more rapidly. This scale is

$$x,\ x^2,\ x^3,\ \dots \quad e^x,\ e^{2x},\ \dots \quad e^{x^2},\ \dots,\ e^{x^3},\ \dots,\ e^{e^x},\ \dots \quad e^{e^{e^x}},\ \dots,$$

where of course $e^{x^2}, \dots, e^{e^x}, \dots$ denote $e^{(x^2)}, \dots, e^{(e^x)}, \dots$.

The reader should try to apply the remarks made in § 184 and Exs. LXXXVI, about the logarithmic scale, to this 'exponential scale' also. The two scales may of course (if the order of one is reversed) be combined into one scale

$$\dots \log\log x,\ \dots \quad \log x,\ \dots \quad x,\ \dots \quad e^x,\ \dots \quad e^{e^x},\ \dots$$

189. The general power a^x. The function a^x has been defined only for rational values of x, except in the particular case when $a = e$. When a is rational and positive, the positive value of the power a^x is given by the equations

$$a^x = (e^{\log a})^x = e^{x \log a}.$$

We take this as our *definition* of a^x when x is irrational. Thus $10^{\sqrt{2}} = e^{\sqrt{2} \cdot \log 10}$. It is to be observed that a^x, when x is irrational, is defined only for positive values of a, and is itself essentially positive. The reader will find no difficulty in verifying the following statements.

(1) Whatever value a may have, $a^x \times a^y = a^{x+y}$ and $(a^x)^y = a^{xy}$. In other words the laws of indices hold for irrational no less than for rational indices.

(2) If $a > 1$, $a^x = e^{x \log a} = e^{\alpha x}$, where α is positive. The graph of a^x is in this case similar to that of e^x, and $a^x \rightarrow +\infty$ with x, more rapidly than any power of x. For if $\alpha > 0$ and $\alpha x = y$, then $x^m e^{-\alpha x} = (y/\alpha)^m e^{-y} \rightarrow 0$ as x and y tend to $+\infty$.

If $a < 1$, $a^x = e^{x \log a} = e^{-\alpha x}$ where α is positive. The graph of a^x is then similar in shape to that of e^x, but reversed as regards right and left, and $a^x \rightarrow 0$ as $x \rightarrow +\infty$, more rapidly than any power of $1/x$.

(3) a^x is a continuous function of x, and $D_x a^x = a^x \log a$.

(4) a^x is also a continuous function of a, and $D_a a^x = xa^{x-1}$.

(5) $(a^x - 1)/x \rightarrow \log a$ as $x \rightarrow 0$. This of course is a mere corollary from the fact that $D_x a^x = a^x \log a$, but the particular form of the result is often useful; it is of course equivalent to the result (Ex. LXXXVII. 3) that $(e^{\alpha x} - 1)/x \rightarrow \alpha$ as $x \rightarrow 0$.

In the course of the preceding chapters a great many results involving the function a^x have been stated with the limitation that x is rational. The definition which we have now given, and the theorems proved above, enable us to remove this restriction.

190. The representation of e^y as a limit. In Ch. IV, § 67, we proved that $\{1 + (1/n)\}^n$ tends, as $n \rightarrow \infty$, to a limit which we provisionally denoted by e. We shall now identify this limit with the number e of the preceding sections. We can however establish a more general result, viz. that expressed by the equations

$$\lim_{n \rightarrow \infty} \left(1 + \frac{y}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 - \frac{y}{n}\right)^{-n} = e^y \ldots\ldots\ldots\ldots(1).$$

As the result is of very great importance, we shall indicate alternative lines of proof.

(1) Since $D_x \log(1 + yx) = y/(1 + yx)$, it follows that

$$\lim \{\log(1 + yh)\}/h = y,$$

as $h \rightarrow 0$; or, putting $h = 1/\xi$,

$$\lim [\xi \log \{1 + (y/\xi)\}] = y$$

as $\xi \rightarrow +\infty$ or $\xi \rightarrow -\infty$. Since the exponential function is continuous it follows that

$$\{1+(y/\xi)\}^{\xi} = e^{\xi \log\{1+(y/\xi)\}} \rightarrow e^{y}$$

as $\xi \rightarrow +\infty$ or $\xi \rightarrow -\infty$: i.e. that

$$\lim_{\xi \rightarrow +\infty} \{1+(y/\xi)\}^{\xi} = \lim_{\xi \rightarrow -\infty} \{1+(y/\xi)\}^{\xi} = e^{y}.$$

If we suppose that $\xi \rightarrow +\infty$ or $-\infty$ through integral values only, we obtain the result expressed by the equations (1).

(2) If n is any positive integer, however large, and $x > 1$, we have

$$\int_1^x \frac{dt}{t^{1+(1/n)}} < \int_1^x \frac{dt}{t} < \int_1^x \frac{dt}{t^{1-(1/n)}},$$

or

$$n(1-x^{-1/n}) < \log x < n(x^{1/n}-1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

from which we deduce

$$\left(1+\frac{y}{n}\right)^{n} < x < \left(1-\frac{y}{n}\right)^{-n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

if $y = \log x > 0$. It is easy to prove, as in § 67, that the first of these two functions of y is an increasing and the second a decreasing function of n, and therefore that each tends to a limit as $n \rightarrow \infty$; and the two limits must be equal. For if $1+(y/n) = \eta_1$ and $1/\{1-(y/n)\} = \eta_2$, we have (Ex. XXXVI. 8), at any rate for sufficiently large values of n,

$$\eta_2^{n} - \eta_1^{n} < n\eta_2^{n-1}(\eta_2 - \eta_1) = y^2\eta_2^{n}/n$$

which evidently tends to 0 as $n \rightarrow \infty$.

191. The representation of $\log x$ as a limit. We can also prove (cf. § 68) that

$$\lim n(1-x^{-1/n}) = \lim n(x^{1/n}-1) = \log x.$$

For $$n(x^{1/n}-1) - n(1-x^{-1/n}) = n(x^{1/n}-1)(1-x^{-1/n})$$

which tends to zero as $n \rightarrow \infty$, since $n(x^{1/n}-1)$ tends to a limit (§ 68) and $x^{-1/n}$ to 1 (Ex. XXX. 10). The result follows from the inequalities (2) of § 190 (2).

Examples LXXXVIII. 1. Prove, by taking $n = 6$ in the inequalities (3) of § 190, that $2.5 < e < 2.9$.

2. Prove that, if $t > 1$, $(t^{1/n} - t^{-1/n})/(t - t^{-1}) < 1/n$, and so that, if $x > 1$,

$$\int_1^x \frac{dt}{t^{1-(1/n)}} - \int_1^x \frac{dt}{t^{1+(1/n)}} < \frac{1}{n}\int_1^x \left(t - \frac{1}{t}\right)\frac{dt}{t} = \frac{1}{n}\left(x + \frac{1}{x} - 2\right).$$

Hence deduce the results of § 190.

3. If ξ_n is a function of n such that $n\xi_n \rightarrow l$ as $n \rightarrow \infty$, then $(1+\xi_n)^n \rightarrow e^l$. [Writing $n \log(1+\xi_n)$ in the form

$$l\left(\frac{n\xi_n}{l}\right)\frac{\log(1+\xi_n)}{\xi_n},$$

and using Ex. LXXXIV. 4, we see that $n \log(1+\xi_n) \rightarrow l$.]

4. If $n\xi_n \rightarrow +\infty$ then $(1+\xi_n)^n \rightarrow +\infty$, and if $n\xi_n \rightarrow -\infty$ then $(1+\xi_n)^n \rightarrow 0$.

192. Common logarithms. The reader is probably familiar with the idea of a logarithm and its use in numerical calculation. He will remember that in elementary algebra $\log_a x$ (the logarithm of x to the base a) is defined by the equations

$$x = a^y,\ y = \log_a x.$$

This definition of course, at the stage of knowledge when it is usually given, can be regarded as applying only to rational values of y, though this point is often passed over in silence.

Our logarithms are therefore logarithms to the base e. For numerical work logarithms to the base 10 are used. If

$$y = \log x = \log_e x,\ z = \log_{10} x,$$

then $x = e^y$ and also $x = 10^z = e^{z \log_e 10}$, so that

$$\log_{10} x = (\log_e x)/(\log_e 10).$$

Thus it is easy to pass from one system to the other when once $\log_e 10$ has been calculated.

It is no part of our purpose in this book to go into details concerning the practical uses of logarithms. If the reader is not familiar with them he should consult some text-book on Elementary Algebra or Trigonometry*.

Examples LXXXIX. 1. Show that

$$D_x e^{\phi(x)} = e^{\phi(x)}\phi'(x),\quad D_x \phi(e^x) = e^x \phi'(e^x).$$

2. Show that

$$D_x e^{ax}\cos bx = re^{ax}\cos(bx+\theta),\quad D_x e^{ax}\sin bx = re^{ax}\sin(bx+\theta)$$

where $r = \surd(a^2+b^2)$, $\cos\theta = a/r$, $\sin\theta = b/r$. Hence determine the nth derivatives of the functions $e^{ax}\cos bx$, $e^{ax}\sin bx$, and show in particular that $D_x^n e^{ax} = a^n e^{ax}$.

3. If $y = e^{ax}u$, $D_x y = e^{ax}(D_x u + au)$, and

$$D_x^n y = e^{ax}\left\{D_x^n u + \binom{n}{1} a D_x^{n-1} u + \binom{n}{2} a^2 D_x^{n-2} u + \dots + a^n u\right\}.$$

[Proceed by induction.]

4. **Integrals containing the exponential function.** Prove that

$$\int e^{ax}\cos bx\,dx = \frac{a\cos bx + b\sin bx}{a^2+b^2}e^{ax},\quad \int e^{ax}\sin bx\,dx = \frac{a\sin bx - b\cos bx}{a^2+b^2}e^{ax}.$$

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* See for example Chrystal's *Algebra*, vol. I., ch. XXI. The value of $\log_e 10$ is 2·302... and that of its reciprocal ·434... .

[Denoting the two integrals by I, J, and integrating by parts, we obtain

$$aI=e^{ax}\cos bx+bJ,\quad aJ=e^{ax}\sin bx-bI.$$

Solve these equations for I and J.]

5. Prove that, if $a>0$, then

$$\int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2},\quad \int_0^\infty e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2}.$$

6. If $I_n=\int e^{ax}x^n\,dx$ then $aI_n=e^{ax}x^n-nI_{n-1}$. [Integrate by parts. It follows that I_n can be calculated for all positive integral values of n.]

7. Prove that, if n is a positive integer, then

$$\int_0^\xi e^{-x}x^n\,dx=n!\,e^{-\xi}\left(e^\xi-1-\xi-\frac{\xi^2}{2!}-\dots-\frac{\xi^n}{n!}\right)$$

and
$$\int_0^\infty e^{-x}x^n\,dx=n!.$$

8. Show how to find the integral of any rational function of e^x. [Put $x=\log u$, $e^x=u$, $dx/du=1/u$, and the integral is transformed into that of a rational function of u.]

9. Prove that we can integrate any function of the form $P(x, e^{ax}, e^{bx}, \dots)$, where P denotes a polynomial. [This follows from the fact that P can be expressed as the sum of a number of terms of the type $Ax^m e^{kx}$, where m is a positive integer.]

10. Integrate $e^x\{\phi(x)+\phi'(x)\}$, $x^{n-1}e^x(x+n)$, and $e^x(1+\sin x)/(1+\cos x)$.

11. Prove that $\int_a^\infty e^{-\lambda x}R(x)\,dx$, where $\lambda>0$ and a is greater than the greatest root of the denominator of $R(x)$, is convergent. [This follows from the fact that $e^{\lambda x}\to+\infty$ more rapidly than any power of x.]

12. Prove that $\int_{-\infty}^\infty e^{-\lambda x^2+\mu x}\,dx$ is convergent, if $\lambda>0$, for all values of μ, and that the same is true of $\int_{-\infty}^\infty e^{-\lambda x^2+\mu x}x^n\,dx$, where n is any positive integer.

13. Draw the graphs of e^{x^2}, e^{-x^2}, xe^x, xe^{-x}, xe^{x^2}, xe^{-x^2}, and $x\log x$, determining any maxima and minima of the functions and any points of inflexion on their graphs.

14. Show that the equation $e^{ax}=bx$, where a and b are positive, has two real roots, one, or none, according as $b>ae$, $b=ae$, or $b<ae$. [The tangent to the curve $y=e^{ax}$ at the point $(\xi, e^{a\xi})$ is

$$y-e^{a\xi}=ae^{a\xi}(x-\xi),$$

which passes through the origin if $a\xi=1$, so that the line $y=aex$ touches the curve at the point $(1/a, e)$. The result now becomes obvious on drawing the line $y=bx$. The reader should discuss the cases in which a or b or both are negative.]

15. Discuss in a similar manner the equation $bxe^{ax}=1$. [Put $x=-z$.]

16. Show that the equation $e^x = 1 + x$ has no real root except $x = 0$, and that $e^x = 1 + x + \frac{1}{2}x^2$ has three real roots.

17. Draw the graphs of the functions

$$\log\{x + \sqrt{(x^2+1)}\}, \quad \log\left(\frac{1+x}{1-x}\right), \quad e^{-ax}\cos(bx+c),$$

$$e^{1/x}, \quad e^{-1/x^2}, \quad e^{[x]}, \quad e^{-x^2}\cos^2 x + x^2 \sin^2 x.$$

18. Determine roughly the positions of the real roots of the equations

$$\log\{x+\sqrt{(x^2+1)}\} = \frac{x}{100}, \quad e^x - \frac{2+x}{2-x} = \frac{1}{10000}, \quad e^x \sin x = 7, \quad e^{x^2}\sin x = 10000.$$

19. **The hyperbolic functions.** The hyperbolic functions $\cosh x$*, $\sinh x$, etc. are defined by the equations

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \quad \sinh x = \tfrac{1}{2}(e^x - e^{-x}),$$

$$\tanh x = (\sinh x)/(\cosh x), \quad \coth x = (\cosh x)/(\sinh x),$$

$$\operatorname{sech} x = 1/(\cosh x), \quad \operatorname{cosech} x = 1/(\sinh x).$$

Draw the graphs of these functions.

20. Establish the formulae

$$\cosh(-x) = \cosh x, \quad \sinh(-x) = -\sinh x, \quad \tanh(-x) = -\tanh x,$$

$$\cosh^2 x - \sinh^2 x = 1, \quad \operatorname{sech}^2 x + \tanh^2 x = 1, \quad \coth^2 x - \operatorname{cosech}^2 x = 1,$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x, \quad \sinh 2x = 2\sinh x \cosh x,$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$$

21. Verify that these formulae may be deduced from the corresponding formulae in $\cos x$ and $\sin x$, by writing $\cosh x$ for $\cos x$ and $i \sinh x$ for $\sin x$.

[It follows that the same is true of all the formulae involving $\cos nx$ and $\sin nx$ which are deduced from the corresponding elementary properties of $\cos x$ and $\sin x$. The reason of this analogy will appear in Ch. X.]

22. Prove that

$$D_x \cosh x = \sinh x, \; D_x \sinh x = \cosh x, \; D_x \tanh x = \operatorname{sech}^2 x, \; D_x \coth x = -\operatorname{cosech}^2 x,$$

$$D_x \operatorname{sech} x = -\operatorname{sech} x \tanh x, \quad D_x \operatorname{cosech} x = -\operatorname{cosech} x \coth x,$$

$$D_x \log\cosh x = \tanh x, \quad D_x \log\sinh x = \coth x,$$

$$D_x(2 \arctan e^x) = \operatorname{sech} x, \quad D_x \log\tanh \tfrac{1}{2}x = \operatorname{cosech} x.$$

[All these formulae may of course be transformed into formulae in integration.]

23. Prove that $\cosh x > 1$ and $-1 < \tanh x < 1$.

24. If $y = \cosh x$, $x = \log\{y \pm \sqrt{(y^2-1)}\}$. If $y = \sinh x$, $x = \log\{y + \sqrt{(y^2-1)}\}$. If $y = \tanh x$, $x = \frac{1}{2}\log\{(1+y)/(1-y)\}$. Account for the ambiguity of sign in the first case.

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* 'Hyperbolic cosine': for an explanation of this phrase see Hobson's *Trigonometry*, ch. XVI.

25. We shall denote the functions inverse to $\cosh x$, $\sinh x$, $\tanh x$ by $\arg\cosh x$*, $\arg\sinh x$, $\arg\tanh x$. Show that $\arg\cosh x$ is defined only for $x \geqq 1$, and is two-valued, while $\arg\sinh x$ is defined for all real values of x, and $\arg\tanh x$ for $-1 < x < 1$, and both of the two latter functions are one-valued. Sketch the graphs of the functions.

26. Show that if $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ and y is positive, and $\cos x \cosh y = 1$, then

$$y = \log(\sec x + \tan x), \quad D_x y = \sec x, \quad D_y x = \operatorname{sech} y.$$

27. Prove that, if $a > 0$, then $\int \frac{dx}{\surd(x^2+a^2)} = \arg\sinh(x/a)$, and

$$\int \frac{dx}{\surd(x^2-a^2)} = \arg\cosh(x/a), \quad -\arg\cosh(-x/a),$$

according as $x > 0$ or $x < 0$.

28. Prove that, if $a > 0$, then

$$\int \frac{dx}{x^2-a^2} = -(1/a)\arg\tanh(x/a), \quad (1/a)\arg\coth(x/a),$$

according as x is numerically less or greater than a. [The results of Exs. 27 and 28 furnish us with an alternative method of writing a good many of the formulae of Ch. VI.]

29. Solve the equation $a\cosh x + b\sinh x = c$, where $c > 0$, showing that it has no real roots if $b^2 + c^2 - a^2 < 0$, while if $b^2 + c^2 - a^2 > 0$ it has two, one, or no real roots according as $a+b$ and $a-b$ are both positive, of opposite signs, or both negative. Discuss the case in which $b^2 + c^2 - a^2 = 0$.

30. Solve the simultaneous equations $\cosh x \cosh y = a$, $\sinh x \sinh y = b$.

31. As $x \to +\infty$, $x^{1/x} \to 1$. [For $x^{1/x} = e^{(\log x)/x}$, and $(\log x)/x \to 0$. Cf. Ex. xxx. 11.]

32. As $x \to +0$, $x^x \to 1$.

33. If $\{f(n+1)\}/\{f(n)\} \to l$, where $l > 0$, as $n \to \infty$, then $\sqrt[n]{\{f(n)\}} \to l$. [For $\log f(n+1) - \log f(n) \to \log l$, and so $(1/n)\log f(n) \to \log l$ (Ch. IV, Misc. Ex. 29).]

34. $\sqrt[n]{(n!)}/n \to 1/e$ as $n \to \infty$.

[If $f(n) = n^{-n} n!$ then $\{f(n+1)\}/\{f(n)\} = \{1 + (1/n)\}^{-n}$.]

35. $\sqrt[n]{\{(2n)!/(n!)^2\}} \to 4$ as $n \to \infty$.

36. Discuss the approximate solution of the equation $e^x = x^{1000000}$.

[It is easy to see by general graphical considerations that the equation has two positive roots, one a little greater than 1 and one very large, and one negative root a little greater (algebraically) than -1. To determine roughly the size of the large positive root we may proceed as follows. If $e^x = x^{1000000}$ then

$$x = 10^6 \log x, \quad \log x = 13{\cdot}82 + \log\log x, \quad \log\log x = 2{\cdot}63 + \log\left(1 + \frac{\log\log x}{13{\cdot}82}\right),$$

roughly, since approximate values of $\log 10^6$ and $\log\log 10^6$ are $13{\cdot}82$ and $2{\cdot}63$ respectively. It is easy to see from these equations that the ratios $\log x : 13{\cdot}82$ and $\log\log x : 2{\cdot}63$ are not very far from ratios of equality, and that

$$x = 10^6(13{\cdot}82 + \log\log x) = 10^6(13{\cdot}82 + 2{\cdot}63) = 16450000$$

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* 'The argument whose hyperbolic cosine is x.'

gives a tolerable approximation to the root, the error involved being roughly measured by $10^6(\log\log x - 2{\cdot}63)$, which is not very far from being in a ratio of equality with $(10^6 \log\log x)/13{\cdot}8$ or $(10^6 \times 2{\cdot}63)/13{\cdot}8$, which is less than 200,000. The approximations are of course very rough, but suffice to give us a good idea of the scale of magnitude of the root.]

37. Discuss similarly the equations $e^x = 1000000\,x^{1000000}$, $e^{x^2} = x^{1000000000}$.

193. Logarithmic tests of convergence for series and integrals.

We showed in Ch. VIII (§§ 158 *et seq.*) that

$$\sum_1^\infty \frac{1}{n^s}, \quad \int_a^\infty \frac{dx}{x^s} \qquad (a > 0)$$

are convergent if $s > 1$ and divergent if $s \leqq 1$. Thus $\Sigma(1/n)$ is divergent, but $\Sigma(1/n^{1+\alpha})$ is convergent, if α is any positive number, however small.

We saw however in § 184 that with the aid of logarithms we can construct functions which tend to zero, as $n \to \infty$, more rapidly than $1/n$, yet less rapidly than $1/n^{1+\alpha}$, however small α may be (provided of course it is positive). For example $1/(n \log n)$ is such a function, and the question as to whether the series $\Sigma(1/n \log n)$ is convergent or divergent cannot be settled by comparison with any series of the type $\Sigma(1/n^s)$.

The same is true of such series as

$$\Sigma\{1/n(\log n)^2\}, \quad \Sigma\{(\log\log n)/n\sqrt{(\log n)}\}.$$

It is a question of some interest to find tests which shall enable us to decide whether series such as these are convergent or divergent: and such tests are easily deduced from the Integral Test of § 157.

For since

$$D_x(\log x)^{1-s} = (1-s)/x(\log x)^s, \quad D_x \log\log x = 1/x\log x,$$

we have

$$\int_a^\xi \frac{dx}{x(\log x)^s} = \frac{(\log\xi)^{1-s} - (\log a)^{1-s}}{1-s}, \quad \int_a^\xi \frac{dx}{x\log x} = \log\log\xi - \log\log a,$$

if $a > 1$. The first integral tends to the limit $-(\log a)^{1-s}/(1-s)$ as $\xi \to +\infty$, if $s > 1$, and to $+\infty$ if $s < 1$. The second integral tends to $+\infty$. Hence *the series and integral*

$$\sum_{n_0}^\infty \frac{1}{n(\log n)^s}, \quad \int_a^\infty \frac{dx}{x(\log x)^s}$$

where n_0 *and* a *are greater than unity, are convergent if* $s > 1$, *divergent if* $s \leqq 1$.

It follows, of course, that $\Sigma\phi(n)$ is convergent if $\phi(n)$ is positive and less than $K/\{n(\log n)^s\}$, where $s > 1$, for all values of n greater than some definite value, and divergent if $\phi(n)$ is positive and greater than $K/(n \log n)$ for all values of n greater than some definite value. And there is a corresponding theorem for integrals which we may leave it to the reader to formulate.

Examples XC. 1. The series

$$\Sigma \frac{1}{n(\log n)^2}, \quad \Sigma \frac{(\log n)^{100}}{n^{101/100}}, \quad \Sigma \frac{n^2-1}{n^2+1} \frac{1}{n(\log n)^{7/6}}$$

are convergent. [The convergence of the first series is a direct consequence of the theorem of the preceding section. That of the second follows from the fact that $(\log n)^{100}$ is less than n^β for sufficiently large values of n, however small β may be (provided it is positive). And so, taking $\beta = 1/200$, $(\log n)^{100} n^{-101/100}$ is less than $n^{-201/200}$ for sufficiently large values of n. The convergence of the third series follows from the comparison test at the end of the last section.]

2. The series

$$\Sigma \frac{1}{n(\log n)^{6/7}}, \quad \Sigma \frac{1}{n^{100/101}(\log n)^{100}}, \quad \Sigma \frac{n \log n}{(n \log n)^2 + 1}$$

are divergent.

3. The series

$$\Sigma \frac{(\log n)^p}{n^{1+s}}, \quad \Sigma \frac{(\log n)^p (\log \log n)^q}{n^{1+s}}, \quad \Sigma \frac{(\log \log n)^p}{n(\log n)^{1+s}},$$

where $s > 0$, are convergent for all values of p and q; similarly the series

$$\Sigma \frac{1}{n^{1-s}(\log n)^p}, \quad \Sigma \frac{1}{n^{1-s}(\log n)^p (\log \log n)^q}, \quad \Sigma \frac{1}{n(\log n)^{1-s}(\log \log n)^p}$$

are divergent.

4. The question of the convergence or divergence of such series as

$$\Sigma \frac{1}{n \log n \log \log n}, \quad \Sigma \frac{\log \log \log n}{n \log n \sqrt{(\log \log n)}}$$

cannot be settled by the theorem of § 193, since in each case the function under the sign of summation tends to zero more rapidly than $1/(n \log n)$ yet less rapidly than $1/\{n(\log n)^{1+\alpha}\}$, where α is any positive number, however small. For such series we need a still more delicate test. The reader should be able, starting from the equations

$$D_x (\log_k x)^{1-s} = (1-s)/\{x \log x \log_2 x \ldots \log_{k-1} x (\log_k x)^s\},$$

$$D_x \log_{k+1} x = 1/(x \log x \log_2 x \ldots \log_{k-1} x \log_k x),$$

where $\log_2 x = \log\log x$, $\log_3 x = \log\log\log x$, ..., to prove the following theorem: *the series and integral*

$$\sum_{n_0}^{\infty} \frac{1}{n\log n \log_2 n \ldots \log_{k-1} n\, (\log_k n)^s}, \quad \int_a^{\infty} \frac{dx}{x\log x \log_2 x \ldots \log_{k-1} x\, (\log_k x)^s}$$

are convergent if $s>1$, *divergent if* $s \leqq 1$, n_0 and a being any numbers sufficiently great to ensure that $\log_k n$ and $\log_k x$ are positive for $n \geqq n_0$ or $x \geqq a$. [These values of n_0 and a increase very rapidly as k increases: thus $\log x>0$ requires $x>1$, $\log_2 x>0$ requires $x>e$, $\log\log x>0$ requires $x>e^e$, and so on; and it is easy to see that $e^e>10$, $e^{e^e}>e^{10}>20{,}000$, $e^{e^{e^e}}>e^{20{,}000}>10^{8000}$. The reader should observe the extreme rapidity with which the higher exponential functions, such as e^{e^x} and $e^{e^{e^x}}$, increase with x. The same remark of course applies to such functions as a^{a^x} and $a^{a^{a^x}}$, where a has any value greater than unity. It has been computed that 9^{9^9} has 369,693,100 figures, while $10^{10^{10}}$ has of course 10,000,000,000.

Conversely, the rate of increase of the higher logarithmic functions is extremely slow. Thus to make $\log\log\log\log x>1$ we have to suppose x a number with over 8000 figures.]

5. Prove that the integral $\int_0^a \frac{1}{x}\left\{\log\left(\frac{1}{x}\right)\right\}^s dx$, where $0<a<1$, is convergent if $s<-1$, divergent if $s \geqq -1$. [Consider the behaviour of

$$\int_{\epsilon}^{a} \frac{1}{x}\left\{\log\left(\frac{1}{x}\right)\right\}^s dx$$

as $\epsilon \rightarrow +0$. This result also may be refined upon by the introduction of higher logarithmic factors.]

6. Prove that $\int_0^1 \frac{1}{x}\left\{\log\left(\frac{1}{x}\right)\right\}^s dx$ has no meaning for any value of s. [The last example shows that $s<-1$ is a necessary condition for convergence at the lower limit: but, if $s<0$, $\{\log(1/x)\}^s$ tends to $+\infty$ as $x \rightarrow 1-0$, like $(1-x)^s$, and so the integral diverges at the upper limit if $s<-1$.]

7. The necessary and sufficient conditions for the convergence of $\int_0^1 x^{a-1}\left\{\log\left(\frac{1}{x}\right)\right\}^s dx$ are $a>0$, $s>-1$.

Examples XCI. 1. **Euler's limit.** Show that

$$\phi(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} - \log n$$

tends to a limit as $n \rightarrow \infty$. [For

$$\phi(n+1) - \phi(n) = \frac{1}{n} - \log\left(\frac{n+1}{n}\right) = \frac{1}{n} - \int_n^{n+1} \frac{dt}{t} > 0.$$

Hence $\phi(n)$ is an increasing function. But

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \int_1^n \frac{dt}{t} = \log n < 1 + \frac{1}{2} + \ldots + \frac{1}{n-1}$$

or $0<\phi(n)<1-(1/n)<1$. Hence $\phi(n) \rightarrow l$ where $0<l \leqq 1$. The value of l is in fact $\cdot 577\ldots$, and l is usually called **Euler's constant.**]

2. Show similarly that if a and b are positive then

$$\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\dots+\frac{1}{a+(n-1)b}-\frac{1}{b}\log(a+nb)$$

tends to a limit as $n\to\infty$.

3. Show similarly that if $0<s<1$ then

$$\phi(n)=1+2^{-s}+3^{-s}+\dots+(n-1)^{-s}-\{n^{1-s}/(1-s)\}$$

tends to a limit as $n\to\infty$.

4. Show that the series

$$\frac{1}{1}+\frac{1}{2(1+\frac{1}{2})}+\frac{1}{3(1+\frac{1}{2}+\frac{1}{3})}+\dots$$

is divergent. [Compare with $\Sigma(1/n\log n)$.] Show also that the series derived from $\Sigma(1/n^s)$, in the same way that the above series is derived from $\Sigma(1/n)$, is convergent if $s>1$ and otherwise divergent.

5. Prove generally that if Σu_n is a series of positive terms, and

$$s_n=u_1+u_2+\dots+u_n,$$

then $\Sigma(u_n/s_n)$ is convergent or divergent according as Σu_n is convergent or divergent. [If Σu_n is convergent, s_n tends to a positive limit l, and so $\Sigma(u_n/s_n)$ is convergent. If Σu_n is divergent, $s_n\to+\infty$, and

$$u_n/s_n>\log\{1+(u_n/s_n)\}=\log(s_{n+1}/s_n)$$

(Ex. LXXXIV. 1); and it is evident that

$$\log(s_2/s_1)+\log(s_3/s_2)+\dots+\log(s_{n+1}/s_n)=\log(s_{n+1}/s_1)$$

tends to $+\infty$ as $n\to\infty$.]

6. Sum the series $1-\frac{1}{2}+\frac{1}{3}-\dots$. [We have

$$1+\frac{1}{2}+\dots+\frac{1}{2n}=\log(2n+1)+\gamma+\epsilon_n,\quad 2\left(\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2n}\right)=\log(n+1)+\gamma+\epsilon_n',$$

by Ex. 1, γ denoting Euler's constant, and ϵ_n, ϵ_n' being numbers which tend to zero as $n\to\infty$. Subtracting and making $n\to\infty$ we see that the sum of the given series is $\log 2$. See also § 195.]

194. Series connected with the exponential and logarithm. Expansion of e^x by Taylor's Theorem. Since all the derivatives of the exponential function are equal to the function itself, we have

$$e^x=1+x+\frac{x^2}{2!}+\dots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}e^{\theta x}$$

where $0<\theta<1$. But $x^n/n!\to 0$ as $n\to\infty$, whatever be the value of x (Ex. XXX. 12); and $e^{\theta x}<e^x$. Hence, making n tend to ∞, we have

$$e^x=1+x+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}+\dots\quad\dots\dots\dots\dots(1).$$

The series on the right-hand side of this equation is known as the **exponential series.** In particular we have

$$e = 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} + \dots \qquad \text{...........(2)};$$

and so

$$\left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} + \dots\right)^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \text{ (3)}$$

a result known as the **exponential theorem.** Also, if $a > 0$,

$$a^x = e^{x \log a} = 1 + (x \log a) + \frac{(x \log a)^2}{2!} + \dots \qquad \text{......(4)}.$$

The reader will observe that the exponential series has the property of reproducing itself when every term is differentiated, and that no other series of powers of x would possess this property (for some further remarks in this connection see Appendix II).

The power series for e^x is so important that it is worth while to establish it by an alternative method which does not depend upon Taylor's Theorem. Let

$$E_n(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!},$$

and suppose that $x > 0$. Then

$$\left(1 + \frac{x}{n}\right)^n = 1 + n\left(\frac{x}{n}\right) + \frac{n(n-1)}{1 . 2}\left(\frac{x}{n}\right)^2 + \dots + \frac{n(n-1)\dots 1}{1 . 2 \dots n}\left(\frac{x}{n}\right)^n$$

which is less than $E_n(x)$. And, provided $n > x$, we have also, by the binomial theorem for a negative integral exponent,

$$\left(1 - \frac{x}{n}\right)^{-n} = 1 + n\left(\frac{x}{n}\right) + \frac{n(n+1)}{1 . 2}\left(\frac{x}{n}\right)^2 + \dots > E_n(x).$$

Thus

$$\left(1 + \frac{x}{n}\right)^n < E_n(x) < \left(1 - \frac{x}{n}\right)^{-n}.$$

But (§ 190) the first and last functions tend to the limit e^x as $n \to \infty$, and therefore $E_n(x)$ must do the same. From this the equation (1) follows when x is positive; its truth when x is negative follows from the fact that the exponential series, as was shown in Ex. LXXXIII. 7, satisfies the functional equation $f(x)f(y) = f(x+y)$, so that $f(x)f(-x) = f(0) = 1$.

Examples XCII. 1. Calculate e, e^2, $1/e$, $1/e^2$ to six places of decimals by means of the exponential series.

2. Show that $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$, $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$.

3. If x is positive, the greatest term in the exponential series is the $([x]+1)$-th, unless x is an integer, when the preceding term is equal to it.

4. Show that $n! > (n/e)^n$. [For $n^n/n!$ is one term in the series for e^n.]

5. Show that e is not a rational number. [If $e=p/q$, where p and q are integers, we must have

$$\frac{p}{q}=1+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{q!}+\dots$$

or, multiplying up by $q!$,

$$q!\left(\frac{p}{q}-1-1-\frac{1}{2!}-\dots-\frac{1}{q!}\right)=\frac{1}{q+1}+\frac{1}{(q+1)(q+2)}+\dots$$

and this is absurd, since the left-hand side is integral, and the right-hand side less than $\{1/(q+1)\}+\{1/(q+1)\}^2+\dots=1/q$.]

6. Sum the series $\sum\limits_0^\infty P_r(n)\frac{x^n}{n!}$, where $P_r(n)$ is a polynomial of degree r in n. [We can express $P_r(n)$ in the form

$$A_0+A_1n+A_2n(n-1)+\dots+A_rn(n-1)\dots(n-r+1),$$

and

$$\sum_0^\infty P_r(n)\frac{x^n}{n!}=A_0\sum_0^\infty\frac{x^n}{n!}+A_1\sum_1^\infty\frac{x^n}{(n-1)!}+\dots+A_r\sum_r^\infty\frac{x^n}{(n-r)!}$$
$$=(A_0+A_1x+A_2x^2+\dots+A_rx^r)e^x.]$$

7. Show that

$$\sum_1^\infty\frac{n^3}{n!}x^n=(x+3x^2+x^3)e^x,\quad \sum_1^\infty\frac{n^4}{n!}x^n=(x+7x^2+6x^3+x^4)e^x;$$

and that if $S_n=1^3+2^3+\dots+n^3$ then

$$\sum_1^\infty S_n\frac{x^n}{n!}=\tfrac{1}{4}(4x+14x^2+8x^3+x^4)e^x.$$

In particular the last series is equal to zero when $x=-2$. (*Math. Trip.* 1904.)

8. Prove that $\Sigma(n/n!)=e$, $\Sigma(n^2/n!)=2e$, $\Sigma(n^3/n!)=5e$, and that $\Sigma(n^k/n!)$, where k any positive integer, is a positive integral multiple of e.

9. Prove that $\sum\limits_1^\infty\frac{(n-1)x^n}{(n+2)n!}=\{(x^2-3x+3)e^x+\tfrac{1}{2}x^2-3\}/x^2$.

[Multiply numerator and denominator by $n+1$, and proceed as in Ex. 6.]

10. Determine a, b, c so that $\{(x+a)e^x+(bx+c)\}/x^3$ tends to a limit as $x\to 0$, evaluate the limit, and draw the graph of the function $e^x+\frac{bx+c}{x+a}$.

11. Draw the graphs of $1+x$, $1+x+\frac{1}{2}x^2$, $1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$, and compare them with that of e^x.

12. Prove that $e^{-x}-1+x-\frac{x^n}{2!}+\dots-(-1)^n\frac{x^n}{n!}$ is positive or negative according as n is odd or even. Deduce the exponential theorem.

13. If

$$Z_0=e^x,\ Z_1=e^x-1,\ Z_2=e^x-1-x,\ Z_3=e^x-1-x-(x^2/2!),\ \dots$$

then $dZ_\nu/dx=Z_{\nu-1}$. Hence prove that if $t>0$ then

$$Z_1(t)=\int_0^t Z_0\,dx<te^t,\ Z_2(t)=\int_0^t Z_1\,dx<\int_0^t xe^x\,dx<e^t\int_0^t x\,dx=\frac{t^2}{2!}e^t,$$

and generally $Z_\nu(t)<t^\nu e^t/\nu!$. Deduce the exponential theorem.

195. The logarithmic series. Another very important expansion in powers of x is that of $\log(1+x)$. Since

$$\log(1+x)=\int_0^x \frac{dt}{1+t},$$

and $1/(1+t)=1-t+t^2-\dots$ if t is numerically less than unity, it is natural to expect* that $\log(1+x)$, when $-1<x<1$, will be equal to the series obtained by integrating each term of the series between 0 and x, i.e. to the series $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\dots$. And this is in fact the case. For

$$1/(1+t)=1-t+t^2-\dots+(-1)^{m-1}t^{m-1}+\{(-1)^m t^m/(1+t)\}$$

and so, if $x>-1$,

$$\log(1+x)=\int_0^x \frac{dt}{1+t}=x-\tfrac{1}{2}x^2+\dots+\frac{(-1)^{m-1}}{m}x^m+(-1)^m R_m,$$

where
$$R_m=\int_0^x \frac{t^m\,dt}{1+t}.$$

We require to show that the limit of R_m, when m tends to ∞, is zero. This is almost obvious when x is positive and less than or equal to unity; for then R_m is positive and less than

$$\int_0^x t^m\,dt=\frac{x^{m+1}}{(m+1)},$$

and therefore less than $1/(m+1)$.

If $-1<x<0$ we put $t=-u$ and $x=-\xi$, so that

$$R_m=(-1)^m\int_0^\xi \frac{u^m\,du}{1-u},$$

which shows that R_m has the sign of $(-1)^m$. Also, since the greatest value of $1/(1-u)$ in the range of integration is $1/(1-\xi)$, we have

$$0<|R_m|<\frac{1}{1-\xi}\int_0^\xi u^m\,du=\frac{\xi^m}{(m+1)(1-\xi)}<\frac{1}{(m+1)(1-\xi)}:$$

and so $\lim R_m=0$. Hence

$$\log(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\dots,$$

provided $-1<x\leqq 1$. If x lies outside these limits the series is not convergent. If $x=1$ we obtain

$$\log 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\dots,$$

a result already proved otherwise (Ex. XCI. 6).

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* See Appendix II for some further remarks on this subject.

196. The series for the inverse tangent. It is easy to prove in a precisely similar manner that

$$\arc\tan x = \int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - t^2 + t^4 - \dots)\,dt$$
$$= x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \dots,$$

provided $-1 \leqq x \leqq 1$. The only difference is that the proof is a little simpler: for since $\operatorname{arc}\tan x$ is an odd function of x we need only consider positive values of x. And the series is convergent for $x = -1$ as well as for $x = +1$. We leave the discussion to the reader. The value of $\operatorname{arc}\tan x$ which is represented by the series is of course that which, when $-1 \leqq x \leqq 1$, lies between $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$, and which we saw in Ch. VII (Ex. LXV. 3) to be the value represented by the integral.

If $x = 1$ we obtain the formula

$$\tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \dots.$$

Examples XCIII. 1. $\log\{1/(1-x)\} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$ if $-1 \leqq x < 1$.

2. $\operatorname{arg}\tanh x = \frac{1}{2}\log\{(1+x)/(1-x)\} = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots$ if $-1 < x < 1$.

3. Obtain the series for $\log(1+x)$ and $\operatorname{arc}\tan x$ by means of Taylor's theorem.

[Since $D_x{}^n \log(1+x) = D_x{}^{n-1}\{1/(1+x)\}$, the coefficients in the first series are easily calculated. A difficulty presents itself in the discussion of the remainder when x is negative, if Lagrange's form $R_n = (-1)^{n-1} x^n/\{n(1+\theta x)^n\}$ is used; Cauchy's form, viz.

$$R_n = (-1)^{n-1}(1-\theta)^{n-1} x^n/(1+\theta x)^n,$$

should be used (cf. the corresponding discussion for the Binomial Series, Ex. LVIII. 2 and § 147).

In the case of the second series we have

$$D_x{}^n \operatorname{arc}\tan x = D_x{}^{n-1}\{1/(1+x^2)\}$$
$$= (-1)^{n-1}(n-1)!\,(x^2+1)^{-n/2} \sin\{n \operatorname{arc}\tan(1/x)\},$$

(Ex. XLVII. 14) which reduces to $(-1)^{n-1}(n-1)!\sin\frac{1}{2}n\pi$ for $x = 0$. In this case there is no difficulty about the remainder, which is obviously not greater in absolute value than $1/n$.]

4. If $y > 0$ then

$$\log y = 2\left\{\frac{y-1}{y+1} + \frac{1}{3}\left(\frac{y-1}{y+1}\right)^3 + \frac{1}{5}\left(\frac{y-1}{y+1}\right)^5 + \dots\right\}.$$

[Use the identity $y = \left(1 + \dfrac{y-1}{y+1}\right)\Big/\left(1 - \dfrac{y-1}{y+1}\right)$. This series may be used to calculate $\log 2$, a purpose for which the series $1 - \frac{1}{2} + \frac{1}{3} - \dots$, owing to the slowness of its convergence, is practically useless. Put $y = 2$ and find $\log 2$ to 3 places of decimals.]

5. Find log 10 to 3 places of decimals from the formula

$$\log 10 = 3\log 2 + \log(1+\tfrac{1}{4}).$$

Deduce the values of $\log_{10} e$, $\log_{10} 2$.

6. Show that

$$\tfrac{1}{2}\log 2 = 7a+5b+3c,\quad \tfrac{1}{2}\log 3 = 11a+8b+5c,\quad \tfrac{1}{2}\log 5 = 16a+12b+7c,$$

where $a = \arg\tanh(1/31)$, $b = \arg\tanh(1/49)$, $c = \arg\tanh(1/161)$.

[These formulae enable us to find log 2, log 3, and log 5 rapidly and with any degree of accuracy.]

7. Show that

$$\tfrac{1}{4}\pi = \arctan(1/2) + \arctan(1/3) = 4\arctan(1/5) - \arctan(1/239),$$

and calculate π to 6 places of decimals.

8. Show that $\dfrac{1}{1-x}\log\left(\dfrac{1}{1-x}\right) = x + (1+\tfrac{1}{2})x^2 + (1+\tfrac{1}{2}+\tfrac{1}{3})x^3 + \dots,$

if $-1<x<1$. [Use Ex. LXXXIII. 2.]

9. Show that $\displaystyle\int_0^x \frac{dt}{1+t^4} = x - \tfrac{1}{5}x^5 + \tfrac{1}{9}x^9 - \dots$ if $-1 \leqq x \leqq 1$.

Deduce that $1 - \tfrac{1}{5} + \tfrac{1}{9} - \dots = \dfrac{1}{4\sqrt{2}}\{\pi + 2\log(\sqrt{2}+1)\}$. (*Math. Trip.* 1896.)

[Proceed as in § 195 and use the result of Ex. LI. 7.]

10. Prove similarly that

$$\tfrac{1}{3} - \tfrac{1}{7} + \tfrac{1}{11} - \dots = \int_0^1 \frac{t^2\,dt}{1+t^4} = \frac{1}{4\sqrt{2}}\{\pi - 2\log(\sqrt{2}+1)\}.$$

11. Prove generally that if a and b are positive integers then

$$\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \dots = \int_0^1 \frac{t^{a-1}\,dt}{1+t^b},$$

and so that the sum of the series can be found. Calculate in this way the sums of $1 - \tfrac{1}{4} + \tfrac{1}{7} - \dots$ and $\tfrac{1}{2} - \tfrac{1}{5} + \tfrac{1}{8} - \dots$.

12. Verify the inequalities of Ex. LXXXIV. 2, viz.

$$u - \tfrac{1}{2}u^2 < \log(1+u) < u - \{\tfrac{1}{2}u^2/(1+u)\},$$

with the help of the logarithmic series, assuming $0<u<1$.

197. The Binomial Series. We have already investigated the Binomial Theorem

$$(1+x)^m = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \dots,$$

assuming $-1 < x < 1$ and m rational. When m is irrational

$$(1+x)^m = e^{m\log(1+x)},$$

$$D_x(1+x)^m = \{m/(1+x)\}\,e^{m\log(1+x)} = m(1+x)^{m-1},$$

so that the rule for the differentiation of $(1+x)^m$ remains the same, and the proof of the theorem given in § 147 retains its validity. We shall not discuss the question of the convergence of the series when $x=1$ or -1 *.

Examples XCIV. 1. Prove that if $-1<x<1$ then

$$\frac{1}{\sqrt{(1+x^2)}}=1-\frac{1}{2}x^2+\frac{1\,.\,3}{2\,.\,4}x^4-\dots,\qquad \frac{1}{\sqrt{(1-x^2)}}=1+\frac{1}{2}x^2+\frac{1\,.\,3}{2\,.\,4}x^4+\dots.$$

2. **Approximation to quadratic and other surds.** Let $\sqrt{N}$ be a quadratic surd whose numerical value is required. Choose the nearest square to N (above or below N), say M^2: and let $N=M^2+d$ or M^2-d, d being positive. Since d cannot be greater than M, d/M^2 is comparatively small, and the surd $\sqrt{N}=M\sqrt{\{1\pm(d/M^2)\}}$ can be expressed in a series

$$=M\left\{1\pm\frac{1}{2}\left(\frac{d}{M^2}\right)-\frac{1\,.\,1}{2\,.\,4}\left(\frac{d}{M^2}\right)^2+\dots\right\},$$

which is at any rate fairly rapidly convergent, and may be very rapidly so.

Thus, e.g.,

$$\sqrt{67}=\sqrt{(64+3)}=8\left\{1+\frac{1}{2}\left(\frac{3}{64}\right)-\frac{1\,.\,1}{2\,.\,4}\left(\frac{3}{64}\right)^2+\dots\right\}.$$

Let us consider the error committed in taking $8\frac{3}{16}$ (the value given by the first two terms) as an approximate value. After the second term the terms alternate in sign and decrease. Hence the error is one of excess, and is numerically less than $3^2/(8\,.\,64^2)<\cdot 0003$.

3. If x be small compared with N^2 then

$$\sqrt{(N^2+x)}=N+(x/4N)+\{Nx/2\,(2N^2+x)\},$$

the error being of the order x^4/N^7. Apply the process to $\sqrt{997}$, $\sqrt[3]{1031}$. [Expanding by the binomial theorem

$$\sqrt{(N^2+x)}=N+\frac{x}{2N}-\frac{x^2}{8N^3}+\frac{x^3}{16N^5},$$

the error being less than the numerical value of the next term, viz. $5x^4/128N^7$. Also

$$\frac{Nx}{2\,(2N^2+x)}=\frac{x}{4N}\left(1+\frac{x}{2N^2}\right)^{-1}=\frac{x}{4N}-\frac{x^2}{8N^3}+\frac{x^3}{16N^5},$$

the error being less than $x^4/32N^7$. The result follows. No difference in method is needed for surds other than quadratic.]

4. If p differs from N^3 by less than 1 per cent. of either then $\sqrt[3]{p}$ differs from $\frac{2}{3}N+(\frac{1}{3}p/N^2)$ by less than $N/90000$. (*Math. Trip.* 1882.)

5. If $p=N^4+x$, and x is small compared with N, a good approximation for $\sqrt[4]{p}$ is

$$\sqrt[4]{p}=\frac{51}{56}N+\frac{5}{56}\frac{p}{N^3}+\frac{27Nx}{14\,(7p+5N^4)}.$$

Show that when $N=10$, $x=1$ this approximation is accurate to 16 places of decimals. (*Math. Trip.* 1886.)

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* See Bromwich, *Infinite Series*, pp. 150 *et seq.*

6. Show how to sum the series

$$\sum_0^\infty \phi_r(n)\binom{m}{n}x^n,$$

where $\phi_r(n)$ is a polynomial of degree r in n.

[Express $\phi_r(n)$ in the form $A_0+A_1n+A_2n(n-1)+\dots$ as in Ex. XCII. 6.]

7. Sum the series $\sum_0^\infty n\binom{m}{n}x^n$, $\sum_0^\infty n^2\binom{m}{n}x^n$ and prove that

$$\sum_0^\infty n^3\binom{m}{n}x^n=\{m^3x^3+m(3m-1)x^2+mx\}(1+x)^{m-3}.$$

198. An alternative method of development of the theory of the exponential and logarithmic functions. We shall now give an outline of a method of investigation of the properties of e^x and $\log x$ entirely different in logical order from that followed in the preceding pages. This method starts from the exponential series $1+x+\frac{x^2}{2!}+\dots$, which we know to be convergent for all values of x. We may therefore define the function $\exp x$ by the equation

$$\exp x=1+x+\frac{x^2}{2!}+\dots \quad\dots\dots\dots\dots(1).$$

We then prove as in Ex. LXXXIII. 7, that

$$\exp x\,.\,\exp y=\exp(x+y)\quad\dots\dots\dots\dots(2).$$

Again $$(\exp h-1)/h=1+\frac{h}{2!}+\frac{h^2}{3!}+\dots=1+\rho_h,$$

where ρ_h is numerically less than

$$|\tfrac{1}{2}h|+|\tfrac{1}{2}h|^2+|\tfrac{1}{2}h|^3+\dots=|\tfrac{1}{2}h|/(1-|\tfrac{1}{2}h|),$$

so that $\rho_h\to 0$ with h. And so

$$\{\exp(x+h)-\exp x\}/h=\exp x\{(\exp h-1)/h\}\to\exp x$$

as $h\to 0$, or

$$D_x\exp x=\exp x\quad\dots\dots\dots\dots(3).$$

Incidentally we have, of course, proved that $\exp x$ is a continuous function.

We have now a choice of procedure. Writing $y=\exp x$ and observing that $\exp 0=1$, we have

$$\frac{dy}{dx}=y,\qquad x=\int_1^y\frac{dt}{t},$$

and if we define the logarithmic function as the function inverse to the exponential function we are brought back to the point of view adopted in this chapter.

But we may proceed differently. From (2) it follows that if n is a positive integer then

$$(\exp x)^n=\exp nx,\quad (\exp 1)^n=\exp n.$$

If x is a positive rational fraction m/n

$$\{\exp(m/n)\}^n=\exp m=(\exp 1)^m,$$

and so $\exp(m/n)$ is equal to the positive value of $(\exp 1)^{m/n}$. By means of the equation $\exp x \cdot \exp(-x) = 1$ this result is extended to negative rational values of x, and so we have

$$\exp x = (\exp 1)^x = e^x,$$

say, where

$$e = \exp 1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots,$$

for all rational values of x. Finally we define e^x, when x is irrational, as being equal to $\exp x$. The logarithm is then defined as the function inverse to $\exp x$ or e^x.

Example. Develop the theory of the Binomial Series

$$f(m, x) = 1 + \binom{m}{1} x + \binom{m}{2} x^2 + \dots,$$

where $-1 < x < 1$, in a similar manner, starting from the equation

$$f(m, x) f(m', x) = f(m + m', x)$$

(Ex. LXXXIII. 6).

MISCELLANEOUS EXAMPLES ON CHAPTER IX*.

1. Given that $\log_{10} e = \cdot 4343$ and that 2^{10}, 3^{21} are nearly equal to powers of 10, calculate $\log_{10} 2$, $\log_{10} 3$ to four places of decimals. (*Math. Trip.* 1905.)

2. Determine which of $(\frac{1}{2}e)^{\sqrt{3}}$, $(\sqrt{2})^{\frac{1}{2}\pi}$ is the greater. [Take logarithms and observe that $\sqrt{3}/(\sqrt{3} + \frac{1}{4}\pi) < \frac{2}{5}\sqrt{3} < \cdot 6929 < \log 2$.]

3. Show that $\log_{10} n$ cannot be a rational number, if n is any positive integer not a power of 10. [If n is not divisible by 10, and $\log_{10} n = p/q$, we have $10^p = n^q$, which is impossible, since 10^p ends with 0 and n^q does not. If $n = 10^a N$, where N is not divisible by 10, $\log_{10} N$ and therefore

$$\log_{10} n = a + \log_{10} N$$

cannot be rational.]

4. For what values of x are the functions $\log x$, $\log\log x$, $\log\log\log x$, ... (*a*) equal to 0, (*b*) equal to 1, (*c*) not defined? Consider also the same question for the functions lx, llx, $lllx$, ..., where $lx = \log|x|$.

5. Show that

$$\log x - \binom{n}{1}\log(x+1) + \binom{n}{2}\log(x+2) - \dots + (-1)^n \log(x+n)$$

is negative and increases steadily towards 0 as x increases from 0 towards $+\infty$. [The derivative of the function is

$$\sum_0^n (-1)^r \binom{n}{r} \frac{1}{x+r} = \frac{n!}{x(x+1)\dots(x+n)},$$

as is easily seen by splitting up the right-hand side into partial fractions. This expression is positive, and the function itself tends to zero as $x \to +\infty$, since

$$\log(x+r) = \log x + \epsilon_x,$$

where $\epsilon_x \to 0$, and $1 - \binom{n}{1} + \binom{n}{2} - \dots = 0$.]

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* A considerable number of these examples are taken from Bromwich's *Infinite Series*.

6. If $x > -1$ then $x^2 > (1+x)\{\log(1+x)\}^2$. (*Math. Trip.* 1906.)

[Put $1+x=e^{\xi}$, and use the fact that $\sinh\xi > \xi$ if $\xi > 0$.]

7. Show that $\{\log(1+x)\}/x$ and $x/\{(1+x)\log(1+x)\}$ both decrease steadily as x increases from 0 towards $+\infty$.

8. Show that $\dfrac{1}{\log(1+x)} - \dfrac{1}{x} \to \dfrac{1}{2}$ as $x \to 0$. [Use Exs. LXXXIV., or the logarithmic series.]

9. Show that $\dfrac{1}{\log(1+x)} - \dfrac{1}{x}$ decreases steadily from 1 to 0 as x increases from -1 towards $+\infty$. [The function is undefined for $x=0$, but if we attribute to it the value $\frac{1}{2}$ for $x=0$ it becomes continuous there. Use Ex. 6 to show that the derivative is negative.]

10. Show that $e^x > Mx^N$, where M and N are large positive numbers, if x is greater than the greater of $2\log M$ and $16N^2$.

[Since $\log x < 2\sqrt{x}$, the inequality given is certainly satisfied if

$$x > \log M + 2N\sqrt{x},$$

and therefore certainly satisfied if $\frac{1}{2}x > \log M$, $\frac{1}{2}x > 2N\sqrt{x}$.]

11. If $f(x)$ and $\phi(x)$ increase steadily as $x \to +\infty$, and $f'(x)/\phi'(x) \to +\infty$, then $f(x)/\phi(x) \to +\infty$. [Use the result of Ch. VI, Misc. Ex. 29.] By taking $f(x)=x^a$, $\phi(x)=\log x$, prove that $(\log x)/x^a \to 0$ for all positive values of a.

12. If p and q are positive integers then

$$\frac{1}{pn+1} + \frac{1}{pn+2} + \dots + \frac{1}{qn} \to \log\left(\frac{q}{p}\right)$$

as $n \to \infty$. [Cf. Exs. LXVI. 5, 6.]

13. Prove that if x is positive then $n\log\{\frac{1}{2}(1+x^{1/n})\} \to -\frac{1}{2}\log x$ as $n \to \infty$. [We have

$$n\log\{\tfrac{1}{2}(1+x^{1/n})\} = n\log\{1-\tfrac{1}{2}(1-x^{1/n})\} = \tfrac{1}{2}n(1-x^{1/n})\frac{\log(1-u)}{u}$$

where $u=\frac{1}{2}(1-x^{1/n})$. Now use § 191 and Ex. LXXXIV. 4.]

14. Prove that, if a and b are positive, then

$$\{\tfrac{1}{2}(a^{1/n}+b^{1/n})\}^n \to \sqrt{(ab)}.$$

[Take logarithms and use Ex. 13.]

15. Show that

$$1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} = \tfrac{1}{2}\log n + \log 2 + \tfrac{1}{2}\gamma + \epsilon_n,$$

where γ is Euler's constant (Ex. XCI. 1) and $\epsilon_n \to 0$ as $n \to \infty$.

16. Show that

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \dots = \tfrac{3}{2}\log 2,$$

the series being formed from the series $1 - \frac{1}{2} + \frac{1}{3} - \dots$ by continually taking two positive terms and then one negative. [The sum of the first $3n$ terms is

$$\begin{aligned} 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{4n-1} - \frac{1}{2}\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) \\ = \tfrac{1}{2}\log(2n) + \log 2 + \tfrac{1}{2}\gamma + \epsilon_n - \tfrac{1}{2}(\log n + \gamma + \epsilon_n'), \end{aligned}$$

where ϵ_n and ϵ_n' tend to 0 as $n \to \infty$.]

17. Show that $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\ldots=\frac{1}{2}\log 2$.

18. Prove that

$$\sum_{1}^{n}\frac{1}{\nu(36\nu^2-1)}=-3+3\Sigma_{3n+1}-\Sigma_n-S_n$$

where $S_n=1+\frac{1}{2}+\ldots+\frac{1}{n}$, $\Sigma_n=1+\frac{1}{3}+\ldots+\frac{1}{2n-1}$. Hence prove that the sum of the series when continued to infinity is

$$-3+\tfrac{3}{2}\log 3+2\log 2. \qquad (\textit{Math. Trip.}\ 1905.)$$

19. Show that

$$\sum_{1}^{\infty}\frac{1}{n(4n^2-1)}=2\log 2-1,\quad \sum_{1}^{\infty}\frac{1}{n(9n^2-1)}=\tfrac{3}{2}(\log 3-1).$$

20. Prove that the sums of the four series

$$\sum_{1}^{\infty}\frac{1}{4n^2-1},\quad \sum_{1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1},\quad \sum_{1}^{\infty}\frac{1}{(2n+1)^2-1},\quad \sum_{1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)^2-1}$$

are $\frac{1}{2}$, $\frac{1}{4}\pi-\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{2}\log 2-\frac{1}{4}$ respectively.

21. Prove that $n!\,(a/n)^n\rightarrow 0$ or $+\infty$ according as $a<e$ or $a>e$. [For $u_{n+1}/u_n=a\{1+(1/n)\}^{-n}\rightarrow a/e$. It can be shown that the function tends to $+\infty$ when $a=e$: for a proof, which is rather beyond the scope of the theorems of this chapter, see Bromwich's *Infinite Series*, pp. 461 *et seq.*]

22. Find the limit as $x\rightarrow+\infty$ of

$$\left(\frac{a_0+a_1x+\ldots+a_rx^r}{b_0+b_1x+\ldots+b_rx^r}\right)^{\lambda_0+\lambda_1x},$$

distinguishing the different cases which may arise. (*Math. Trip.* 1886.)

23. Determine α and β so that the series $\Sigma\left(e^{1/n}-\alpha-\dfrac{\beta}{n}\right)$ is convergent.

24. If α, β, ... are different numbers, no equation of the type

$$Ae^{\alpha x}+Be^{\beta x}+\ldots=0,$$

where A, B, ... are polynomials, can hold for all values of x. [If α is the algebraically greatest of α, β, ..., the term $Ae^{\alpha x}$ outweighs all the rest as $x\rightarrow+\infty$.]

25. Show that the sequence

$$a_1=e,\quad a_2=e^{e^2},\quad a_3=e^{e^{e^3}},\ \ldots$$

tends to infinity more rapidly than any member of the exponential scale. [Let $e_1(x)=e^x$, $e_2(x)=e^{e_1(x)}$, and so on. Then, if $e_k(x)$ is any member of the exponential scale, $a_n>e_k(n)$ if $n>k$.]

26. Prove that

$$\frac{d}{dx}\{\phi(x)\}^{\psi(x)}=\frac{d}{dx}\{\phi(x)\}^{\alpha}+\frac{d}{dx}\{\beta^{\psi(x)}\}$$

where α is to be put equal to $\psi(x)$ and β to $\phi(x)$ after differentiation. Establish a similar rule for the differentiation of $\phi(x)^{[\{\psi(x)\}^{\chi(x)}]}$.

27. Prove that if $D_x{}^n e^{-x^2} = e^{-x^2}\phi_n(x)$ then (i) $\phi_n(x)$ is a polynomial of degree n, (ii) $\phi_{n+1} = -2x\phi_n + \phi_n'$, and (iii) all the roots of $\phi_n = 0$ are real and distinct, and separated by those of $\phi_{n-1} = 0$. [To prove (iii) assume the truth of the result for $n = 1, 2, \dots \kappa$, and consider the signs of $\phi_{\kappa+1}$ for the n values of x for which $\phi_\kappa = 0$ and for large (positive or negative) values of x.]

28. Show that the most general function $\phi(x)$, such that $\phi'' - a^2\phi = 0$ for all values of x, is $Ae^{ax} + Be^{-ax}$, where A and B are constants (cf. Ch. VI, Misc. Ex. 43). [Let $\phi = \psi e^{-ax}$; then $\psi'' - 2a\psi' = 0$ or $\psi''/\psi' = 2a$. If $\psi' > 0$ it follows that $\log\psi' = 2ax + C$, where C is a constant, or $\psi' = De^{2ax}$, where D is a positive constant. If $\psi' < 0$ we obtain $(-\psi'')/(-\psi') = 2a$, or

$$\log(-\psi') = 2ax + C,$$

or $\psi' = -De^{2ax}$. Hence we may write generally $\psi' = Ae^{2ax}$, where A is any constant, positive or negative. Thus

$$\psi = Ae^{2ax} + B, \quad \phi = Ae^{ax} + Be^{-ax}.$$

An equivalent form of solution is $G\cosh ax + H\sinh ax$, where G and H are constants. On the other hand the form $L\cosh(ax - \alpha)$ is not sufficiently general, for it is equivalent to $Ae^{ax} + Be^{-ax}$, where $A = \frac{1}{2}Le^{-\alpha}$, $B = \frac{1}{2}Le^{\alpha}$, so that A and B have the same sign, instead of being perfectly arbitrary. The reader should also attempt to solve the equation $\phi'' - a^2\phi = 0$ as in Ch. VI, Misc. Ex. 43, viz. by multiplication by $2\phi'$.]

29. If $\phi'' - a^2\phi = 0$ for all values of x, and $\phi(0) = \lambda$, $\phi'(0) = \mu$, then

$$\phi = \lambda\cosh ax + (\mu/a)\sinh ax.$$

30. The general solution of $f(xy) = f(x)f(y)$, where f is a differentiable function, is x^a, where a is a constant: and that of

$$f(x+y) + f(x-y) = 2f(x)f(y)$$

is $\cosh ax$ or $\cos ax$, according as $f''(0)$ is positive or negative. [In proving the second result assume that f has derivatives of the first three orders. Then

$$2f(x) + y^2\{f''(x) + \epsilon_y\} = 2f(x)[f(0) + yf'(0) + \tfrac{1}{2}y^2\{f''(0) + \epsilon_y'\}],$$

where ϵ_y and ϵ_y' tend to zero with y. It follows that $f(0) = 1$, $f'(0) = 0$, $f''(x) = f''(0)f(x)$, so that $a = \sqrt{\{f''(0)\}}$ or $\sqrt{\{-f''(0)\}}$.]

31. How do the functions $x^{\sin(1/x)}$, $x^{\sin^2(1/x)}$, $x^{\operatorname{cosec}(1/x)}$ behave as $x \to +0$?

32. Trace the curves $y = \tan x\, e^{\tan x}$, $y = \sin x \log\tan\frac{1}{2}x$.

33. Sketch the forms of the graphs of the functions

$$e^{xe^{-x}}, \quad e^{-xe^{x}}, \quad xe^{-e^{x}/x}, \quad (1/x)\,e^{-(e^{-x}/x)}.$$

34. The equation $e^x = ax + b$ has one real root if $a < 0$ or $a = 0$, $b > 0$. If $a > 0$ it has two real roots or none according as $a\log a \gtrless b - a$.

35. Show by graphical considerations that the equation $e^x = ax^2 + 2bx + c$ has one, two, or three real roots if $a > 0$, none, one, or two if $a < 0$; and show how to distinguish between the different cases.

36. Trace the curve $y=\frac{1}{x}\log\left(\frac{e^x-1}{x}\right)$, showing that the point $(0, \frac{1}{2})$ is a centre of symmetry, and that as x increases from $-\infty$ to $+\infty$, y steadily increases from 0 to 1. Deduce that the equation

$$\frac{1}{x}\log\left(\frac{e^x-1}{x}\right)=a$$

has no real root unless $0<a<1$, and then one, whose sign is the same as that of $a-\frac{1}{2}$. [In the first place

$$y-\tfrac{1}{2}=\frac{1}{x}\left\{\log\left(\frac{e^x-1}{x}\right)-\log\left(e^{\frac{1}{2}x}\right)\right\}=\frac{1}{x}\log\left(\frac{\sinh\frac{1}{2}x}{\frac{1}{2}x}\right)$$

is clearly an odd function of x. Also

$$\frac{dy}{dx}=\frac{1}{x^2}\left\{\tfrac{1}{2}x\coth\tfrac{1}{2}x-1-\log\left(\frac{\sinh\frac{1}{2}x}{\frac{1}{2}x}\right)\right\}.$$

The function inside the large bracket tends to zero as $x\rightarrow 0$; and its derivative is

$$\frac{1}{x}\left\{1-\left(\frac{\frac{1}{2}x}{\sinh\frac{1}{2}x}\right)^2\right\},$$

which has the sign of x. Hence $\frac{dy}{dx}>0$ for all values of x.]

37. Trace the curve $y=e^{1/x}\sqrt{(x^2+2x)}$, and show that the equation

$$e^{1/x}\sqrt{(x^2+2x)}=a$$

has no real roots if a is negative, one negative root if $0<a<\alpha=e^{1/\sqrt{2}}\sqrt{(2+2\sqrt{2})}$, and two positive roots and one negative if $a>\alpha$.

38. Show that the equation $f_n(x)=1+x+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}=0$ has one real root if n is odd and none if n is even.

[Assume this proved for $n=1, 2, \dots 2k$. Then $f_{2k+1}(x)=0$ has at least one real root, since its degree is odd, and it cannot have more since, if it had, $f'_{2k+1}(x)=f_{2k}(x)$ would have to vanish once at least. Hence $f_{2k+1}(x)=0$ has just one root, and so $f_{2k+2}(x)=0$ cannot have more than two. If it has two, say α and β, $f'_{2k+2}(x)=f_{2k+1}(x)$ must vanish once at least between α and β, say at γ. And

$$f_{2k+2}(\gamma)=f_{2k+1}(\gamma)+\frac{\gamma^{2k+2}}{(2k+2)!}>0.$$

But $f_{2k+2}(x)$ is also positive when x is large (positively or negatively), and a glance at a figure will show that these results are contradictory. Hence $f_{2k+2}(x)=0$ has no roots.]

39. If a and b are positive and nearly equal, prove that

$$\log\frac{a}{b}=\frac{1}{2}(a-b)\left(\frac{1}{a}+\frac{1}{b}\right),$$

approximately, the error being about $\frac{1}{6}\{(a-b)/a\}^3$. [Use the logarithmic series. This formula is interesting historically as having been employed by Napier for the numerical calculation of logarithms.]

40. Show by means of the exponential series that $x^k e^{-x} \to 0$ as $x \to +\infty$, for all values of k, and that $(\log x)/x^\alpha \to 0$ for all positive values of α. [Use the fact that $e^x > x^n/n!$ for all positive values of x and all positive integral values of n: then put $x = \log y$.]

41. Prove by multiplication of series that if $-1 < x < 1$ then

$$\tfrac{1}{2}\{\log(1+x)\}^2 = \tfrac{1}{2}x^2 - \tfrac{1}{3}(1+\tfrac{1}{2})x^3 + \tfrac{1}{4}(1+\tfrac{1}{2}+\tfrac{1}{3})x^4 - \dots,$$

$$-\tfrac{1}{2}\log(1+x)\log(1-x) = \tfrac{1}{2}x^2 + \tfrac{1}{4}(1-\tfrac{1}{2}+\tfrac{1}{3})x^4 + \tfrac{1}{6}(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5})x^6 + \dots,$$

$$\tfrac{1}{2}(\arctan x)^2 = \tfrac{1}{2}x^2 - \tfrac{1}{4}(1+\tfrac{1}{3})x^4 + \tfrac{1}{6}(1+\tfrac{1}{3}+\tfrac{1}{5})x^6 - \dots,$$

$$\tfrac{1}{4}\arctan x \log\left(\frac{1+x}{1-x}\right) = \tfrac{1}{2}x^2 + \tfrac{1}{6}(1-\tfrac{1}{3}+\tfrac{1}{5})x^6 + \tfrac{1}{10}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9})x^{10} + \dots,$$

$$\tfrac{1}{2}\arctan x \log(1+x^2) = \tfrac{1}{3}(1+\tfrac{1}{2})x^3 - \tfrac{1}{5}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})x^5 + \dots.$$

42. Prove that $\log(1+x) = x(1-x) + \tfrac{1}{2}x^2(1-x^2) + \tfrac{1}{3}x^3(1-x^3) + \dots$, if $-1 < x < 1$.

43. Prove that if $-\tfrac{1}{4}\pi < \theta < \tfrac{1}{4}\pi$ then

$$\tan^2\theta - \tfrac{1}{2}\tan^4\theta + \tfrac{1}{3}\tan^6\theta - \dots = \sin^2\theta + \tfrac{1}{2}\sin^4\theta + \tfrac{1}{3}\sin^6\theta + \dots.$$

44. Prove that

$$(1+ax)^{1/x} = e^a\{1 - \tfrac{1}{2}a^2x + \tfrac{1}{24}(8+3a)a^3x^2(1+\epsilon_x)\},$$

where $\epsilon_x \to 0$ with x.

45. The first $n+2$ terms in the expansion of $\log\left(1+x+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}\right)$ in powers of x are

$$x - \frac{x^{n+1}}{n!}\left\{\frac{1}{n+1} - \frac{x}{1!\,(n+2)} + \frac{x^2}{2!\,(n+3)} - \dots + (-1)^n\frac{x^n}{n!\,(2n+1)}\right\}.$$

(*Math. Trip.* 1899.)

[Let $1+x+\frac{x^2}{2!}+\dots+\frac{x^n}{n!} = s_n(x)$, $e^x = s_n(x) + r_n(x)$, and $f(x) = \log s_n(x)$. Then

$$f'(x) = \frac{d}{dx}\log\{e^x - r_n(x)\} = 1 + \frac{d}{dx}\log\{1 - e^{-x}r_n(x)\} = 1 - \frac{x^n}{n!}\,\frac{e^{-x}}{1-e^{-x}r_n(x)}$$

$$= 1 - \frac{x^n}{n!}\left\{1 - x + \frac{x^2}{2!} - \dots + (-1)^n\frac{x^n}{n!}\right\} + \rho(x),$$

where $\rho(x)$ is of the $(n+1)$th order of smallness when x is small. This equation gives the first $n+2$ terms of the Taylor's Series for $f'(x)$, and we can at once deduce those of the expansion of $f(x)$. For we must have

$$f'(0) = 1, \quad f''(0) = f'''(0) = \dots = f^{(n)}(0) = 0,$$

$$f^{(n+1)}(0) = -1, \quad f^{(n+2)}(0) = n+1, \quad f^{(n+3)}(0) = -\tfrac{1}{2}(n+1)(n+2), \dots$$

$$f^{(2n+1)}(0) = (-1)^{n+1}\frac{2n!}{(n!)^2};$$

and clearly $f(0) = 0$. The same argument enables us to prove generally that if the Taylor's Series for $f(x)$ is $a_0 + a_1x + a_2x^2 + \dots$ then the Taylor's Series for $f'(x)$ is $a_1 + 2a_2x + 3a_3x^2 + \dots$, the series obtained by differentiating separately each of the terms of the former series; and conversely.]

46. Show that, if $-1<x<1$, then

$$\frac{1}{3}x+\frac{1\,.\,4}{3\,.\,6}2^2x^2+\frac{1\,.\,4\,.\,7}{3\,.\,6\,.\,9}3^2x^3+\ldots=\frac{x(x+3)}{9(1-x)^{7/3}},$$

$$\frac{1}{3}x+\frac{1\,.\,4}{3\,.\,6}2^3x^2+\frac{1\,.\,4\,.\,7}{3\,.\,6\,.\,9}3^3x^3+\ldots=\frac{x(x^2+18x+9)}{27(1-x)^{10/3}}.$$

[Use the method of Ex. XCIV. 6: or start from the equation

$$1+\frac{1}{3}x+\frac{1\,.\,4}{3\,.\,6}x^2+\frac{1\,.\,4\,.\,7}{3\,.\,6\,.\,9}x^3+\ldots=\frac{1}{(1-x)^{1/3}},$$

and differentiate each side as explained above; then multiply the resulting equation by x and differentiate again; and so on.]

47. Prove that $\displaystyle\int_0^\infty \frac{dx}{(x+a)(x+b)}=\frac{1}{a-b}\log\left(\frac{a}{b}\right)$,

$$\int_0^\infty \frac{dx}{(x+a)(x+b)^2}=\frac{1}{(a-b)^2b}\left\{a-b-b\log\left(\frac{a}{b}\right)\right\},$$

$$\int_0^\infty \frac{x\,dx}{(x+a)(x+b)^2}=\frac{1}{(a-b)^2}\left\{a\log\left(\frac{a}{b}\right)-(a-b)\right\},$$

$$\int_0^\infty \frac{dx}{(x+a)(x^2+b^2)}=\frac{1}{(a^2+b^2)b}\left\{\tfrac{1}{2}\pi a-b\log\left(\frac{a}{b}\right)\right\},$$

$$\int_0^\infty \frac{x\,dx}{(x+a)(x^2+b^2)}=\frac{1}{a^2+b^2}\left\{\tfrac{1}{2}\pi b+a\log\left(\frac{a}{b}\right)\right\},$$

provided a and b are positive. Deduce, and verify independently, that each of the functions

$$a-1-\log a,\quad a\log a-a+1,\quad \tfrac{1}{2}\pi a-\log a,\quad \tfrac{1}{2}\pi+a\log a$$

is positive for all positive values of a.

48. Prove that if α, β, γ are all positive and $\beta^2>\alpha\gamma$, then

$$\int_0^\infty \frac{dx}{\alpha x^2+2\beta x+\gamma}=\frac{1}{\sqrt{(\beta^2-\alpha\gamma)}}\log\left\{\frac{\beta+\sqrt{(\beta^2-\alpha\gamma)}}{\sqrt{\alpha\gamma}}\right\};$$

while if α is positive and $\alpha\gamma>\beta^2$ the value of the integral is

$$\frac{1}{\sqrt{(\alpha\gamma-\beta^2)}}\arctan\left\{\frac{\sqrt{(\alpha\gamma-\beta^2)}}{\beta}\right\},$$

that value of the inverse tangent being chosen which lies between 0 and π. Are there any other really different cases in which the integral is convergent?

49. Prove that, if $a>-1$, then

$$\int_1^\infty \frac{dx}{(x+a)\sqrt{(x^2-1)}}=\int_0^\infty \frac{dt}{\cosh t+a}=2\int_1^\infty \frac{du}{u^2+2au+1}$$

and deduce that the value of the integral is

$$\frac{2}{\sqrt{(1-a^2)}}\arctan\sqrt{\left(\frac{1-a}{1+a}\right)}$$

if $-1<a<1$, and

$$\frac{1}{\sqrt{(a^2-1)}}\log\frac{\sqrt{(a+1)}+\sqrt{(a-1)}}{\sqrt{(a+1)}-\sqrt{(a-1)}}=\frac{2}{\sqrt{(a^2-1)}}\operatorname{arg\,tanh}\sqrt{\left(\frac{a-1}{a+1}\right)}$$

if $a>1$. Discuss the case in which $a=1$.

50. Transform the integral $\int_0^\infty \frac{dx}{(x+a)\sqrt{(x^2+1)}}$, where $a>0$, in the same ways, showing that its value is

$$\frac{1}{\sqrt{(a^2+1)}}\log\frac{a+1+\sqrt{(a^2+1)}}{a+1-\sqrt{(a^2+1)}}=\frac{2}{\sqrt{(a^2+1)}}\operatorname{arg\,tanh}\left\{\frac{\sqrt{(a^2+1)}}{a+1}\right\}.$$

51. Prove that $$\int_0^1 \arctan x\,dx=\tfrac{1}{4}\pi-\tfrac{1}{2}\log 2.$$

52. If $0<\alpha<1$, $0<\beta<1$, then

$$\int_{-1}^1 \frac{dx}{\sqrt{\{(1-2\alpha x+\alpha^2)(1-2\beta x+\beta^2)\}}}=\frac{1}{\sqrt{(\alpha\beta)}}\log\frac{1+\sqrt{(\alpha\beta)}}{1-\sqrt{(\alpha\beta)}}.$$

53. Prove that, if $a>b>0$, then

$$\int_{-\infty}^\infty \frac{d\theta}{a\cosh\theta+b\sinh\theta}=\frac{\pi}{\sqrt{(a^2-b^2)}}.$$

54. Prove that

$$\int_0^1 \frac{\log x}{1+x^2}\,dx=-\int_1^\infty \frac{\log x}{1+x^2}\,dx,\quad \int_0^\infty \frac{\log x}{1+x^2}\,dx=0\,;$$

and deduce that, if $a>0$, then

$$\int_0^\infty \frac{\log x}{a^2+x^2}\,dx=\frac{\pi}{2a}\log a.$$

[Use the substitutions $x=1/t$ and $x=au$.]

55. Prove that $\int_0^\infty \log\left(1+\frac{a^2}{x^2}\right)dx=\pi a$ if $a>0$. [Integrate by parts.]

56. Prove that the successive areas bounded by the curve $y=e^{-ax}\sin bx$, where a and b are positive, and the positive half of the axis of x, form a geometrical progression, and that their sum is

$$\frac{b}{a^2+b^2}\,\frac{1+e^{-a\pi/b}}{1-e^{-a\pi/b}}.$$

CHAPTER X.

THE GENERAL THEORY OF THE LOGARITHMIC, EXPONENTIAL, AND CIRCULAR FUNCTIONS.

199. Functions of a complex variable. In Ch. III we defined the complex variable

$$z = x + yi,$$

and we considered a few simple properties of some classes of expressions involving z, such as the polynomial $P(z)$. It is natural to describe such expressions as **functions** of z, and in fact we did describe the quotient $P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials, as a 'rational function.' We have however given no general definition of what is meant by a function of z.

It might seem natural to define a function of z in the same way as that in which we defined a function of the real variable x; i.e. to say that Z is a function of z if any relation subsists between z and Z in virtue of which a value or values of Z corresponds to some or all values of z. But it will be found, on closer examination, that this definition is not one from which any profit can be derived. For if z is given, so are x and y, and conversely: to assign a value of z is precisely the same thing as to assign a pair of values of x and y. Thus the 'function of z,' according to the definition suggested, is precisely the same thing as *a complex function*

$$f(x, y) + ig(x, y),$$

of the two real variables x and y. For example

$$x - yi, \quad xy, \quad |z| = \sqrt{(x^2 + y^2)}, \quad \operatorname{am} z = \arctan(y/x)$$

are 'functions of z.' The definition, although perfectly legitimate, is futile because it does not really define a new idea at all. It is therefore more convenient to use the expression 'function of the complex variable z' in a more restricted sense: in other words to pick out, from the general class of complex functions of the two real variables x and y, a special class to which the expression shall be restricted. But if we were to attempt to explain how this selection is made, and what are the characteristic properties of the special class of functions selected, we should be led far beyond the limits of this book. We shall therefore not attempt to give any general definitions, but shall confine ourselves entirely to special functions defined directly.

200. We have already defined *polynomials* in z (§ 34), *rational functions* of z (§ 37), and *roots* of z (§ 40). There is no difficulty in extending to the complex variable the definitions of *algebraical functions*, explicit and implicit, which we gave (§§ 16, 17) in the case of the real variable x. In all these cases we shall call the complex number z, the argument (§ 35) of the point z, the *argument* of the function $f(z)$ under consideration. The question which will occupy us in this chapter is that of defining, and determining the principal properties of, the logarithmic, exponential, and trigonometrical or circular functions of z. These functions are of course so far defined for real values only, the logarithm indeed for positive values only.

We shall begin with the logarithmic function. It is natural to attempt to define it by means of some extension of the definition

$$\log x = \int_1^x \frac{dt}{t} \quad (x > 0);$$

and in order to do this we shall find it necessary to consider briefly some extensions of the notion of an integral.

201. Integrals along a curve. Let AB be an arc C of a curve defined by the equations

$$x = \phi(t), \quad y = \psi(t);$$

and suppose that as t varies from t_0 to t_1 the point (x, y) moves along the curve, in the same direction, from A to B.

Then we define the *curvilinear integral*

$$\int_C \{g(x, y)\,dx + h(x, y)\,dy\} \quad \text{..............(1)},$$

as being equivalent to the ordinary integral obtained by effecting the formal substitutions $x=\phi(t)$, $y=\psi(t)$, i.e. to

$$\int_{t_0}^{t_1}\{g(\phi,\psi)\phi'+h(\phi,\psi)\psi'\}\,dt.$$

In giving this definition we tacitly assume that all the functions involved satisfy the restrictions of §§ 137 *et seq.*—viz. that they are continuous and have only a finite number of maxima and minima. We call C the *path of integration.*

The functions ϕ, ψ are of course essentially real. But we may suppose g and h complex, if we please, the integral of a complex function of a real variable t being naturally defined by the equation

$$\int_{t_0}^{t_1}\{G(t)+iH(t)\}\,dt=\int_{t_0}^{t_1}G(t)\,dt+i\int_{t_0}^{t_1}H(t)\,dt.$$

We now define

$$\int_C f(z)\,dz \quad\text{...........................}(2),$$

where $z=x+iy$, $f(z)=u+iv$, as being equivalent to

$$\int_C (u+iv)(dx+idy),$$

which is itself defined as being equivalent to

$$\int_C (udx-vdy)+i\int_C (vdx+udy),$$

each of these latter integrals being defined as above.

202. It will not be necessary for us to consider in any detail the properties of curvilinear integrals in general or of those of the type (2) in particular. If we were to attempt to do so we should be led beyond the range of this volume into what is known as the Theory of Functions of a Complex Variable. But there are one or two simple properties of such integrals which we shall have occasion to use.

In the first place, if $F(t)=G(t)+iH(t)$ is a complex function of the real variable t, and $t_0<t_1$, we have

$$\left|\int_{t_0}^{t_1}F(t)\,dt\right| \leqq \int_{t_0}^{t_1}|F(t)|\,dt.$$

For the left-hand side is the limit (§ 139) of a sum of the type

$$\Sigma\{G(t_\nu)+iH(t_\nu)\}(t_{\nu+1}-t_\nu),$$

and the right-hand side is the limit of

$$\Sigma\sqrt{[\{G(t_\nu)\}^2+\{H(t_\nu)\}^2]}\;(t_{\nu+1}-t_\nu),$$

and the second sum is at least as great as the modulus of the first (§ 36, (6)).

We can now prove a relation of inequality satisfied by the modulus of the integral (2). For

$$\left|\int_C f(z)\,dz\right| = \left|\int_{t_0}^{t_1} (u+iv)(\phi'+i\psi')\,dt\right| \leqq \int_{t_0}^{t_1} \surd\{(u^2+v^2)(\phi'^2+\psi'^2)\}\,dt$$
$$\leqq M\int_{t_2}^{t_1} \surd(\phi'^2+\psi'^2)\,dt,$$

where M is the maximum of $|f| = \surd(u^2+v^2)$ for all points on the path of integration. But the length of the path of integration is given by the integral

$$\int \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx = \int \surd(\phi'^2+\psi'^2)\,dt,$$

taken between the appropriate limits, and so the modulus of the integral (2) is certainly not greater than Ml, where l is the length of the path.

We shall also require the value of the integral $\int z^m\,dz$, where m is a positive integer, taken along any path in the plane of z. In order to determine this we observe first that if ϕ and ψ are real functions of the real variable t, then

$$\frac{d}{dt}(\phi+i\psi)^m = m(\phi+i\psi)^{m-1}(\phi'+i\psi') \quad \ldots\ldots\ldots\ldots\ldots\ldots(1);$$

or in other words that the formulae

$$\frac{d}{dt}\{f(t)\}^m = m\{f(t)\}^{m-1}f'(t), \quad \int \{f(t)\}^m f'(t)\,dt = \frac{\{f(t)\}^{m+1}}{m+1}$$

still hold when $f(t)$ is a complex function of t. For suppose that (1) holds for $m = 1, 2, \ldots, k$. We know already (§ 95) that the rule for the differentiation of a product (§ 94) still holds when the functions involved are complex. Thus

$$D_t(\phi+i\psi)^{m+1} = D_t\{(\phi+i\psi)(\phi+i\psi)^m\} = (m+1)(\phi+i\psi)^m(\phi'+i\psi');$$

and so (1), which certainly holds for $m=0$ and $m=1$, is established by induction. Hence

$$\int_C z^m\,dz = \int_{t_0}^{t_1} (\phi+i\psi)^m(\phi'+i\psi')\,dt = \left[\frac{(\phi+i\psi)^{m+1}}{m+1}\right]_{t_0}^{t_1} = \frac{z_1^{m+1}-z_0^{m+1}}{m+1},$$

where z_0 and z_1 are the arguments of the ends of the path C. The formula is the same as when the path is real, and it can be shown that all the ordinary formulae of integration are capable of similar extension: but this need not concern us at present.

Finally we shall have occasion to use the equation

$$\int_C f(z)\,dz = \int_{C_1} f(Z+a)\,dZ,$$

where a is any complex number and C_1 is the path in the plane of Z obtained by copying the path C in the plane of z and then giving it the displacement represented by the complex number $-a$. The truth of this equation follows immediately from the definitions. In particular

$$\int_1^{1+\zeta} \frac{dz}{z} = \int_0^{\zeta} \frac{dZ}{1+Z},$$

the paths of integration being the straight lines from 1 to $1+\zeta$ and from 0 to ζ respectively.

203. The definition of Log ζ. Now let $\zeta=\xi+i\eta$ be any complex number. We define Log ζ, the general logarithm of ζ, by the equation

$$\operatorname{Log}\zeta=\int_1^{\zeta}\frac{dz}{z},$$

the path of integration being any curve $x=\phi(t)$, $y=\psi(t)$ in the plane (x, y), subject only to the restrictions (i) that it does not pass through the origin, (ii) that ϕ and ψ satisfy the conditions of §§ 137 *et seq.* The first condition is of course required because of the discontinuity of $1/z$ for $z=0$. Thus (Fig. 69) the paths (a), (b), (c) are paths such as are contemplated in the definition. The value of Log z is thus defined when the particular path of integration has been chosen. But at present it is not clear how far the value of Log z resulting from the definition depends upon what path is chosen. Suppose for example that ζ is real and positive, say equal to ξ. Then one possible path of integration is the straight line from 1 to ξ, a path which we may suppose to be defined by the equations $x=t$, $y=0$. In this case and with this particular choice of the path of integration

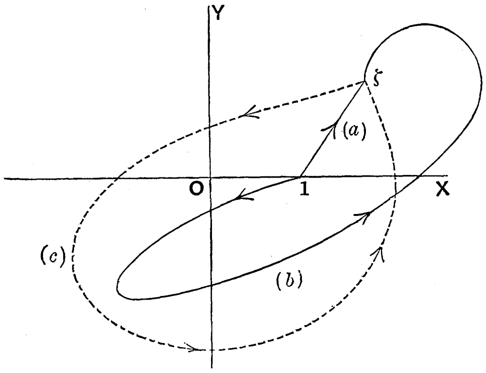

Fig. 69.

$$\operatorname{Log}\xi=\int_1^{\xi}\frac{dt}{t},$$

i.e. $\operatorname{Log}\xi=\log\xi$, the ordinary arithmetical logarithm of ξ defined in the last chapter. Thus one value at any rate of Log ξ, when ξ

is real and positive, is $\log \xi$. But in this case, as in the general case, there is an infinite variety of possible choice of the path of integration. There is nothing to show that *every* value of $\operatorname{Log} \xi$ is equal to $\log \xi$; and in point of fact we shall see that this is not the case. This is why we have adopted the notation $\operatorname{Log} \zeta$, $\operatorname{Log} \xi$ instead of $\log \zeta$, $\log \xi$. $\operatorname{Log} \xi$ is (possibly at any rate) a many valued function, and $\log \xi$ is only one of its values.

To return to the general case: so far as we can see at present, three alternatives are equally possible:—

(1) we may always get the same value of $\operatorname{Log} \zeta$, by whatever path we go from 1 to ζ;

(2) we may get a different value corresponding to every different path;

(3) we may get a number of different values each of which corresponds to a whole class of paths:—

and the truth or falsehood of any one of these alternatives is in no way implied by our definition.

204. The values of Log ζ. Let us suppose that the polar coordinates of the point $z = \zeta$ are (ρ, ϕ), so that

$$\zeta = \rho(\cos\phi + i\sin\phi).$$

We suppose for the present that $-\pi < \phi < \pi$, while ρ may have any positive value. Thus ζ may have any value other than real negative values.

Further, we suppose the coordinates (x, y) of any point on the path expressed as in § 23 in terms of t. Then r and θ, the polar coordinates of (x, y), may also be expressed in terms of t, by such equations as $r = f(t)$, $\theta = F(t)$. Now

$$\operatorname{Log} \zeta = \int_1^{\zeta} \frac{dz}{z} = \int_{t_0}^{t_1} \frac{1}{x + iy}\left(\frac{dx}{dt} + i\frac{dy}{dt}\right) dt,$$

by the definition. But if $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\begin{aligned}\frac{dx}{dt} + i\frac{dy}{dt} &= \left(\cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}\right) + i\left(\sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt}\right)\\ &= (\cos\theta + i\sin\theta)\left(\frac{dr}{dt} + ir\frac{d\theta}{dt}\right),\end{aligned}$$

and so
$$\operatorname{Log} \zeta = \int_{t_0}^{t_1} \frac{1}{r}\frac{dr}{dt}\,dt + i\int_{t_0}^{t_1} \frac{d\theta}{dt}\,dt.$$

Since
$$\int \frac{1}{r}\frac{dr}{dt}\,dt = \log r,$$
(r being positive), we obtain
$$\operatorname{Log}\zeta = [\log r] + i\,[\theta],$$
where $[\log r]$ and $[\theta]$ denote the differences between the final and initial values of $\log r$ and θ respectively.

It is clear that
$$[\log r] = \log\rho - \log 1 = \log\rho\,;$$
but the value of $[\theta]$ requires a little more consideration. Let us suppose first that the path of integration is the straight line from 1 to ζ. The initial value of θ is the amplitude of 1, or rather *one* of the amplitudes of 1 (viz. 0 and $2k\pi$, where k is any integer). Let us suppose that initially $\theta = 2k\pi$. It is evident from the figure that as t moves along the line θ increases from $2k\pi$ to $2k\pi + \phi$. Thus
$$[\theta] = (2k\pi + \phi) - 2k\pi = \phi,$$
and, when the path of integration is a straight line, $\operatorname{Log}\zeta = \log\rho + i\phi$.

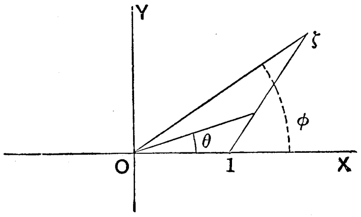

FIG. 70.

We shall denote this particular value of $\operatorname{Log}\zeta$ the **principal value**. When ζ is real and positive, $\zeta = \rho$ and $\phi = 0$, so that the principal value of $\operatorname{Log}\zeta$ is the ordinary logarithm $\log\zeta$. Hence it will be convenient in general to denote the principal value of $\operatorname{Log}\zeta$ by $\log\zeta$. Thus
$$\log\zeta = \log\rho + i\phi.$$

Next let us consider any path (such as those shown in Fig. 71) such that the area or areas included between the path and the straight line from 1 to ζ does not include the origin.

It is easy to see that $[\theta]$ is still equal to ϕ. Along the thick curve shown in the figure, for example, θ, initially equal to $2k\pi$, first decreases to the value
$$2k\pi - X\hat{O}P$$

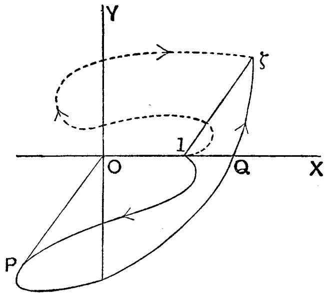

FIG. 71.

and then increases again, being equal to $2k\pi$ at Q, and finally to $2k\pi + \phi$. The dotted curve shows a similar but slightly more complicated case in which the straight line and the curve bound *two* areas, neither of which includes the origin.

Thus *if the path of integration is such that the closed curve formed by it and the line from* 1 *to* ζ *does not include the origin, then*

$$\operatorname{Log}\zeta = \log\zeta = \log\rho + i\phi.$$

On the other hand it is easy to construct paths of integration such that $[\theta]$ is not equal to ϕ. Consider, for example, the thick curve shown in Fig. 72.

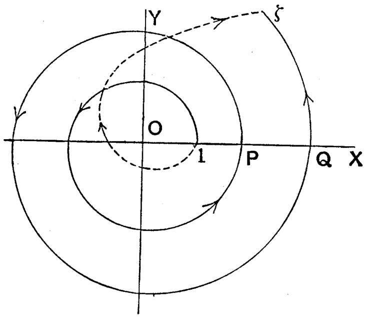

FIG. 72.

If θ is initially equal to $2k\pi$, it will evidently have increased by 2π when we get to P and by 4π when we get to Q: its final value will be $2k\pi + 4\pi + \phi$, and so $[\theta] = 4\pi + \phi$ and

$$\operatorname{Log}\zeta = \log\rho + i(4\pi + \phi).$$

In this case the path of integration *winds twice round the origin in the positive sense.* If we had taken a path winding m times round the origin we should have found, in a precisely similar way, that $[\theta] = 2m\pi + \phi$ and

$$\operatorname{Log}\zeta = \log\rho + i(2m\pi + \phi).$$

Here m is positive. By making the path wind round the origin in the opposite direction (as shown in the dotted path in Fig. 72), we obtain a similar series of values in which m is negative. Since $|\zeta| = \rho$, and the different angles $2m\pi + \phi$ are the different values of $\operatorname{am}\zeta$, we conclude that every value of $\log|\zeta| + i\operatorname{am}\zeta$ is a value of $\operatorname{Log}\zeta$: and it is clear from the preceding discussion that every value of $\operatorname{Log}\zeta$ must be of this form.

We may summarise our conclusions as follows: *the general value of* $\operatorname{Log}\zeta$ *is*

$$\log|\zeta| + i\operatorname{am}\zeta = \log\rho + i(2m\pi + \phi),$$

where m is any positive or negative integer. The value of m is determined by the path of integration chosen. If this path is a *straight line* $m = 0$, *and* $\operatorname{Log}\zeta = \log\rho + i\phi$. This particular value of $\operatorname{Log}\zeta$, which we denote by $\log\zeta$, is called the *principal value*; it is characterised by the fact that its imaginary part lies between $-\pi$ and $+\pi$.

In what precedes we have used ζ to denote the argument of the function $\operatorname{Log}\zeta$, and (ξ, η) or (ρ, ϕ) for the coordinates of ζ, and z, (x, y), (r, θ) for an arbitrary point on the path of integration and its coordinates. There is however no reason now why we should not revert to the natural notation in which z is used as the argument of the function $\operatorname{Log} z$, and this we shall do in the following examples.

Examples XCV. 1. We supposed above that $-\pi<\theta<\pi$, and so excluded the case in which z is *real and negative.* In this case the straight line from 1 to z passes through 0, and is therefore not admissible as a path of integration. Both π and $-\pi$ are values of $\operatorname{am} z$, and θ is equal to one or other of them: also $r=-z$. The values of $\operatorname{Log} z$ are still the values of $\log|z| + i \operatorname{am} z$, viz.

$$\log(-z)+(2m+1)\pi i$$

where m is an integer. The values $\log(-z)\pm\pi i$ correspond to paths from 1 to z lying respectively entirely above and below the real axis. Either of them may be taken as the principal value of $\operatorname{Log} z$, as convenience dictates. We shall choose the value $\log(-z)+\pi i$ corresponding to the first path.

2. Any value of $\operatorname{Log} z$ is a continuous function of both x and y, except for $x=0$, $y=0$.

3. **The functional equation satisfied by** $\operatorname{Log} z$. The function $\operatorname{Log} z$ satisfies the equation

$$\operatorname{Log}(z_1z_2)=\operatorname{Log} z_1+\operatorname{Log} z_2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

that is to say every value of either side of this equation is *one* of the values of the other side. This follows at once by putting

$$z_1=r_1(\cos\theta_1+i\sin\theta_1),\quad z_2=r_2(\cos\theta_2+i\sin\theta_2),$$

and applying the formula of § 204. It is not however true that

$$\log(z_1z_2)=\log z_1+\log z_2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

in all circumstances. If, e.g.,

$$z_1=z_2=\tfrac{1}{2}(-1+i\sqrt{3})=\cos\tfrac{2}{3}\pi+i\sin\tfrac{2}{3}\pi,$$

then $\log z_1=\log z_2=\frac{2}{3}\pi i$, and $\log z_1+\log z_2=\frac{4}{3}\pi i$, which is one of the values of $\operatorname{Log}(z_1z_2)$, but not the principal value. In fact $\log(z_1z_2)=-\frac{2}{3}\pi i$.

An equation such as (1), in which every value of either side is a value of the other, we shall call a *complete* equation, or an equation which is *completely true.*

4. The equation $\operatorname{Log} z^m=m\operatorname{Log} z$, where m is an integer, is not completely true: every value of the right-hand side is a value of the left-hand side, but the converse is not true.

5. The equation $\operatorname{Log}(1/z) = -\operatorname{Log} z$ is completely true. It is also true that $\log(1/z) = -\log z$, except when z is real and negative.

6. Show that the equation (2) of Ex. 3 is true unless the line joining the points $z=z_1$, $z=z_2$ cuts the negative half of the real axis.

7. The equation

$$\log\left(\frac{z-a}{z-b}\right) = \log(z-a) - \log(z-b)$$

is true if z lies outside the region bounded by the line joining the points $z=a$, $z=b$, and lines through these points parallel to OX and extending to infinity in the negative direction.

8. The equation

$$\log\left(\frac{a-z}{b-z}\right) = \log\left(1-\frac{a}{z}\right) - \log\left(1-\frac{b}{z}\right)$$

is true if z lies outside the triangle formed by the three points 0, a, b.

9. The function of the real variable x defined by

$$\pi f(x) = p\pi + (q-p)\, I(\log x),$$

(where $I(u)$, as usual, denotes the imaginary part of u) is equal to p when x is positive and to q when x is negative.

10. The function of x defined by

$$\pi f(x) = p\pi + (q-p)\, I\{\log(x-1)\} + (r-q)\, I(\log x)$$

is equal to p $(1<x)$, to q $(0<x<1)$, and to r $(x<0)$.

11. Draw the graph of the function $I(\operatorname{Log} x)$ of the real variable x. [The graph consists of the positive halves of the lines $y=2k\pi$ and the negative halves of the lines $y=(2k+1)\pi$.]

12. For what values of z is (i) $\log z$ (ii) any value of $\operatorname{Log} z$ (a) real or (b) purely imaginary?

13. If $z=x+iy$ then $\operatorname{Log}\operatorname{Log} z = \log R + (\Theta + 2k'\pi)\, i$, where

$$R^2 = (\log r)^2 + (\theta + 2k\pi)^2$$

and Θ is the least positive angle determined by the equations

$$\cos\Theta : \sin\Theta : 1 :: \log r : \theta + 2k\pi : \sqrt{\{(\log r)^2 + (\theta + 2k\pi)^2\}}.$$

Plot roughly the doubly infinite set of values of $\operatorname{Log}\operatorname{Log}(1+i\sqrt{3})$, indicating which of them are values of $\log\operatorname{Log}(1+i\sqrt{3})$ and which of $\operatorname{Log}\log(1+i\sqrt{3})$.

205. Note on the logarithm of a real negative number. According to the definitions adopted in this chapter a negative number x has no real logarithm, but an infinite number of complex ones, whose values may be expressed in any one of the equivalent forms

$$\log(-x) + (2k+1)\pi i, \quad \log|x| + (2k+1)\pi i, \quad \tfrac{1}{2}\log x^2 + (2k+1)\pi i,$$

where k is an integer. The question remains as to how this is to be reconciled with the definition which we gave in Ch. VI. We there defined $\log x$, or rather $\log x + C$, where C is an arbitrary constant, as the function whose derivative is $1/x$.

It follows from §§ 137 *et seq.* that when x is positive $\int_1^x \frac{dt}{t}$ or $\int_a^x \frac{dt}{t}$, where a is any positive number, is such a function. This function is, in virtue of the definition of $\log x$ given in Ch. IX, equal to $\log x - \log a$ and so included in the form $\log x + C$. If x is negative the integral ceases to have any meaning, but the integral $\int_b^x \frac{dt}{t}$, where b is any *negative* number, has a meaning, and its derivative is $1/x$. Now, making the substitution $t = -u$, we obtain

$$\int_b^x \frac{dt}{t} = \int_{-b}^{-x} \frac{du}{u} = \log(-x) - \log(-b),$$

which is included in the form $\log(-x) + C$. We therefore conclude that *the most general real function whose derivative is* $1/x$ *is*

$$\log x + C \quad (x > 0)\ , \quad \log(-x) + C \quad (x < 0)\ ;$$

and we may write generally

$$\int \frac{dx}{x} = \log(\pm x) + C = \log|x| + C = \tfrac{1}{2}\log x^2 + C,$$

the last two forms applying equally to positive or negative values of x, while in the first that sign is to be chosen which makes $\pm x$ positive.

In other words $\log x$, as defined in Ch. VI, is really $\log|x|$ or $\frac{1}{2}\log x^2$ as defined in Ch. IX. The results of this chapter show that, when x is negative, $\log x$ differs from $\log|x|$ or $\frac{1}{2}\log x^2$ by an imaginary constant, and so the two definitions are equivalent so far as the determination of integral functions is concerned. In order, however, to preserve an absolute consistency between the notation of Ch. VI and that of these later chapters, we should modify some of the formulae of Ch. VI by the introduction of an ambiguous sign, or of the sign of the absolute value, before the arguments of the logarithms involved. Thus we should write

$$\int \frac{dx}{x} = \log(\pm x), \quad \int \frac{dx}{ax+b} = \frac{1}{a}\log\{\pm(ax+b)\},$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left\{\pm\left(\frac{x-a}{x+a}\right)\right\}, \quad \int \sec x\, dx = \log\{\pm(\sec x + \tan x)\},$$

$$\int \operatorname{cosec} x\, dx = \log(\pm\tan\tfrac{1}{2}x), \quad \int \log(\pm x)\, dx = x\log(\pm x) - x,$$

the ambiguous sign being in each case chosen so as to make the contents of the bracket positive: or, instead of $\log(\pm x)$, $\log\{\pm(ax+b)\}$, etc. we may write $\log|x|$, $\log|ax+b|$, etc. or $\frac{1}{2}\log x^2$, $\frac{1}{2}\log(ax+b)^2$, etc. In § 118 we did actually adopt this course (see equations (2) *et seq.* on pp. 228–9).

206. The exponential function. In Ch. IX we defined a function e^y of the real variable y as the inverse of the function $y = \log x$. It is naturally suggested that we should define a function which is the inverse of the function $\operatorname{Log} z$.

DEFINITION. *If any value of* Log z *is equal to* ζ, *we shall call* z *the exponential of* ζ *and write*

$$z = \exp \zeta.$$

Thus $z = \exp \zeta$ if $\zeta = \operatorname{Log} z$. It is certain that to any given value of z correspond infinitely many different values of ζ. It would not be unnatural to suppose that, conversely, to any given value of ζ correspond infinitely many values of z, or in other words that $\exp \zeta$ is an infinitely many-valued function of ζ. This is however not the case, as is proved by the following theorem.

THEOREM. *The exponential function* $\exp \zeta$ *is a one-valued function of* ζ.

For suppose that $z_1 = \exp \zeta$ and $z_2 = \exp \zeta$. Then

$$\zeta = \operatorname{Log} z_1 = \operatorname{Log} z_2.$$

If $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$ we deduce

$$\log r_1 + (2m\pi + \theta_1) i = \log r_2 + (2n\pi + \theta_2) i,$$

where m and n are integers. This involves

$$\log r_1 = \log r_2, \quad 2m\pi + \theta_1 = 2n\pi + \theta_2.$$

Thus $r_1 = r_2$ and θ_1 and θ_2 differ by a multiple of 2π. Hence $z_1 = z_2$.

COROLLARY. *When* ζ *is real* $\exp \zeta = e^{\zeta}$, *the real exponential function of* ζ *defined in Ch. IX.*

For if $z = e^{\zeta}$, $\log z = \zeta$, i.e. one of the values of $\operatorname{Log} z$ is ζ. Hence $z = \exp \zeta$.

207. The value of $\exp \zeta$. Let $\zeta = \xi + i\eta$ and

$$z = \exp \zeta = r (\cos \theta + i \sin \theta).$$

Then

$$\xi + i\eta = \operatorname{Log} z = \log r + (\theta + 2m\pi) i,$$

where m is an integer. Hence $\xi = \log r$, $\eta = \theta + 2m\pi$, or

$$r = e^{\xi}, \ \theta = \eta - 2m\pi;$$

and accordingly

$$\exp (\xi + i\eta) = e^{\xi} (\cos \eta + i \sin \eta).$$

If $\eta = 0$, $\exp \xi = e^{\xi}$, as we have already inferred (§ 206). It is clear that $\exp (\xi + i\eta)$ is a continuous function of ξ and η for all values of ξ and η.

208. The functional equation satisfied by $\exp \zeta$. Let $\zeta_1 = \xi_1 + i\eta_1$, $\zeta_2 = \xi_2 + i\eta_2$. Then

$$\begin{aligned}\exp \zeta_1 \times \exp \zeta_2 &= e^{\xi_1} (\cos \eta_1 + i \sin \eta_1) \times e^{\xi_2} (\cos \eta_2 + i \sin \eta_2)\\ &= e^{\xi_1 + \xi_2} \{\cos (\eta_1 + \eta_2) + i \sin (\eta_1 + \eta_2)\}\\ &= \exp (\zeta_1 + \zeta_2).\end{aligned}$$

The exponential function therefore satisfies the functional relation $f(\zeta_1 + \zeta_2) = f(\zeta_1) f(\zeta_2)$, as we have already seen (Ch. IX, § 187) to be the case for real values of ζ_1 and ζ_2.

209. The general power a^ζ. It might seem natural, as $\exp \zeta = e^\zeta$ when ζ is real, to adopt the same notation when ζ is complex and to drop the notation $\exp \zeta$ altogether. We shall not follow this course because we shall have to give a more general definition of the meaning of the symbol e^ζ: we shall find then that e^ζ represents a function with infinitely many values of which $\exp \zeta$ is only one.

We have already defined the meaning of the symbol a^ζ in a considerable variety of cases. It is defined in elementary Algebra in the case in which a is real and positive and ζ rational, or a real and negative and ζ a rational fraction whose denominator is odd. According to the definitions there given a^ζ has at most two values. In Ch. III we extended our definitions to cover the case in which a is any real or complex number and ζ any rational number p/q: according to the new definition a^ζ had in all cases (unless $a = 0$) exactly q values, and we showed that this definition is consistent with and includes the previous definition. In Ch. IX we gave a new definition, expressed by the equation

$$a^\zeta = e^{\zeta \log a},$$

which applies whenever ζ is real and a real and positive.

Thus we have, in one way or another, attached a meaning to such expressions as

$$3^{7/12},\ (-1)^{1/3},\ (\sqrt{3} + \tfrac{1}{2}i)^{-1/2},\ (3{\cdot}125)^{1+\sqrt{2}},$$

but we have as yet given no definitions which enable us to attach one to such expressions as

$$(1 + i)^{\sqrt{2}},\ i^i,\ (3 + 2i)^{2+3i}.$$

We shall now give a general definition of a^ζ which applies to all values of a and ζ, real or complex, with the one limitation that a must not be equal to 0.

DEFINITION. *The function* a^ζ *is defined by the equation*

$$a^\zeta = \exp(\zeta \operatorname{Log} a)$$

where Log a *is any value of the logarithm of* a.

We must first satisfy ourselves that this definition is consistent with the previous definitions and includes them all as particular cases.

(1) If a is positive and ζ real one value of $\zeta \operatorname{Log} a$, viz. $\zeta \log a$, is real: and $\exp(\zeta \log a) = e^{\zeta \log a}$, which agrees with the definition adopted in Ch. IX. The definition of Ch. IX was, as we then saw, consistent with the definition given in elementary Algebra; and so our new definition is so too.

(2) If $a = e^\alpha(\cos A + i \sin A)$, then

$$\operatorname{Log} a = \alpha + (A + 2m\pi)i,$$

$$\exp\{(p/q)\operatorname{Log} a\} = e^{p\alpha/q} \operatorname{Cis}\{(p/q)(A + 2m\pi)\},$$

where m may have any integral value. It is easy to see that, if m assumes all possible integral values, this expression assumes q and only q different values, which are precisely the values of $a^{p/q}$ found in Ch. III, § 41. Hence our new definition is consistent also with that of Ch. III.

210. The general value of a^ζ. Let

$$\zeta = \xi + i\eta, \quad a = \rho(\cos A + i \sin A)$$

where $-\pi < A \leqq \pi$, so that, in the notation of § 209, $\rho = e^\alpha$ or $\alpha = \log \rho$.

Then

$$\zeta \operatorname{Log} a = (\xi + i\eta)\{\log\rho + (A + 2m\pi)i\} = L + iM,$$

where

$$L = \xi\log\rho - \eta(A + 2m\pi), \quad M = \eta\log\rho + \xi(A + 2m\pi);$$

and

$$a^\zeta = \exp(\zeta \operatorname{Log} a) = e^L(\cos M + i \sin M).$$

Thus the general value of a^ζ is

$$e^{\xi\log\rho - \eta(A+2m\pi)}[\cos\{\eta\log\rho + \xi(A + 2m\pi)\} + i\sin\{\eta\log\rho + \xi(A + 2m\pi)\}].$$

In general a^ζ is an infinitely many-valued function. For

$$|a^\zeta| = e^{\xi\log\rho - \eta(A + 2m\pi)}$$

has a different value for every value of m unless $\eta = 0$. If $\eta = 0$, on the other hand, the moduli of all the different values of a^ζ are the same. But any two values differ unless their amplitudes also are the same. This requires that $\xi(A + 2m\pi)$ and $\xi(A + 2n\pi)$, where m and n are different integers, shall differ, if at all, by a multiple of 2π. But if

$$\xi(A + 2m\pi) - \xi(A + 2n\pi) = 2k\pi,$$

it is clear that $\xi = k/(m-n)$ is rational. We conclude that *a^ζ is infinitely many-valued unless ζ is real and rational.* If ζ is complex the values have all different moduli: if ζ is real and irrational there are infinitely many values, all of which have the same modulus, so that their representative points lie on a circle. On the other hand we have already seen that when ζ is real and rational a^ζ has but a finite number of values.

The *principal value* of $a^\zeta = \exp(\zeta \operatorname{Log} a)$ is obtained by giving $\operatorname{Log} a$ its principal value, i.e. by supposing $m=0$ in the general formula. Thus the principal value of a^ζ is

$$e^{\xi \log \rho - \eta A}\{\cos(\eta \log \rho + \xi A) + i \sin(\eta \log \rho + \xi A)\}.$$

Two particular cases are of especial interest: (i) if a is real and positive and ζ real, then $\rho = a$, $A = 0$, $\xi = \zeta$, $\eta = 0$, and the principal value of a^ζ is $e^{\zeta \log a}$, which is the value defined in the last chapter: (ii) if $|a| = 1$ and ζ is real, then $\rho = 1$, $\xi = \zeta$, $\eta = 0$, and the principal value of $(\cos A + i \sin A)^\zeta$ is $\cos \zeta A + i \sin \zeta A$. This is a further generalisation of De Moivre's Theorem (Ch. III, §§ 36, 42).

Examples XCVI. 1. Find all the values of i^i. [By definition

$$i^i = \exp(i \operatorname{Log} i).$$

But $$i = \cos \tfrac{1}{2}\pi + i \sin \tfrac{1}{2}\pi, \quad \operatorname{Log} i = (2k\pi + \tfrac{1}{2}\pi)\, i$$

where k is any integer. Hence

$$i^i = \exp\{-(2k + \tfrac{1}{2})\pi\} = e^{-(2k+\frac{1}{2})\pi}.$$

All the values of i^i, therefore, are real and positive.]

2. Find all the values of $(1+i)^i$, i^{1+i}, $(1+i)^{1+i}$.

3. The values of a^ζ, when plotted in the Argand diagram, are the vertices of an equiangular polygon inscribed in an equiangular spiral whose angle is independent of a. (*Math. Trip.* 1899.)

[If $a^\zeta = r(\cos\theta + i \sin\theta)$ we find

$$r = e^{\xi \log \rho - \eta(A + 2m\pi)}, \quad \theta = \eta \log \rho + \xi(A + 2m\pi);$$

and all the points lie on the spiral $r = \rho^{(\xi^2+\eta^2)/\xi}\, e^{-\eta\theta/\xi}$.]

4. **The function e^ζ.** If we write e for a in the general formula, so that $\log \rho = 1$, $A = 0$, we obtain

$$e^\zeta = e^{\xi - 2m\pi\eta}\{\cos(\eta + 2m\pi\xi) + i \sin(\eta + 2m\pi\xi)\}$$

The principal value of e^{ζ} is $e^{\xi}(\cos\eta + i\sin\eta)$, which is the same as $\exp\zeta$ (§ 207). In particular, if ζ is real, so that $\eta=0$, we obtain

$$e^{\zeta}(\cos 2m\pi\zeta + i\sin 2m\pi\zeta)$$

as the general and e^{ζ} as the principal value, e^{ζ} denoting here the positive value of the exponential defined in Ch. IX.

5. Show that $\operatorname{Log} e^{\zeta}=(1+2m\pi i)\zeta+2n\pi i$, m and n being any integers, and that in general $\operatorname{Log} a^{\zeta}$ has a double infinity of values.

6. The equation $1/a^{\zeta}=a^{-\zeta}$ is completely true (Ex. XCV. 3): it is also true of the principal values.

7. The equation $a^{\zeta}\times b^{\zeta}=(ab)^{\zeta}$ is completely true but not always true of the principal values.

8. The equation $a^{\zeta}\times a^{\zeta'}=a^{\zeta+\zeta'}$ is not completely true, but is true of the principal values. [Every value of the right-hand side is a value of the left-hand side, but the general value of $a^{\zeta}\times a^{\zeta'}$, viz.

$$\exp\{\zeta(\log a+2m\pi i)+\zeta'(\log a+2n\pi i)\},$$

is not as a rule a value of $a^{\zeta+\zeta'}$ unless $m=n$.]

9. What are the corresponding results as regards the equations

$$\operatorname{Log} a^{\zeta}=\zeta\operatorname{Log} a,\quad (a^{\zeta})^{\zeta'}=(a^{\zeta'})^{\zeta}=a^{\zeta\zeta'}\,?$$

10. For what values of ζ is (*a*) any value (*b*) the principal value of e^{ζ} (i) real (ii) purely imaginary (iii) of unit modulus?

11. The necessary and sufficient conditions that all the values of a^{ζ} should be real are that

$$\xi,\quad \{\eta\log|a|+\xi\operatorname{am} a\}/\pi,$$

should both be integral. What are the corresponding conditions that all the values should be of unit modulus?

12. The general value of $|x^{i}+x^{-i}|$, where $x>0$, is

$$e^{-(m-n)\pi}\sqrt{[2\{\cosh 2(m+n)\pi+\cos(2\log x)\}]}.$$

13. Explain the fallacy in the following argument: since $e^{2m\pi i}=e^{2n\pi i}=1$, where m and n are any integers, therefore, raising each side to the power i, we obtain $e^{-2m\pi}=e^{-2n\pi}$.

14. In what circumstances are any of the values of x^{x}, where x is real, themselves real? [If $x>0$ then

$$x^{x}=\exp(x\operatorname{Log} x)=\exp(x\log x)\operatorname{Cis} 2m\pi x,$$

the first factor being real. The principal value, for which $m=0$, is always real.

If x is a rational fraction $p/(2q+1)$, or is irrational, there is no other real value. But if x is of the form $p/2q$ there is one other value, viz. $-\exp(x\log x)$, given by $m=q$.

If $x = -\xi < 0$ then

$$x^x = \exp\{-\xi \operatorname{Log}(-\xi)\} = \exp(-\xi \log \xi) \operatorname{Cis}\{-(2m+1)\pi\xi\}.$$

The only case in which any value is real is that in which $\xi = p/(2q+1)$, when $m = q$ gives the real value

$$\exp(-\xi \log \xi) \operatorname{Cis}(-p\pi) = (-1)^p \xi^{-\xi}.$$

The cases of reality are illustrated by the examples

$$(\tfrac{1}{3})^{\frac{1}{3}} = 1/\sqrt[3]{3}, \quad (\tfrac{1}{2})^{\frac{1}{2}} = \pm 1/\sqrt{2}, \quad (-\tfrac{2}{3})^{-\frac{2}{3}} = \sqrt[3]{(\tfrac{3}{2})^2}, \quad (-\tfrac{1}{3})^{-\frac{1}{3}} = -\sqrt[3]{3}.]$$

15. **Logarithms to any base.** We may define $\zeta = \operatorname{Log}_a z$ in two different ways: (i) we may say that $\zeta = \operatorname{Log}_a z$ if the *principal* value of a^ζ is equal to z; (ii) we may say that $\zeta = \operatorname{Log}_a z$ if *any* value of a^ζ is equal to z.

Thus if $a = e$, then $\zeta = \operatorname{Log}_e z$, according to the first definition, if the principal value of e^ζ is equal to z, or if $\exp \zeta = z$; and so $\operatorname{Log}_e z$ is identical with $\operatorname{Log} z$. But, according to the second definition, $\zeta = \operatorname{Log}_e z$ if

$$e^\zeta = \exp(\zeta \operatorname{Log} e) = z, \quad \zeta \operatorname{Log} e = \operatorname{Log} z,$$

or $\zeta = (\operatorname{Log} z)/(\operatorname{Log} e)$, any values of the logarithms being taken. Thus

$$\zeta = \operatorname{Log}_e z = \frac{\log|z| + (\operatorname{am} z + 2m\pi)i}{1 + 2n\pi i},$$

so that ζ is a doubly infinitely many valued function of z. And generally, according to this definition, $\operatorname{Log}_a z = (\operatorname{Log} z)/(\operatorname{Log} a)$.

16. $\operatorname{Log}_e 1 = 2m\pi i/(1 + 2n\pi i)$, $\operatorname{Log}_e(-1) = (2m+1)\pi i/(1 + 2n\pi i)$, where m and n are any integers.

17. We know that, as $\zeta \to 0$, ζ being real, $\{\log(1+\zeta)\}/\zeta \to 1$. This result may be extended to complex values of ζ. For

$$\log(1+\zeta) = \int_1^{1+\zeta} \frac{dz}{z},$$

the path of integration being the straight line from 1 to $1+\zeta$. This line may be represented by the equations

$$x = 1 + t\rho\cos\phi, \quad y = t\rho\sin\phi, \qquad (0 \leqq t \leqq 1),$$

where ρ is the modulus and ϕ the amplitude of ζ. Thus

$$\log(1+\zeta) = \int_0^1 \frac{\rho \operatorname{Cis}\phi\, dt}{1 + t\rho \operatorname{Cis}\phi},$$

and

$$\frac{1}{\zeta}\log(1+\zeta) = \int_0^1 \frac{dt}{1 + t\rho\operatorname{Cis}\phi} = 1 - \rho\int_0^1 \frac{t\operatorname{Cis}\phi\, dt}{1 + t\rho\operatorname{Cis}\phi}.$$

The modulus of the last term is less than

$$\rho\int_0^1 \frac{t\,dt}{\sqrt{(1 + 2t\rho\cos\phi + t^2\rho^2)}} < \rho\int_0^1 \frac{t\,dt}{1 - t\rho} < \frac{\rho}{1-\rho}\int_0^1 t\,dt = \frac{\rho}{2(1-\rho)},$$

which tends to zero with ρ, and so $\{\log(1+\zeta)\}/\zeta \to 1$.

If $\zeta = \xi + i\eta$, and ξ and η are each made to tend to zero, then ζ approaches the origin along a path the nature of which depends on the way in which

ξ and η tend to zero, or on the relations which hold between them in the course of this process. Thus if ξ were always equal to η this path would be a straight line bisecting the angle between the axes. If a function $f(\zeta)$ has the property of tending to a limit l as $\xi \to 0$ and $\eta \to 0$, whatever these relations may be (cf. § 90) or along whatever path ζ approaches the origin, we shall say that $\lim f(\zeta)=l$ or $f(\zeta)\to l$ as $\zeta \to 0$. Thus $\{\log(1+\zeta)\}/\zeta \to 1$ as $\zeta \to 0$. Similarly we define the meaning of '$f(\zeta)\to l$ as $\zeta \to \zeta_0$'.

18. **Generalisation of the exponential limit.** In Ch. IX we proved that

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \exp z$$

when z is real. We shall now prove that the result remains true for complex values of z. For, by the last example, $\{\log(1+\zeta)\}/\zeta \to 1$ as $\zeta \to 0$. If we put $\zeta = z/n$, where n is a positive integer, we see that, as $n\to\infty$, $n\log\left(1+\frac{z}{n}\right)\to z$; and

$$\left(1+\frac{z}{n}\right)^n = \exp\left\{n\log\left(1+\frac{z}{n}\right)\right\}\to \exp z,$$

since $\exp z$, where $z=x+iy$, is a continuous function of x and y.

19. More generally, if Z_n is a function of z, such that $nZ_n \to z$ as $n\to\infty$, then $(1+Z_n)^n \to \exp z$. [For

$$n\log(1+Z_n) = (nZ_n)\left\{\frac{\log(1+Z_n)}{Z_n}\right\}\to z.]$$

20. Prove that if t is real then $D_t \log(1+tz) = z/(1+tz)$, for all values of z real or complex. [This is of course the same formula as is given by Th. (8) of § 94 when z is real. In the general case we have

$$\frac{d}{dt}\log(1+tz) = \lim_{h\to 0}\frac{1}{h}\log\left(1+\frac{hz}{1+tz}\right) = \frac{z}{1+tz}\lim_{h\to 0}\left\{\frac{1}{u}\log(1+u)\right\},$$

where $u = hz/(1+tz)$, and the result follows from Ex. 17.]

21. Show that the formula $D_t \log\phi(t) = \phi'(t)/\phi(t)$ holds generally when ϕ is a complex function of the real variable t. [Write

$$\phi = \psi + i\chi, \quad \log\phi = \tfrac{1}{2}\log(\psi^2+\chi^2) + i\arctan(\chi/\psi)$$

and differentiate according to the ordinary rules.]

22. Prove the formula $D_t \exp\phi(t) = \phi'(t)\exp\phi(t)$. [Write

$$\phi = \psi + i\chi, \quad \exp\phi = \exp\psi\,(\cos\chi + i\sin\chi)$$

and differentiate according to the ordinary rules.]

23. Prove that $D_t(1+tz)^m = mz(1+tz)^{m-1}$, where t is real, z and m have any values, and the principal value of each side is taken. [Write

$$(1+tz)^m = \exp\{m\log(1+tz)\}$$

and apply the results of Exs. 20 and 22.]

211. The exponential values of the sine and cosine. From the formula

$$\exp(\xi + i\eta) = \exp\xi\,(\cos\eta + i\sin\eta),$$

we can deduce a number of extremely important subsidiary formulae. Taking $\xi = 0$ we obtain $\exp(i\eta) = \cos\eta + i\sin\eta$; and changing the sign of η, $\exp(-i\eta) = \cos\eta - i\sin\eta$. Hence

$$\cos\eta = \tfrac{1}{2}\{\exp(i\eta) + \exp(-i\eta)\},$$
$$\sin\eta = -\tfrac{1}{2}i\{\exp(i\eta) - \exp(-i\eta)\}.$$

We can of course deduce expressions for any of the trigonometrical ratios of η in terms of $\exp(i\eta)$.

212. Definition of $\sin\zeta$ and $\cos\zeta$ for all values of ζ. We saw in the last section that, when ζ is real,

$$\left.\begin{aligned}\cos\zeta &= \tfrac{1}{2}\{\exp(i\zeta) + \exp(-i\zeta)\}\\ \sin\zeta &= -\tfrac{1}{2}i\{\exp(i\zeta) - \exp(-i\zeta)\}\end{aligned}\right\}\ldots\ldots\ldots\ldots(1).$$

The left-hand sides of these equations are defined, by the ordinary geometrical definitions adopted in elementary Trigonometry, only for real values of ζ. The right-hand sides have, on the other hand, been defined for all values of ζ, real or complex. We are therefore naturally led to adopt the formulae (1) as the *definitions* of $\cos\zeta$ and $\sin\zeta$ for all values of ζ. These definitions agree, in virtue of the results of § 211, with the elementary definitions for real values of ζ.

Having defined $\cos\zeta$ and $\sin\zeta$ we define the other trigonometrical ratios by the equations

$$\tan\zeta = \frac{\sin\zeta}{\cos\zeta},\quad \cot\zeta = \frac{\cos\zeta}{\sin\zeta},\quad \sec\zeta = \frac{1}{\cos\zeta},\quad \operatorname{cosec}\zeta = \frac{1}{\sin\zeta}\ldots(2).$$

It is evident that $\cos\zeta$ and $\sec\zeta$ are even functions of ζ, and $\sin\zeta$, $\tan\zeta$, $\cot\zeta$, and $\operatorname{cosec}\zeta$ odd functions. Also, if $\exp(i\zeta) = t$, we have

$$\cos\zeta = \tfrac{1}{2}\{t + (1/t)\},\quad \sin\zeta = -\tfrac{1}{2}i\{t - (1/t)\},$$
$$\cos^2\zeta + \sin^2\zeta = \tfrac{1}{4}[\{t + (1/t)\}^2 - \{t - (1/t)\}^2] = 1\ \ldots\ldots(3).$$

Moreover we can express the trigonometrical functions of $\zeta + \zeta'$ in terms of those of ζ and ζ' by precisely the same formulae as those which hold in elementary trigonometry. For if $\exp(i\zeta) = t$, $\exp(i\zeta') = t'$, we have

$$\cos(\zeta+\zeta') = \tfrac{1}{2}\left\{tt' + \frac{1}{tt'}\right\} = \tfrac{1}{4}\left\{\left(t+\frac{1}{t}\right)\left(t'+\frac{1}{t'}\right) + \left(t-\frac{1}{t}\right)\left(t'-\frac{1}{t'}\right)\right\}$$

$$= \cos\zeta\cos\zeta' - \sin\zeta\sin\zeta' \ldots\ldots\ldots\ldots(4);$$

and similarly we can prove that

$$\sin(\zeta+\zeta') = \sin\zeta\cos\zeta' + \cos\zeta\sin\zeta' \ldots\ldots\ldots\ldots(5).$$

In particular

$$\cos(\zeta+\tfrac{1}{2}\pi) = -\sin\zeta, \qquad \sin(\zeta+\tfrac{1}{2}\pi) = \cos\zeta \ldots\ldots(6).$$

All the ordinary formulae of elementary trigonometry are algebraical corollaries of the equations (2)—(6); and so all such relations hold also for the generalised trigonometrical functions defined in this section.

213. The generalised hyperbolic functions. In Ch. IX, Ex. LXXXIX. 19, we defined $\cosh\zeta$ and $\sinh\zeta$, for real values of ζ, by the equations

$$\cosh\zeta = \tfrac{1}{2}\{\exp\zeta + \exp(-\zeta)\}, \quad \sinh\zeta = \tfrac{1}{2}\{\exp\zeta - \exp(-\zeta)\} \ldots\ldots(1).$$

We can now extend this definition to complex values of the variable; i.e. we can agree that the equations (1) are to define $\cosh\zeta$ and $\sinh\zeta$ for all values of ζ real or complex. The reader will easily verify the following relations:

$$\cos i\zeta = \cosh\zeta, \quad \sin i\zeta = i\sinh\zeta, \quad \cosh i\zeta = \cos\zeta, \quad \sinh i\zeta = i\sin\zeta.$$

We have seen that any elementary trigonometrical formula, such as the formula $\cos 2\zeta = \cos^2\zeta - \sin^2\zeta$, remains true when ζ is allowed to assume complex values. It remains true therefore if we write $\cos i\zeta$ for $\cos\zeta$, $\sin i\zeta$ for $\sin\zeta$ and $\cos 2i\zeta$ for $\cos 2\zeta$; or, in other words, if we write $\cosh\zeta$ for $\cos\zeta$, $\cosh 2\zeta$ for $\cos 2\zeta$, and $i\sinh\zeta$ for $\sin\zeta$. Hence

$$\cosh 2\zeta = \cosh^2\zeta + \sinh^2\zeta.$$

The same process of transformation may be applied to any trigonometrical identity. It is of course this fact which explains the correspondence noted in Ex. LXXXIX. 21 between the formulae for the hyperbolic and those for the ordinary trigonometrical functions.

214. Formulae for $\cos(\xi+i\eta)$, $\sin(\xi+i\eta)$, etc. It follows from the addition formulae that

$$\cos(\xi+i\eta) = \cos\xi\cos i\eta - \sin\xi\sin i\eta = \cos\xi\cosh\eta - i\sin\xi\sinh\eta,$$

$$\sin(\xi+i\eta) = \sin\xi\cos i\eta + \cos\xi\sin i\eta = \sin\xi\cosh\eta + i\cos\xi\sinh\eta.$$

These formulae are true for all values of ξ and η. The interesting case is that in which ξ and η are real. They then give expressions for the real and imaginary parts of the cosine and sine of a complex number.

Examples XCVII. 1. Determine the values of ζ for which $\cos\zeta$ and $\sin\zeta$ are (i) real, (ii) purely imaginary. [For example $\cos\zeta$ is real when $\eta=0$ or when ξ is any multiple of π.]

2. $|\cos(\xi+i\eta)| = \sqrt{(\cos^2\xi+\sinh^2\eta)} = \sqrt{\{\tfrac{1}{2}(\cosh 2\eta+\cos 2\xi)\}}$,

$|\sin(\xi+i\eta)| = \sqrt{(\sin^2\xi+\sinh^2\eta)} = \sqrt{\{\tfrac{1}{2}(\cosh 2\eta-\cos 2\xi)\}}$.

[Use the equation $|\cos(\xi+i\eta)| = \sqrt{\{\cos(\xi+i\eta)\cos(\xi-i\eta)\}}$.]

3. $\tan(\xi+i\eta) = \dfrac{\sin 2\xi + i\sinh 2\eta}{\cosh 2\eta+\cos 2\xi}$, $\quad \cot(\xi+i\eta) = \dfrac{\sin 2\xi - i\sinh 2\eta}{\cosh 2\eta - \cos 2\xi}$.

[For example

$$\tan(\xi+i\eta) = \frac{\sin(\xi+i\eta)\cos(\xi-i\eta)}{\cos(\xi+i\eta)\cos(\xi-i\eta)} = \frac{\sin 2\xi+\sin 2i\eta}{\cos 2\xi+\cos 2i\eta},$$

which leads at once to the result given.]

4.
$$\sec(\xi+i\eta) = \frac{\cos\xi\cosh\eta + i\sin\xi\sinh\eta}{\tfrac{1}{2}(\cosh 2\eta+\cos 2\xi)},$$

$$\operatorname{cosec}(\xi+i\eta) = \frac{\sin\xi\cosh\eta - i\cos\xi\sinh\eta}{\tfrac{1}{2}(\cosh 2\eta-\cos 2\xi)}.$$

5. If $|\cos(\xi+i\eta)|=1$ then $\sin^2\xi=\sinh^2\eta$, and if $|\sin(\xi+i\eta)|=1$ then $\cos^2\xi=\sinh^2\eta$.

6. If $|\cos(\xi+i\eta)|=1$, then

$$\sin\{\operatorname{am}\cos(\xi+i\eta)\} = \pm\sin^2\xi = \pm\sinh^2\eta.$$

7. Prove that $\operatorname{Log}\cos(\xi+i\eta) = A+iB$, where

$$A = \tfrac{1}{2}\log\{\tfrac{1}{2}(\cosh 2\eta+\cos 2\xi)\}$$

and B is any angle such that

$$\frac{\cos B}{\cos\xi\cosh\eta} = -\frac{\sin B}{\sin\xi\sinh\eta} = \frac{1}{\sqrt{\{\tfrac{1}{2}(\cosh 2\eta+\cos 2\xi)\}}}.$$

Find a similar formula for $\operatorname{Log}\sin(\xi+i\eta)$.

8. **Solution of the equation $\cos\zeta=a$, where a is real.** Putting $\zeta=\xi+i\eta$, and equating real and imaginary parts, we obtain

$$\cos\xi\cosh\eta=a,\quad \sin\xi\sinh\eta=0.$$

Hence either $\eta=0$ or ξ is a multiple of π. If (i) $\eta=0$ then $\cos\xi=a$, which is impossible unless $-1\leqq a\leqq 1$. This hypothesis leads to the solution

$$\zeta = 2k\pi \pm \operatorname{arc}\cos a,$$

where $\operatorname{arc}\cos a$ lies between 0 and π. If (ii) $\xi=m\pi$ then $\cosh\eta=(-1)^m a$, so that either $a\geqq 1$ and m is even, or $a\leqq -1$ and m is odd. If $a=\pm 1$ then $\eta=0$, and we are led back to our first case. If $|a|>1$ then $\cosh\eta=|a|$, and we are led to the solutions

$$\zeta = 2k\pi \pm i\log\{a+\sqrt{(a^2-1)}\}\quad (a>1),$$

$$\zeta = (2k+1)\pi \pm i\log\{-a+\sqrt{(a^2-1)}\}\quad (a<-1).$$

For example, the general solution of $\cos\zeta=-\tfrac{5}{3}$ is $\zeta=(2k+1)\pi\pm i\log 3$.

9. Solve $\sin\zeta=a$, where a is real.

10. **Solution of $\cos\zeta=\alpha+i\beta$, where $\beta\neq 0$.** We may suppose $\beta>0$, since the results when $\beta<0$ may be deduced by merely changing the sign of i. In this case

$$\cos\xi\cosh\eta=\alpha,\quad \sin\xi\sinh\eta=-\beta \text{ .... (1)},$$

and
$$(\alpha/\cosh\eta)^2+(\beta/\sinh\eta)^2=1.$$

If we put $\cosh^2 \eta = x$ we find that

$$x^2 - (1 + a^2 + \beta^2)\, x + a^2 = 0$$

or $x = (A_1 \pm A_2)^2$, where

$$A_1 = \tfrac{1}{2}\sqrt{\{(a+1)^2 + \beta^2\}}, \qquad A_2 = \tfrac{1}{2}\sqrt{\{(a-1)^2 + \beta^2\}}.$$

Suppose $a > 0$. Then $A_1 > A_2 > 0$ and $\cosh \eta = A_1 \pm A_2$. Also

$$\cos \xi = a/(\cosh \eta) = A_1 \mp A_2,$$

and since $\cosh \eta > \cos \xi$ we must take

$$\cosh \eta = A_1 + A_2, \qquad \cos \xi = A_1 - A_2.$$

The general solutions of these equations are

$$\xi = 2k\pi \pm \operatorname{arc\,cos} M, \qquad \eta = \pm \log \{L + \sqrt{(L^2 - 1)}\} \;\ldots\ldots\ldots\ldots(2),$$

where $L = A_1 + A_2$, $M = A_1 - A_2$, and $\operatorname{arc\,cos} M$ lies between 0 and $\tfrac{1}{2}\pi$.

The values of η and ξ thus found above include, however, the solutions of the equations

$$\cos \xi \cosh \eta = a, \qquad \sin \xi \sinh \eta = \beta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

as well as those of the equations (1), since we have only used the second of the latter equations after squaring it. To distinguish the two sets of solutions we observe that the sign of $\sin \xi$ is the same as the ambiguous sign in the first of the equations (2), and the sign of $\sinh \eta$ is the same as the ambiguous sign in the second. Since $\beta > 0$ these two signs must be different. Hence the general solution required is

$$\zeta = 2k\pi \pm [\operatorname{arc\,cos} M - i \log \{L + \sqrt{(L^2 - 1)}\}].$$

11. Work out the cases in which $a < 0$ and $a = 0$ in the same way.

12. If $\beta = 0$, $L = \tfrac{1}{2}|a+1| + \tfrac{1}{2}|a-1|$, $M = \tfrac{1}{2}|a+1| - \tfrac{1}{2}|a-1|$. Verify that the results thus obtained agree with those of Ex. 8.

13. Show that, if a and β are positive, the general solution of $\sin \zeta = a + i\beta$ is

$$\zeta = k\pi + (-1)^k [\operatorname{arc\,sin} M + i \log \{L + \sqrt{(L^2 - 1)}\}],$$

where $\operatorname{arc\,sin} M$ lies between 0 and $\tfrac{1}{2}\pi$. Obtain the solution in the other possible cases.

14. Solve $\tan \zeta = a$, where a is real. [All the roots are real.]

15. Show that the general solution of $\tan \zeta = a + i\beta$, where $\beta \neq 0$, is

$$\zeta = k\pi + \tfrac{1}{2}\theta + \tfrac{1}{4} i \log \left\{\frac{a^2 + (1+\beta)^2}{a^2 + (1-\beta)^2}\right\},$$

where θ is the numerically least angle such that

$$\cos \theta : \sin \theta : 1 :: 1 - a^2 - \beta^2 : 2a : \sqrt{\{(1 - a^2 - \beta^2)^2 + 4a^2\}}.$$

16. If $z = \xi \exp(\tfrac{1}{4}\pi i)$, where ξ is real, and c is also real, then the modulus of $\cos 2\pi z - \cos 2\pi c$ is

$$\sqrt{[\tfrac{1}{2}\{1 + \cos 4\pi c + \cos(2\pi\xi\sqrt{2}) + \cosh(2\pi\xi\sqrt{2}) \\ - 4\cos 2\pi c \cos(\pi\xi\sqrt{2}) \cosh(\pi\xi\sqrt{2})\}]}.$$

17. Prove that $$|\exp\exp(\xi+i\eta)|=\exp(\exp\xi\cos\eta),$$
$$R\{\cos\cos(\xi+i\eta)\}=\cos(\cos\xi\cosh\eta)\cosh(\sin\xi\sinh\eta),$$
$$I\{\sin\sin(\xi+i\eta)\}=\cos(\sin\xi\cosh\eta)\sinh(\cos\xi\sinh\eta).$$

18. Prove that $|\exp\zeta|$ tends to $+\infty$ if ζ moves away towards infinity along any straight line through the origin making an angle less than $\frac{1}{2}\pi$ with OX, and to 0 if ζ moves away along a similar line making an angle greater than $\frac{1}{2}\pi$ with OX.

19. Prove that $|\cos\zeta|$ and $|\sin\zeta|$ tend to $+\infty$ if ζ moves away towards infinity along any line through the origin other than either half of the real axis.

20. Prove that $\tan\zeta\rightarrow\mp i$, according as the line lies above or below OX.

215. The connection between the logarithmic and the inverse trigonometrical functions. We found in Ch. VI that the integral of a rational or algebraical function $\phi(x, \alpha, \beta, \dots)$, where $\alpha, \beta, \dots$ are constants, often assumes different forms according to the values of $\alpha, \beta, \dots$; sometimes it can be expressed by means of logarithms, and sometimes by means of inverse trigonometrical functions. Thus

$$\int\frac{dx}{x^2+a}=\frac{1}{\sqrt{a}}\arctan\frac{x}{\sqrt{a}},\quad \int\frac{dx}{x^2+a}=\frac{1}{2\sqrt{(-a)}}\log\left\{\pm\frac{x-\sqrt{(-a)}}{x+\sqrt{(-a)}}\right\}\ \dots\dots(1),$$

according as $a>0$ or $a<0$, and

$$\int\frac{dx}{\sqrt{(x^2+a)}}=\log\{x+\sqrt{(x^2+a)}\},\quad \int\frac{dx}{\sqrt{(x^2-a)}}=\arcsin\left\{\frac{x}{\sqrt{(-a)}}\right\}\ \dots(2),$$

according as $a>0$ or $a<0$. These facts suggest the existence of some functional connection between the logarithmic and the inverse circular functions. That there is such a connection may also be inferred from the facts that we have expressed the circular functions of ζ in terms of $\exp i\zeta$, and that the logarithm is the inverse of the exponential function.

Let us consider more particularly the equation

$$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right),$$

which holds when a is real and $(x-a)/(x+a)$ is positive. If we could write ia instead of a in this equation we should be led to the formula

$$\arctan\left(\frac{x}{a}\right)=\frac{1}{2i}\log\left(\frac{x-ia}{x+ia}\right)+\text{const.}\ \dots\dots\dots\dots\dots\dots(3),$$

and the question is suggested whether, now that we have defined the logarithm of a complex number, this equation will not be found to be actually true.

Now (§ 204)

$$\operatorname{Log}(x\pm ia)=\tfrac{1}{2}\log(x^2+a^2)\pm(\phi+2k\pi)i,$$

where k is an integer and ϕ is the numerically least angle such that $\cos\phi=x/\sqrt{(x^2+a^2)}$ and $\sin\phi=a/\sqrt{(x^2+a^2)}$. Thus

$$\frac{1}{2i}\operatorname{Log}\left(\frac{x-ia}{x+ia}\right)=-(\phi+l\pi),$$

where l is an integer, and this does in fact differ by a constant from any value of $\arctan(x/a)$.

The standard formula connecting the logarithmic and inverse circular functions is

$$\operatorname{arc}\tan x = \frac{1}{2i}\operatorname{Log}\left(\frac{1+ix}{1-ix}\right) \quad\ldots\ldots\ldots\ldots(4),$$

where x is real. It is most easily verified by putting $x = \tan y$, when the right-hand side reduces to

$$\frac{1}{2i}\operatorname{Log}\left(\frac{\cos y + i\sin y}{\cos y - i\sin y}\right) = \frac{1}{2i}\operatorname{Log}(\exp 2iy) = y + k\pi,$$

k being any integer, so that the equation (4) is 'completely' true. The reader should also verify the formulae

$$\operatorname{arc}\cos x = -i\operatorname{Log}\{x \pm i\sqrt{(1-x^2)}\}, \quad \operatorname{arc}\sin x = -i\operatorname{Log}\{ix \pm \sqrt{(1-x^2)}\} \ldots(5),$$

where $-1 \leqq x \leqq 1$, each of which is also 'completely' true.

Example. Solving the equation

$$c = \cos x = \tfrac{1}{2}\{y + (1/y)\},$$

where $y = \exp(ix)$, with respect to y, we obtain $y = c \pm i\sqrt{(1-c^2)}$. Thus

$$x = -i\operatorname{Log} y = -i\operatorname{Log}\{c \pm i\sqrt{(1-c^2)}\},$$

which is equivalent to the first of the equations (5). Obtain the remaining equations (4), (5) by similar reasoning.

216. The power series for $\exp z$*. We saw in Ch. IX, § 194, that when z is real

$$\exp z = 1 + z + \frac{z^2}{2!} + \ldots \quad\ldots\ldots\ldots\ldots(1).$$

Moreover we saw in Ch. VIII, § 173, that the series on the right-hand side remains convergent (indeed absolutely convergent) when z is complex. It is naturally suggested that the equation (1) remains true, and we shall now prove that this is the case.

Let the series (1) be denoted by $F(z)$. The series being absolutely convergent, it follows by direct multiplication (as in Ex. LXXXIII. 7) that $F(z)$ satisfies the functional equation

$$F(z)\,F(z') = F(z+z') \quad\ldots\ldots\ldots\ldots(2).$$

Now let $z = iy$, where y is real, and $F(z) = f(y)$. Then

$$f(y)\,f(y') = f(y+y');$$

and so

$$\frac{f(y+y') - f(y)}{y'} = f(y)\left\{\frac{f(y') - 1}{y'}\right\}.$$

---

* It will be convenient now to use z instead of ζ as the argument of the exponential function.

But $$\frac{f(y')-1}{y'} = i\left\{1 + \frac{iy'}{2!} + \frac{(iy')^2}{3!} + \dots\right\};$$
and so, if $|y'| < 1$,
$$\left|\frac{f(y')-1}{y'} - i\right| < |y'|\left(\frac{1}{2!} + \frac{1}{3!} + \dots\right) < |y'|(e-2).$$

Hence $\{f(y')-1\}/y' \to i$ as $y' \to 0$, and so
$$\frac{d}{dy}f(y) = \lim_{y'\to 0}\frac{f(y+y')-f(y)}{y'} = if(y).$$

Incidentally we have proved that $f(y)$ is a continuous function of y. Now
$$|f(y)| = |F(iy)| = \sqrt{\{F(iy)F(-iy)\}} = \sqrt{\{F(0)\}} = 1,$$
and therefore
$$f(y) = \cos Y + i\sin Y$$
where Y is an angle depending on y. Since $f(y)$ is continuous and has a differential coefficient, its real and imaginary parts must satisfy the same conditions; and so Y must satisfy them.

Hence $$f'(y) = (-\sin Y + i\cos Y)\,D_y Y.$$
But we have already seen that
$$f'(y) = if(y) = -\sin Y + i\cos Y.$$
Hence $$D_y Y = 1,\ \ Y = y + C,$$
where C is a constant, and
$$f(y) = \cos(y+C) + i\sin(y+C).$$

But when $y = 0$, $f(0) = 1$; hence C is a multiple of 2π and $f(y) = \cos y + i\sin y$. Thus $F(iy) = \cos y + i\sin y$ for all real values of y. And if x also is real
$$F(x+iy) = F(x)F(iy) = \exp x\,(\cos y + i\sin y) = \exp(x+iy),$$
or $$\exp z = 1 + z + \frac{z^2}{2!} + \dots,$$
for all values of z.

217. The power series for $\cos z$ **and** $\sin z$. From the result of the last section and the equations (1) of § 212 it follows at once that
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots,\quad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

for all values of z. These results were proved for real values of z in Ex. LVIII. 1. This is of course by far the most interesting case.

Examples XCVIII. 1. Calculate $\cos i$, $\sin(1+i)$, and $\sin(\pi-1-i)$ to two places of decimals by means of the power series for $\cos z$ and $\sin z$.

2. Prove that $|\cos z| \leqq \cosh|z|$ and $|\sin z| \leqq \sinh|z|$.

3. Prove that if $|z|<1$ then $|\cos z|<2$ and $|\sin z|<\frac{6}{5}|z|$.

4. Since $\sin 2z = 2\sin z\cos z$ we have

$$(2z)-\frac{(2z)^3}{3!}+\frac{(2z)^5}{5!}-\dots=2\left(z-\frac{z^3}{3!}+\dots\right)\left(1-\frac{z^2}{2!}+\dots\right).$$

Prove by multiplying the two series on the right-hand side (§ 177) and equating coefficients (§ 176) that

$$\binom{2n+1}{1}+\binom{2n+1}{3}+\dots+\binom{2n+1}{2n+1}=2^{2n}.$$

Verify the result by means of the binomial theorem. Derive similar identities from the equations

$$\cos^2 z+\sin^2 z=1,\quad \cos 2z=2\cos^2 z-1=1-2\sin^2 z.$$

5. Show that $\exp\{(1+i)z\}=\sum_0^\infty 2^{\frac{1}{2}n}\exp\left(\tfrac{1}{4}n\pi i\right)\dfrac{z^n}{n!}$.

6. Expand $\cos z\cosh z$ in powers of z. [We have

$$\cos z\cosh z-i\sin z\sinh z=\cos(1+i)z=\tfrac{1}{2}\left[\exp\{(1+i)z\}+\exp\{-(1+i)z\}\right]$$

$$=\tfrac{1}{2}\sum_0^\infty 2^{\frac{1}{2}n}\{1+(-1)^n\}\exp\left(\tfrac{1}{4}n\pi i\right)\frac{z^n}{n!},$$

and similarly $\qquad \cos z\cosh z+i\sin z\sinh z=\cos(1-i)z$

$$=\tfrac{1}{2}\sum_0^\infty 2^{\frac{1}{2}n}\{1+(-1)^n\}\exp\left(-\tfrac{1}{4}n\pi i\right)\frac{z^n}{n!}.$$

Hence $\qquad \cos z\cosh z=\tfrac{1}{2}\sum_0^\infty 2^{\frac{1}{2}n}\{1+(-1)^n\}\cos\tfrac{1}{4}n\pi\dfrac{z^n}{n!}=1-\dfrac{2^2z^4}{4!}+\dfrac{2^4z^8}{8!}-\dots\dots$.

7. Expand $\sin z\sinh z$, $\cos z\sinh z$, and $\sin z\cosh z$ in powers of z.

8. Expand $\sin^2 z$ and $\sin^3 z$ in powers of z. [Use the formulae

$$\sin^2 z=\tfrac{1}{2}(1-\cos 2z),\quad \sin^3 z=\tfrac{1}{4}(3\sin z-\sin 3z),\ \dots.$$

It is clear that the same method may be used to expand $\cos^n z$ and $\sin^n z$, where n is any integer.]

9. Sum the series

$$C=1+\frac{\cos z}{1!}+\frac{\cos 2z}{2!}+\frac{\cos 3z}{3!}+\dots,\quad S=\frac{\sin z}{1!}+\frac{\sin 2z}{2!}+\frac{\sin 3z}{3!}+\dots.$$

[Here $C+iS=1+\frac{\exp(iz)}{1!}+\frac{\exp(2iz)}{2!}+\dots=\exp\{\exp(iz)\}$

$$=\exp(\cos z)\{\cos(\sin z)+i\sin(\sin z)\},$$

and similarly

$$C-iS=\exp\{\exp(-iz)\}=\exp(\cos z)\{\cos(\sin z)-i\sin(\sin z)\}.$$

Hence $C=\exp(\cos z)\cos(\sin z),\ S=\exp(\cos z)\sin(\sin z)$.]

10. Sum $1+\frac{a\cos z}{1!}+\frac{a^2\cos 2z}{2!}+\dots,\ \frac{a\sin z}{1!}+\frac{a^2\sin 2z}{2!}+\dots$.

11. Sum $1-\frac{\cos 2z}{2!}+\frac{\cos 4z}{4!}-\dots,\ \frac{\cos z}{1!}-\frac{\cos 3z}{3!}+\dots$

and the corresponding series involving sines.

12. Show that

$$1+\frac{\cos 4z}{4!}+\frac{\cos 8z}{8!}+\dots=\tfrac{1}{2}\{\cos(\cos z)\cosh(\sin z)+\cos(\sin z)\cosh(\cos z)\}.$$

13. Show that the expansions of $\cos(x+h)$ and $\sin(x+h)$ in powers of h (Ex. LVIII. 1) are valid for all values of x and h, real or complex.

218. The logarithmic series. We found that when z is real and numerically less than unity

$$\log(1+z)=z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\dots \qquad (1).$$

The series on the right-hand side is convergent (indeed absolutely convergent) when z has any complex value whose modulus is less than unity. It is naturally suggested that the equation (1) remains true for such complex values of z.

That this is in fact true may be proved by an easy modification of the argument of § 195. It will be remembered that $\log(1+z)$ is the principal value of $\operatorname{Log}(1+z)$, and that

$$\log(1+z)=\int_1^{1+z}\frac{dt}{t}=\int_0^z\frac{du}{1+u},$$

the paths of integration in the planes of t and u being the straight lines from 1 to $1+z$ and from 0 to z respectively. But, as in § 195, we have

$$\frac{1}{1+u}=1-u+u^2-\dots+(-1)^{m-1}u^{m-1}+\frac{(-1)^m u^m}{1+u},$$

and from this (using the results of § 202) we deduce

$$\log(1+z)=z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\dots+\frac{(-1)^m}{m}z^m+R_m,$$

where

$$R_m=(-1)^m\int_0^z\frac{u^m\,du}{1+u}.$$

Now $|1+u|$, which is equal to the length of the line drawn from the point $z=-1$ to the position of u, is, as a reference to Fig. 73 shows, always greater than ϖ, the perpendicular from A on to the line OP. Hence, if $z=r\operatorname{Cis}\theta$ and $u=\rho\operatorname{Cis}\phi$, we have, by § 202,

$$|R_m| < \frac{1}{\varpi}\int_0^r \rho^m\,d\rho = \frac{r^{m+1}}{(m+1)\varpi}.$$

Hence $R_m \to 0$ as $m\to\infty$, and therefore

$$\log(1+z)=z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\dots \quad (2).$$

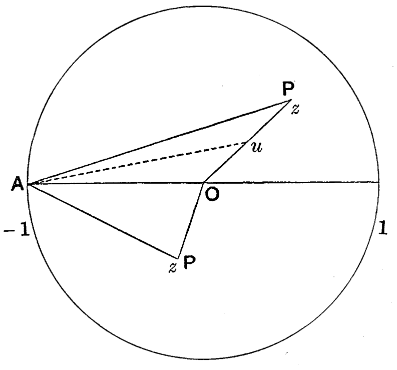

FIG. 73.

If $|z|=1$ the real and imaginary parts of the logarithmic series are each conditionally convergent (unless $z=-1$). The proof given above is applicable in this case also. Thus the equation (2) holds for all points inside or on the circle $|z|=1$, except $z=-1$.

Changing z into $-z$ we obtain

$$\log\left(\frac{1}{1-z}\right)=-\log(1-z)=z+\tfrac{1}{2}z^2+\tfrac{1}{3}z^3+\dots \quad \dots\dots(3).$$

219. Now let $z=r(\cos\theta+i\sin\theta)$ where $0\leqq r\leqq 1$. Then

$$\begin{aligned}\log(1+z)&=\log\{(1+r\cos\theta)+ir\sin\theta\}\\ &=\tfrac{1}{2}\log(1+2r\cos\theta+r^2)+i\arctan\left(\frac{r\sin\theta}{1+r\cos\theta}\right).\end{aligned}$$

That value of the inverse tangent must be taken which lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. For since $1+z$ is the vector represented by the line from -1 to z, the principal value of $\operatorname{am}(1+z)$ always lies between these limits when z lies within the unit circle.

Since $z^m=r^m(\cos m\theta+i\sin m\theta)$ we obtain, on equating the real and imaginary parts in (2),

$$\tfrac{1}{2}\log(1+2r\cos\theta+r^2)=r\cos\theta-\tfrac{1}{2}r^2\cos 2\theta+\tfrac{1}{3}r^3\cos 3\theta-\dots,$$

$$\arctan\left(\frac{r\sin\theta}{1+r\cos\theta}\right)=r\sin\theta-\tfrac{1}{2}r^2\sin 2\theta+\tfrac{1}{3}r^3\sin 3\theta-\dots.$$

These equations hold for $0\leqq r\leqq 1$ and all values of θ, except that, when $r=1$, θ must not be equal to an odd multiple of π. It is easy to see that they also hold for $-1\leqq r\leqq 0$ and all values of θ, except that, when $r = -1$, θ must not be equal to an even multiple of π.

A particularly interesting case is that in which $r = 1$. In this case, if $-\pi < \theta < \pi$, we have

$$\log(1+z) = \log(1 + \operatorname{Cis}\theta) = \tfrac{1}{2}\log(2 + 2\cos\theta) + i \arctan\left(\frac{\sin\theta}{1+\cos\theta}\right)$$
$$= \tfrac{1}{2}\log(4\cos^2 \tfrac{1}{2}\theta) + \tfrac{1}{2}i\theta,$$

and so

$$\cos\theta - \tfrac{1}{2}\cos 2\theta + \tfrac{1}{3}\cos 3\theta - \dots = \tfrac{1}{2}\log(4\cos^2\tfrac{1}{2}\theta)$$
$$\sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \dots = \tfrac{1}{2}\theta.$$

The sums of the series, for other values of θ, are easily found from the consideration that they are periodic functions of θ with the period 2π. Thus the sum of the cosine series is $\frac{1}{2}\log(4\cos^2\frac{1}{2}\theta)$ for all values of θ save odd multiples of π (for which the series is divergent), while the sum of the sine series is $\frac{1}{2}(\theta - 2k\pi)$ if $(2k-1)\pi < \theta < (2k+1)\pi$, and zero if θ is an odd multiple of π. The graph of the function represented by the sine series is shown in Fig. 74. It is discontinuous for $\theta = (2k+1)\pi$.

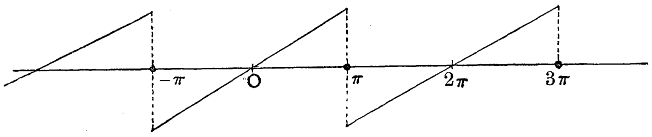

FIG. 74.

If we write $\pm iz$ for z in (2) we obtain

$$\log(1 \pm iz) = \pm iz - \tfrac{1}{2}z^2 \mp \tfrac{1}{3}iz^3 + \tfrac{1}{4}z^4 \pm \tfrac{1}{5}iz^5 - \dots.$$

Hence $$-\tfrac{1}{2}i\{\log(1+iz) - \log(1-iz)\} = z - \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 - \dots.$$

If z is real and numerically less than unity we are led, by the results of § 215, to the formula

$$\arctan z = z - \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 - \dots,$$

already proved in a different manner in § 196.

Examples XCIX. 1. Prove that, in any triangle for which $a > b$,

$$\log c = \log a - \frac{b}{a}\cos C - \frac{b^2}{2a^2}\cos 2C - \dots.$$

[Use the formula $\log c = \frac{1}{2}\log(a^2 + b^2 - 2ab\cos C)$.]

2. Prove that if $-1 < r < 1$ and $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ then

$$r\sin 2\theta - \tfrac{1}{2}r^2\sin 4\theta + \tfrac{1}{3}r^3\sin 6\theta - \dots = \theta - \arctan\left\{\left(\frac{1-r}{1+r}\right)\tan\theta\right\},$$

the inverse tangent lying between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. Determine the sum of the series for all other values of θ.

3. Prove, by considering the expansions of $\log(1+iz)$ and $\log(1-iz)$ in powers of z, that if $-1<r<1$ then

$$r\sin\theta \pm \tfrac{1}{2}r^2\cos 2\theta - \tfrac{1}{3}r^3\sin 3\theta \mp \tfrac{1}{4}r^4\cos 4\theta + \dots = \pm\tfrac{1}{2}\log(1 \pm 2r\sin\theta + r^2),$$

$$r\cos\theta \pm \tfrac{1}{2}r^2\sin 2\theta - \tfrac{1}{3}r^3\cos 3\theta \mp \tfrac{1}{4}r^4\sin 4\theta + \dots = \operatorname{arc\,tan}\left(\frac{r\cos\theta}{1 \mp r\sin\theta}\right),$$

$$r\sin\theta - \tfrac{1}{3}r^3\sin 3\theta + \dots = \tfrac{1}{4}\log\left(\frac{1+2r\sin\theta+r^2}{1-2r\sin\theta+r^2}\right),$$

$$r\cos\theta - \tfrac{1}{3}r^3\cos 3\theta + \dots = \tfrac{1}{2}\operatorname{arc\,tan}\left(\frac{2r\cos\theta}{1-r^2}\right),$$

the inverse tangents lying between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$[*].

4. Prove that

$$\cos\theta\cos\theta - \tfrac{1}{2}\cos 2\theta\cos^2\theta + \tfrac{1}{3}\cos 3\theta\cos^3\theta - \dots = \tfrac{1}{2}\log(1+3\cos^2\theta),$$

$$\sin\theta\sin\theta - \tfrac{1}{2}\sin 2\theta\sin^2\theta + \tfrac{1}{3}\sin 3\theta\sin^3\theta - \dots = \operatorname{arc\,cot}(1+\cot\theta+\cot^2\theta),$$

the inverse cotangent lying between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; and find similar expressions for the sums of the series

$$\cos\theta\sin\theta - \tfrac{1}{2}\cos 2\theta\sin^2\theta + \dots,\quad \sin\theta\cos\theta - \tfrac{1}{2}\sin 2\theta\cos^2\theta + \dots.$$

220. The general form of the Binomial Theorem. We have already proved (§ 197) that the sum of the series

$$1 + \binom{m}{1}z + \binom{m}{2}z^2 + \dots$$

is $(1+z)^m = \exp\{m\log(1+z)\}$, for all real values of m and all real values of z between -1 and $+1$. If a_n is the coefficient of z^n then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{m-n}{n+1}\right| \to 1$$

whether m is real or complex. Hence (Ex. LXXXII. 3) the series is always convergent if the modulus of z is less than unity, and we shall now prove that its sum is still $\exp\{m\log(1+z)\}$, i.e. the *principal value* of $(1+z)^m$.

For (Ex. XCVI. 23) we know that if t is real then

$$D_t(1+tz)^m = mz(1+tz)^{m-1},$$

z and m having any real or complex values and each side having its principal value. Hence, if $\phi(t) = (1+tz)^m$, we have

$$\phi^{(n)}(t) = m(m-1)\dots(m-n+1)z^n(1+tz)^{m-n}.$$

This formula still holds if $t=0$, so that $\phi^{(n)}(0)/n! = \binom{m}{n}z^n$.

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[*] Some examples of the cases in which $r=\pm 1$ will be found in the Miscellaneous Examples at the end of the chapter.

Now

$$\phi(1) = \phi(0) + \phi'(0) + \frac{\phi''(0)}{2!} + \dots + \frac{\phi^{(n-1)}(0)}{(n-1)!} + R_n,$$

where $$R_n = \frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1}\phi^{(n)}(t)\,dt;$$

for (§ 146) this formula holds for either the real or the imaginary parts of $\phi(t)$, and therefore, by addition, for $\phi(t)$ itself. But, if $z = r \operatorname{Cis}\theta$, we have

$$|1+tz| = \sqrt{(1+2tr\cos\theta + t^2r^2)} \geqq (1-tr),$$

and therefore

$$|R_n| < \left|\frac{m(m-1)\dots(m-n+1)}{(n-1)!}\right| r^n \int_0^1 \frac{(1-t)^{n-1}}{(1-tr)^{n-m}}\,dt$$
$$< \left|\frac{m(m-1)\dots(m-n+1)}{(n-1)!}\right| \frac{r^n(1-\theta)^{n-1}}{(1-\theta r)^{n-m}},$$

where $0<\theta<1$; and it follows, exactly as in § 147, that $R_n \to 0$ as $n \to \infty$.

Hence we arrive at the following theorem:

THEOREM. *The sum of the binomial series*

$$1 + \binom{m}{1}z + \binom{m}{2}z^2 + \dots$$

is $\exp\{m\log(1+z)\}$, *where the logarithm has its principal value, for all values of* m, *real or complex, and all values of* z *such that* $|z|<1$.

A more complete discussion of the binomial series, including a discussion of the more difficult case in which $|z|=1$, will be found on pp. 225 *et seq.* of Mr Bromwich's *Infinite Series.*

Examples C. 1. Suppose m real. Then since

$$\log(1+z) = \tfrac{1}{2}\log(1+2r\cos\theta+r^2) + i\arctan\left(\frac{r\sin\theta}{1+r\cos\theta}\right),$$

we obtain

$$\sum_0^\infty \binom{m}{n} z^n = \exp\{\tfrac{1}{2}m\log(1+2r\cos\theta+r^2)\}\operatorname{Cis}\left\{m\arctan\left(\frac{r\sin\theta}{1+r\cos\theta}\right)\right\}$$
$$= (1+2r\cos\theta+r^2)^{\frac{1}{2}m}\operatorname{Cis}\left\{m\arctan\left(\frac{r\sin\theta}{1+r\cos\theta}\right)\right\},$$

all the inverse tangents lying between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. In particular, if we suppose $\theta=\frac{1}{2}\pi$, $z=ir$, and equate the real and imaginary parts, we obtain

$$1-\binom{m}{2}r^2+\binom{m}{4}r^4-\dots=(1+r^2)^{\frac{1}{2}m}\cos(m\arctan r),$$

$$\binom{m}{1}r-\binom{m}{3}r^3+\binom{m}{5}r^5-\dots=(1+r^2)^{\frac{1}{2}m}\sin(m\arctan r).$$

2. Verify the formulae of Ex. 1 when $m=1, 2, 3$. [Of course when m is a positive integer the series is finite.]

3. Prove that if $-1<r<1$ then

$$1-\frac{1.3}{2.4}r^2+\frac{1.3.5.7}{2.4.6.8}r^4-\dots=\sqrt{\left\{\frac{\sqrt{(1+r^2)}+1}{2(1+r^2)}\right\}},$$

$$\frac{1}{2}r-\frac{1.3.5}{2.4.6}r^3+\frac{1.3.5.7.9}{2.4.6.8.10}r^5-\dots=\sqrt{\left\{\frac{\sqrt{(1+r^2)}-1}{2(1+r^2)}\right\}}.$$

[Take $m=-\frac{1}{2}$ in the last two formulae of Ex. 1.]

4. Prove that if $-\frac{1}{4}\pi<\theta<\frac{1}{4}\pi$ then

$$\cos m\theta=\cos^m\theta\left\{1-\binom{m}{2}\tan^2\theta+\binom{m}{4}\tan^4\theta-\dots\right\},$$

$$\sin m\theta=\cos^m\theta\left\{\binom{m}{1}\tan\theta-\binom{m}{3}\tan^3\theta+\dots\right\},$$

for all real values of m. [These results follow at once from the equations

$$\cos m\theta+i\sin m\theta=(\cos\theta+i\sin\theta)^m=\cos^m\theta\,(1+i\tan\theta)^m.]$$

5. We proved (Ex. LXXXIII. 6), by direct multiplication of series, that $f(m, z)=\Sigma\binom{m}{n}z^n$ (where $|z|<1$) satisfies the functional equation

$$f(m, z)f(m', z)=f(m+m', z).$$

Deduce, by an argument similar to that of § 198, and without assuming the general result of § 220, that if m is real and rational then

$$f(m, z)=\exp\{m\log(1+z)\}.$$

6. If z and μ are real, and $-1<z<1$, then

$$\Sigma\binom{i\mu}{n}z^n=\cos\{\mu\log(1+z)\}+i\sin\{\mu\log(1+z)\}.$$

MISCELLANEOUS EXAMPLES ON CHAPTER X.

1. Show that the real part of $i^{\log(1+i)}$ is

$$e^{-\frac{1}{8}(4k+1)\pi^2}\cos\{\tfrac{1}{4}(4k+1)\pi\log 2\},$$

where k is any integer. Find a general formula for the real part of $i^{\operatorname{Log}(1+i)}$.

2. If $a\cos\theta + b\sin\theta + c = 0$, where a, b, c are real and $c^2 > a^2 + b^2$, then

$$\theta = m\pi + \alpha \pm i\log\frac{|c| + \sqrt{(c^2 - a^2 - b^2)}}{\sqrt{(a^2 + b^2)}},$$

where m is any odd or any even integer, according as c is positive or negative, and α is the least angle whose cosine and sine are $a/\sqrt{(a^2+b^2)}$ and $b/\sqrt{(a^2+b^2)}$.

3. Prove that if θ is real and $\sin\theta\sin\phi = 1$ then

$$\phi = (k + \tfrac{1}{2})\pi + i\log\cot\tfrac{1}{2}(k\pi + \theta),$$

where k is any even or any odd integer, according as $\sin\theta$ is positive or negative.

4. Show that if x is real then

$$\frac{d}{dx}\exp\{(a+ib)x\} = (a+ib)e^{(a+ib)x}, \qquad \int \exp\{(a+ib)x\}\,dx = \frac{\exp\{(a+ib)x\}}{a+ib}.$$

Deduce the results of Ex. LXXXIX. 4.

5. Show that if $a > 0$ then $\int_0^\infty \exp\{-(a+ib)x\}\,dx = \frac{1}{a+ib}$, and deduce the results of Ex. LXXXIX. 5.

6. Show that if x is real then

$$e^{ax}\cos bx = \sum_0^\infty \frac{x^n}{n!}\left\{a^n - \binom{n}{2}a^{n-2}b^2 + \binom{n}{4}a^{n-4}b^4 - \dots\right\},$$

where there are $\frac{1}{2}(n+1)$ or $\frac{1}{2}(n+2)$ terms in the brackets. Find a similar series for $e^{ax}\sin bx$.

7. If $(x/a)^2 + (y/b)^2 = 1$ is the equation of an ellipse, and $f(x, y)$ denotes the terms of highest degree in the equation of any other algebraic curve, show that the sum of the eccentric angles of the points of intersection of the ellipse and the curve differs by a multiple of 2π from

$$-i\{\log f(a, ib) - \log f(a, -ib)\}.$$

[The eccentric angles are given by $f(a\cos\alpha, b\sin\alpha) + \dots = 0$ or by

$$f\left\{\tfrac{1}{2}a\left(u + \frac{1}{u}\right), -\tfrac{1}{2}ib\left(u - \frac{1}{u}\right)\right\} + \dots = 0,$$

if $u = \exp i\alpha$; and $\Sigma\alpha$ is equal to one of the values of $-i\operatorname{Log} P$, where P is the product of the roots of this equation.]

8. Determine the number and approximate positions of the roots of the equation $\tan z = az$, where a is real.

[We know already (Ex. XVIII. 5) that the equation has infinitely many real roots. Now let $z = x + iy$, and equate real and imaginary parts. We obtain

$$\sin 2x/(\cos 2x + \cosh 2y) = ax, \qquad \sinh 2y/(\cos 2x + \cosh 2y) = ay,$$

so that, unless x or y is zero, we have

$$(\sin 2x)/2x = (\sinh 2y)/2y.$$

This is impossible, the left-hand side being numerically less, and the right-hand side numerically greater than unity. Thus $x = 0$ or $y = 0$. If $y = 0$ we come back to the real roots of the equation, which have already been discussed.

If $x=0$, $\tanh y=ay$. It is easy to see by graphical methods that this equation has no real root other than zero if $a \leqq 0$ or $a \geqq 1$, and two such roots if $0<a<1$. Thus there are two purely imaginary roots if $0<a<1$; otherwise all the roots are real.]

9. The equation $\tan z=az+b$, where a and b are real and $b \neq 0$, has no complex roots if $a \leqq 0$. If $a>0$ the real parts of all the complex roots are numerically greater than $|b/2a|$.

10. The equation $\tan z=a/z$, where a is real, has no complex roots, but has one purely imaginary root if $a<0$.

11. The equation $\tan z=a \tanh (cz)$, where a and c are real, has an infinity of real and of purely imaginary roots, but no complex roots.

12. **Transformations.** In Ch. III (Exs. 21 *et seq.*, and Misc. Exs. 29 *et seq.*) we considered some simple cases of the geometrical relations between figures in the planes of two variables z, Z connected by a relation $z=f(Z)$. We shall now consider some cases in which the relation involves logarithmic, exponential, or circular functions.

Suppose firstly that

$$z=\exp(\pi Z/a), \qquad Z=(a/\pi)\operatorname{Log} z$$

where a is positive. To one value of Z corresponds one of z, but to one of z infinitely many of Z. If x, y, r, θ are the coordinates of z and X, Y, R, Θ those of Z, we have the relations

$$\left.\begin{aligned} x&=e^{\pi X/a}\cos(\pi Y/a) \\ y&=e^{\pi X/a}\sin(\pi Y/a) \end{aligned}\right\} \qquad \left.\begin{aligned} X&=(a/\pi)\log r \\ Y&=(a\theta/\pi)+2ka \end{aligned}\right\},$$

where k is any integer. If we suppose that $-\pi<\theta \leqq \pi$ and that $\operatorname{Log} z$ has its principal value $\log z$, then $k=0$, and Z is confined to a strip of its plane parallel to the axis OX and extending to a distance a from it on each side, one point of this strip corresponding to one of the whole z-plane, and conversely. By taking a value of $\operatorname{Log} z$ other than the principal value we obtain a similar relation between the z-plane and another strip of breadth $2a$ in the Z-plane.

To the lines in the Z-plane for which X and Y are constant correspond the circles and radii vectores in the z-plane for which r and θ are constant. To one of the latter lines corresponds the whole of a parallel to OX, but to a circle for which r is constant corresponds only a part (of length $2a$) of a parallel to OY. To make Z describe the whole of the latter line we must make z move continually round and round the circle.

13. Show that to a straight line in the Z-plane corresponds an equiangular spiral in the z-plane.

14. Discuss similarly the transformation $z=c\cosh(\pi Z/a)$, showing in particular that the whole z-plane corresponds to any one of an infinite number of strips in the Z-plane, each parallel to the axis OX and of breadth $2a$. Show also that to the line $X=X_0$ corresponds the ellipse

$$\left\{\frac{x}{c\cosh(\pi X_0/a)}\right\}^2+\left\{\frac{y}{c\sinh(\pi X_0/a)}\right\}^2=1,$$

and that for different values of X_0 these ellipses form a confocal system; and that the lines $Y=Y_0$ correspond to the associated system of confocal hyperbolas. Trace the variation of z as Z describes the whole of a line $X=X_0$ or $Y=Y_0$. How does Z vary as z describes the degenerate ellipse and hyperbola formed by the segment between the foci of the confocal system and the remaining segments of the axis of x?

15. Verify that the results of Ex. 14 are in agreement with those of Ex. 12 and those of Ch. III, Misc. Ex. 34. [The transformation $z=c\cosh(\pi Z/a)$ may be regarded as compounded from the transformations

$$z=cz_1,\quad z_1=\tfrac{1}{2}\{z_2+(1/z_2)\},\quad z_2=\exp(\pi Z/a).]$$

16. Discuss similarly the transformation $z=c\tanh(\pi Z/a)$, showing that to the lines $X=X_0$ correspond the coaxal circles

$$\{x-c\coth(\pi X_0/a)\}^2+y^2=c^2\operatorname{cosech}^2(\pi X_0/a)$$

and to the lines $Y=Y_0$ the orthogonal system of coaxal circles.

17. **The Stereographic and Mercator's Projections.** The points of a unit sphere whose centre is the origin are projected from the S. pole (whose coordinates are 0, 0, -1) on to the tangent plane at the N. pole. The coordinates of a point on the sphere are ξ, η, ζ, and Cartesian axes OX, OY are taken on the tangent plane, parallel to the axes of ξ and η. Show that the coordinates of the projection of the point are

$$x=2\xi/(1+\zeta),\quad y=2\eta/(1+\zeta),$$

and that $x+iy=2\tan\frac{1}{2}\theta\operatorname{Cis}\phi$, where ϕ is the longitude (measured from the plane $\eta=0$) and θ the north polar distance of the point on the sphere.

This projection gives a map of the sphere on the tangent plane, generally known as the *Stereographic Projection.* If now we introduce a new complex variable

$$Z=X+iY=-i\log\tfrac{1}{2}z=-i\log\tfrac{1}{2}(x+iy)$$

so that $X=\phi$, $Y=\log\cot\frac{1}{2}\theta$, we obtain another map in the plane of Z, usually called *Mercator's Projection.* In this map parallels of latitude and longitude are represented by straight lines parallel to the axes of X and Y respectively.

18. Discuss the transformation given by the equation

$$z=\operatorname{Log}\{(Z-a)/(Z-b)\},$$

showing that the straight lines for which x and y are constant correspond to two orthogonal systems of coaxal circles in the Z-plane.

19. Discuss the transformation

$$z=\operatorname{Log}\left\{\frac{\sqrt{(Z-a)}+\sqrt{(Z-b)}}{\sqrt{(b-a)}}\right\},$$

showing that the straight lines for which x and y are constant correspond to sets of confocal ellipses and hyperbolas whose foci are the points $Z=a$ and $Z=b$.

[We have
$$\surd(Z-a)+\surd(Z-b)=\surd(b-a)\exp(x+iy),$$
$$\surd(Z-a)-\surd(Z-b)=\surd(b-a)\exp(-x-iy);$$
and it will be found that
$$|Z-a|+|Z-b|=|b-a|\cosh 2x,\quad |Z-a|-|Z-b|=|b-a|\cos 2y.]$$

20. **The transformation** $z=Z^i$. If $z=Z^i$, where the imaginary power has its principal value, we have
$$\exp(\log r+i\theta)=z=\exp(i\log Z)=\exp(i\log R-\Theta),$$
so that $\log r=-\Theta$, $\theta=\log R+2k\pi$, where k is an integer. As all values of k give the same point z we shall suppose $k=0$, so that
$$\log r=-\Theta,\quad \theta=\log R \quad\text{.............................}(1).$$

The whole plane of Z is covered if we allow R to vary from 0 to ∞ and Θ from $-\pi$ to π: then r has the range $e^{-\pi}$ to e^{π} and θ ranges through all real values. Thus to the Z-plane corresponds the ring in the z-plane bounded by the circles $r=e^{-\pi}$, $r=e^{\pi}$; but this ring is covered infinitely often.

If however we restrict the variation of θ to the range $(-\pi, +\pi)$, so that the ring is covered only once, then R is restricted to the range $(e^{-\pi}, e^{\pi})$, so that the variation of Z is restricted to a ring similar in all respects to that within which z varies. Each ring, moreover, must be regarded as having a barrier along the negative real axis which z (or Z) must not cross, as its amplitude must not transgress the limits $-\pi$, $+\pi$.

We thus obtain a correspondence between two rings, given by the pair of equations
$$z=Z^i,\quad Z=z^{-i},$$
where each power has its principal value. To circles whose centre is the origin in one plane correspond straight lines through the origin in the other.

21. If Z, starting at the point e^{π} moves round the larger circle in the positive direction to $-e^{\pi}$, along the barrier, round the smaller circle in the negative direction, back along the barrier, and round the remainder of the larger circle to its original position, trace the corresponding variation of z.

22. Suppose each plane to be divided up into an infinite series of rings by circles of radii
$$\dots,\ e^{-(2n+1)\pi},\ \dots,\ e^{-\pi},\ e^{\pi},\ e^{3\pi},\ \dots,\ e^{(2n+1)\pi},\ \dots.$$

Show how, by taking suitable values of the powers in the equations $z=Z^i$, $Z=z^{-i}$, to make any ring in one plane correspond to any ring in the other.

23. If $z=Z^i$ (any value of the power being taken) and Z moves along an equiangular spiral whose pole is the origin in its plane, then z moves along an equiangular spiral whose pole is the origin in its plane.

24. How does $Z=z^{ai}$, where a is real, behave as z approaches the origin along the real axis. [Z moves round and round a circle whose centre is the origin (the unit circle if z^{ai} has its principal value), and the real and imaginary parts of Z both oscillate finitely.]

25. Discuss the same question for $Z = z^{a+bi}$, where a and b are any real numbers.

26. Show that the region of convergence of a series of the type $\sum_{-\infty}^{\infty} a_n z^{nai}$, where a is real, is an angle, i.e. a region bounded by inequalities of the type $\theta_0 < \mathrm{am}\, z < \theta_1$. [The angle may vanish, or include all possible values of $\mathrm{am}\, z$.]

27. **Level Curves.** If $f(z)$ is any function of the complex variable z, we shall call the curves for which $|f(z)|$ is constant the *level curves* of $f(z)$. Sketch the forms of the level curves of

$z-a$ (*concentric circles*), $(z-a)(z-b)$ (*Cartesian ovals*),
$(z-a)/(z-b)$ (*coaxal circles*), $\exp z$ (*straight lines*).

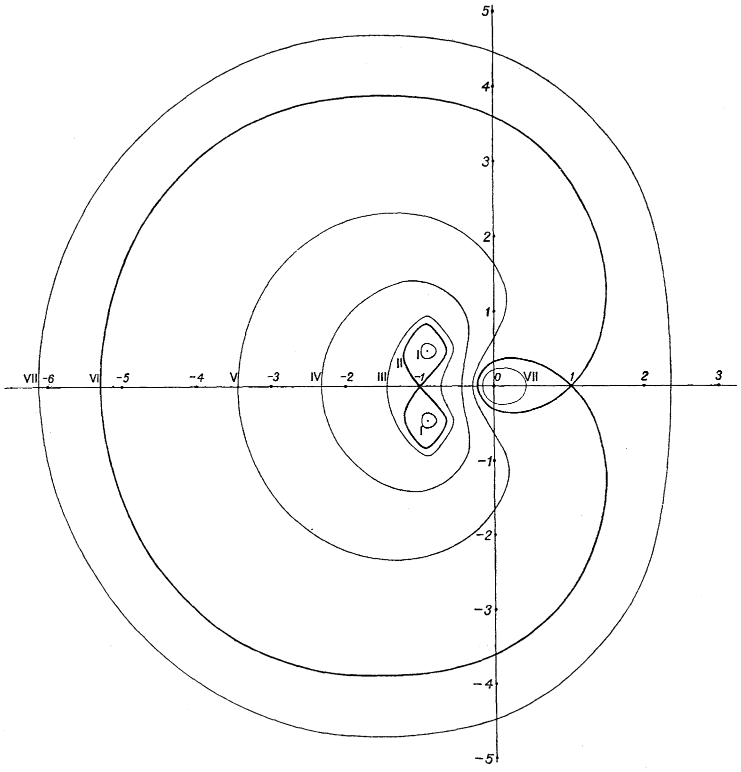

Fig. 75.

28. Sketch the forms of the level curves of $(z-a)(z-b)(z-c)$, $(1 + z\sqrt{3} + z^2)/z$. [Some of the level curves of the latter function are drawn in Fig. 75, the curves marked I–VII corresponding to the values

$$\cdot 10,\ 2-\sqrt{3} = \cdot 27,\ \cdot 40,\ 1\cdot 00,\ 2\cdot 00,\ 2+\sqrt{3} = 3\cdot 73,\ 4\cdot 53$$

of $|f(z)|$. The reader will probably find but little difficulty in arriving at a general idea of the forms of the level curves of any given rational function; but to enter into details would carry us into the general theory of functions of a complex variable.]

29. Sketch the forms of the level curves of (i) $z \exp z$, (ii) $\sin z$. [See Fig. 76, which represents the level curves of $\sin z$. The curves marked I–VIII correspond to $k = \cdot 35, \cdot 50, \cdot 71, 1\cdot 00, 1\cdot 41, 2\cdot 00, 2\cdot 83, 4\cdot 00$.]

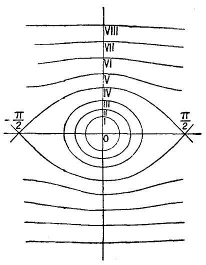

FIG. 76.

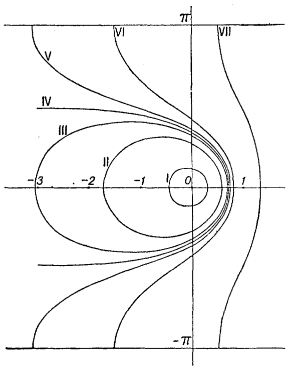

FIG. 77.

30. Sketch the forms of the level curves of $\exp z - c$, where c is a real constant. [Fig. 77 shows the level curves of $|\exp z - 1|$, the curves I–VII corresponding to the values of k given by $\log k = -1\cdot 00, -\cdot 20, -\cdot 05, 0\cdot 00, \cdot 05, \cdot 20, 1\cdot 00$.]

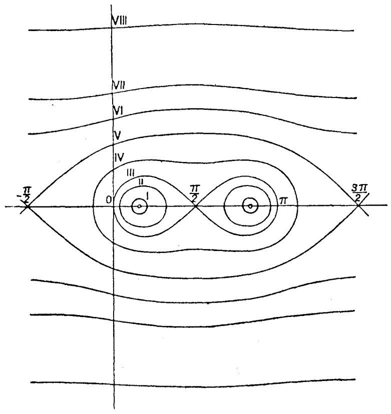

FIG. 78.

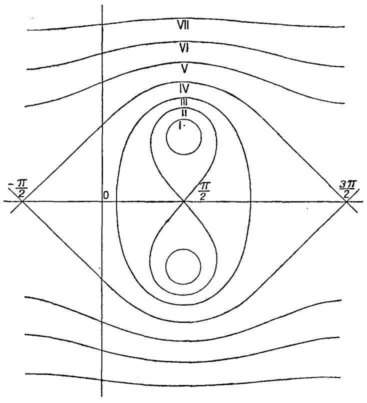

FIG. 79.

31. The level curves of $\sin z - c$, where c is a positive constant, are sketched in Figs. 78, 79. [There are two kinds of curves corresponding to the cases in which $c<1$ and $c>1$. In Fig. 78 $c=\cdot 5$, and the curves I–VIII correspond to $k=\cdot 29$, $\cdot 37$, $\cdot 50$, $\cdot 87$, $1\cdot 50$, $2\cdot 60$, $4\cdot 50$, $7\cdot 79$. In Fig. 79 $c=2$, and the curves I–VII correspond to $k=\cdot 58$, $1\cdot 00$, $1\cdot 73$, $3\cdot 00$, $5\cdot 20$, $9\cdot 00$, $15\cdot 59$. If $c=1$ the curves are the same as those of Fig. 76, except that the origin and scale are different.]

32. Prove that if $0<\theta<\pi$ then

$$\cos\theta+\tfrac{1}{3}\cos 3\theta+\tfrac{1}{5}\cos 5\theta+\dots=\tfrac{1}{4}\log\cot^2\tfrac{1}{2}\theta,$$

$$\sin\theta+\tfrac{1}{3}\sin 3\theta+\tfrac{1}{5}\sin 5\theta+\dots=\tfrac{1}{4}\pi,$$

and determine the sums of the series for all other values of θ for which they are convergent. [Use the equation

$$z+\tfrac{1}{3}z^3+\tfrac{1}{5}z^5+\dots=\tfrac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

where $z=\cos\theta+i\sin\theta$. When θ is increased by π the sum of each series simply changes its sign. It follows that the first formula holds for all values of θ save multiples of π (for which the series diverges), while the sum of the second series is $\frac{1}{4}\pi$ if $2k\pi<\theta<(2k+1)\pi$, $-\frac{1}{4}\pi$ if $(2k+1)\pi<\theta<(2k+2)\pi$, and 0 if θ is a multiple of π.]

33. Prove that if $0<\theta<\frac{1}{2}\pi$ then

$$\cos\theta-\tfrac{1}{3}\cos 3\theta+\tfrac{1}{5}\cos 5\theta-\dots=\tfrac{1}{4}\pi,$$

$$\sin\theta-\tfrac{1}{3}\sin 3\theta+\tfrac{1}{5}\sin 5\theta-\dots=\tfrac{1}{4}\log(\sec\theta+\tan\theta)^2;$$

and determine the sums of the series for all other values of θ for which they are convergent.

34. Prove that

$$\cos\theta\cos\alpha+\tfrac{1}{2}\cos 2\theta\cos 2\alpha+\tfrac{1}{3}\cos 3\theta\cos 3\alpha+\dots=-\tfrac{1}{4}\log\{4(\cos\theta-\cos\alpha)^2\},$$

unless $\theta-\alpha$ or $\theta+\alpha$ is a multiple of 2π.

35. Prove that if neither a nor b is real then

$$\int_0^\infty \frac{dx}{(x-a)(x-b)}=\frac{\log a-\log b}{a-b},$$

each logarithm having its principal value. Verify the result when $a=ci$, $b=-ci$, where c is positive. Discuss also the cases in which a or b or both are real and negative.

36. Prove that if α and β are real, and $\beta>0$, then

$$\int_0^\infty \frac{dx}{x^2-(\alpha+i\beta)^2}=\frac{\pi i}{2(\alpha+i\beta)}.$$

What is the value of the integral when $\beta<0$?

37. Prove that

$$\int_{-\infty}^\infty \frac{dx}{Ax^2+2Bx+C}=\frac{\pi i}{\sqrt{(B^2-AC)}},$$

the sign of $\sqrt{(B^2-AC)}$ being so chosen that the real part of $\{\sqrt{(B^2-AC)}\}/Ai$ is positive.

APPENDIX I.

(To Chapters III, IV, V.)

The Proof that every Equation has a Root.

Let $$Z = P(z) \equiv a_0 z^n + a_1 z^{n-1} + \ldots + a_n$$ be any polynomial in z, with real or complex coefficients. We can represent the values of z and Z by points in two planes, which we may call the z-plane and the Z-plane respectively. If z describes a closed path γ in the z-plane it is evident that Z describes a corresponding closed path Γ in the Z-plane. When Z returns to its original position its amplitude may be the same as before, as will certainly be the case if Γ does not enclose the origin, like path (a) in Fig. B, or it may differ from its original value by any multiple of 2π. Thus if its path is like (b) in Fig. B, winding once round the origin in the positive direction, its amplitude will have increased by 2π.

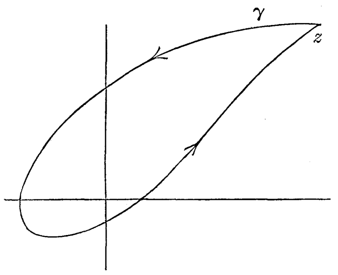

Fig. A.

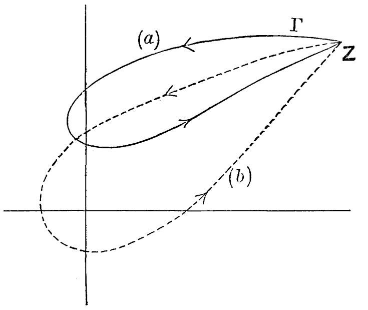

Fig. B.

We shall now prove that *if the amplitude of Z is not the same when Z returns to its original position, the path of z must contain inside or on it at least one point at which $Z=0$.*

We can divide γ into a number of smaller contours by drawing parallels to the axes at a distance δ_1 from one another (see Fig. C). If there is, on the boundary of any one of these contours, any point for which $Z=0$, what we wish to prove is already established. We may therefore suppose that this is not the case. Then the increment of am Z, when z describes γ, is equal to the sum of all the increments of am Z obtained by supposing z to describe each of these smaller contours separately in the same sense as γ

For if z describes each of the smaller contours in turn, in the same sense, it will ultimately (see Fig. D) have described the boundary of γ once, and each

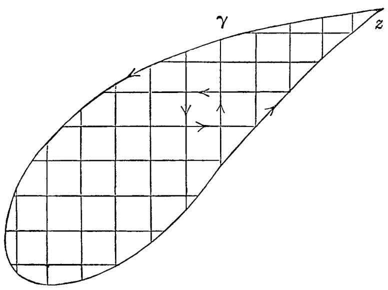

Fig. C.

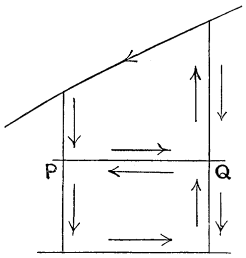

Fig. D.

part of each of the dividing parallels twice and in opposite directions. Thus PQ will have been described twice, once from P to Q and once from Q to P. As z moves from P to Q, am Z varies continuously (since Z does not pass through the origin); and if the increment of am Z is θ its increment when z moves from Q to P will be $-\theta$; so that when we add up the increments of am Z due to the description of the various parts of the smaller contours all cancel one another save the increments due to the description of parts of γ itself.

Hence, if am Z is changed when z describes γ, there must be *at least one* of the smaller contours (say γ_1) such that am Z is changed when z describes γ_1. This contour may be a square bounded wholly by the auxiliary parallels or may be bounded in part by the boundary of γ. In any case it is entirely included within a square Δ_1 whose sides are of length δ_1 and parallel to the axes.

We can now further subdivide γ_1 by the help of parallels to the axes at a smaller distance δ_2 from one another, and we can find a contour γ_2, entirely included in a square Δ_2 of side δ_2 (itself included inside Δ_1), and such that am Z is changed when z describes it.

Now let us take an infinite sequence of decreasing numbers $\delta_1, \delta_2, \ldots, \delta_m, \ldots$, whose limit is zero. By repeating the argument used above we can determine a series of squares $\Delta_1, \Delta_2, \ldots, \Delta_m, \ldots$ and a series of contours $\gamma_1, \gamma_2, \ldots, \gamma_m, \ldots$ such that (i) Δ_{m+1} lies entirely inside Δ_m, (ii) γ_m lies entirely inside Δ_m, (iii) am Z is changed when z describes γ_m.

If (x_m, y_m) and $(x_m+\delta_m, y_m+\delta_m)$ are the lower left-hand and upper right-hand corners of Δ_m, it is clear that $x_1, x_2, \ldots, x_m, \ldots$ is an increasing and $x_1+\delta_1, x_2+\delta_2, \ldots, x_m+\delta_m, \ldots$ a decreasing sequence, and that they have a common limit x_0. Similarly y_m and $y_m+\delta_m$ approach a common limit y_0, and (x_0, y_0) is the one and only point situated inside every square. However small be δ, we can draw a square which includes (x_0, y_0), and whose sides are parallel to the axes and of length δ, and inside this square a closed contour such that am Z is changed when z describes the contour.

It can now be shown that

$$P(x_0+iy_0)=0.$$

For if not let $P(x_0+iy_0)=a$, where $a\neq 0$, and let $|a|=\rho$. Since $P(x+iy)$ is a continuous function of x and y we can draw a square whose centre is (x_0, y_0) and whose sides are parallel to the axes, and which is such that

$$|P(x+iy)-P(x_0+iy_0)|<\tfrac{1}{2}\rho$$

for all points $x+iy$ inside it or on its boundary. For all such points

$$P(x+iy)=a+\phi,$$

where $|\phi|<\frac{1}{2}\rho$. Now let us take any closed contour lying entirely inside this square. As z describes it, $Z=a+\phi$ describes a closed contour. But this contour evidently lies inside the circle whose centre is a and whose radius is $\frac{1}{2}\rho$, and this circle does not include the origin. Hence the amplitude of Z is unchanged.

But this contradicts what was proved above, viz. that inside each square Δ_m we can find a closed contour the description of which by z changes am Z. Hence $P(x_0+iy_0)=0$.

All that remains now is to show that we can always find *some* contour the description of which by z changes am Z. Now

$$Z=a_0z^n\left(1+\frac{a_1}{a_0z}+\frac{a_2}{a_0z^2}+\dots+\frac{a_n}{a_0z^n}\right).$$

We can choose R so that

$$\frac{|a_1|}{|a_0|R}+\frac{|a_2|}{|a_0|R^2}+\dots+\frac{|a_n|}{|a_0|R^n}<\epsilon,$$

where ϵ is any positive number, however small; and then, if γ is the circle whose centre is the origin and whose radius is R,

$$Z=a_0z^n(1+\rho),$$

where $|\rho|<\epsilon$, for all points on γ. But so long as $|\rho|<1$ it follows, by an argument similar to that used above, that $\operatorname{am}(1+\rho)$ is unchanged as z describes γ in the positive sense, while am z^n on the other hand is increased by $2n\pi$. Hence am Z is increased by $2n\pi$, and the proof that $Z=0$ has a root is completed.

We leave it as an exercise to the reader to infer, from the discussion which precedes and that of § 34, that *when z describes any contour γ in the positive sense the increment of am Z is $2k\pi$, where k is the number of roots of $Z=0$ inside γ, multiple roots being counted multiply.*

There is another proof, proceeding on different lines, which is often given. It depends, however, on an extension to functions of two or more variables of the results of § 89.

We define, precisely on the lines of § 89, the *upper and lower limits* of a function $f(x, y)$, for all pairs of values of x and y corresponding to any point of any region in the plane of (x, y) bounded by an ordinary closed curve. And we can prove, much as in § 89, that a continuous function $f(x, y)$ attains its upper and lower limits in any such region.

Now $$|Z| = |P(x+iy)|$$
is a continuous and positive function of x and y. If m is its lower limit for points on and inside γ, there must be a point z_0 for which $|Z| = m$, and this must be the *least* value assumed by $|Z|$.

If $m=0$, $P(z_0)=0$, and we have proved what we want. Suppose, therefore, that $m>0$.

If we put $z=z_0+\zeta$, and rearrange $P(z)$ according to powers of ζ, we obtain
$$P(z)=P(z_0)+A_1\zeta+A_2\zeta^2+\dots+A_n\zeta^n,$$
say. Let A_k be the first of the coefficients which does not vanish, and let $|A_k|=\mu$, $|\zeta|=\rho$. We can choose ρ so small that
$$|A_{k+1}|\rho+|A_{k+2}|\rho^2+\dots+|A_n|\rho^{n-k}<\tfrac{1}{2}\mu.$$
Then $$|P(z)-P(z_0)-A_k\zeta^k|<\tfrac{1}{2}\mu\rho^k,$$
and $$|P(z)|<|P(z_0)+A_k\zeta^k|+\tfrac{1}{2}\mu\rho^k.$$

Now suppose that z moves round the circle whose centre is z_0 and radius ρ. Then
$$P(z_0)+A_k\zeta^k$$
moves k times round the circle whose centre is $P(z_0)$ and radius $|A_k\zeta^k|=\mu\rho^k$, and passes k times through the point in which this circle is intersected by the line joining $P(z_0)$ to the origin. Hence there are k points on the circle described by z for which $|P(z_0)+A_k\zeta^k|=|P(z_0)|-\mu\rho^k$ and so
$$|P(z)|<|P(z_0)|-\tfrac{1}{2}\mu\rho^k=m-\tfrac{1}{2}\mu\rho^k;$$
and this contradicts the hypothesis that m is the lower limit of $|P(z)|$.

The preceding argument would fail if z_0 were on the boundary of γ: but if γ is a circle whose centre is the origin, and whose radius R is large enough, the last hypothesis is untenable, since $|P(z)|\to+\infty$ with $|z|$. In this case it must therefore be true that $m=0$, and our conclusion follows.

EXAMPLES ON APPENDIX I.

1. Show that the number of roots of $f(z)=0$, which lie within a closed contour which does not pass through any root, is equal to the increment of
$$\{\log f(z)\}/2\pi i$$
when z describes the contour.

2. Show that if R is any number such that
$$\frac{|a_1|}{R}+\frac{|a_2|}{R^2}+\dots+\frac{|a_n|}{R^n}<1,$$
then all the roots of $x^n+a_1x^{n-1}+\dots+a_n=0$ are in absolute value less than R. In particular show that all the roots of $x^5-13x-7=0$ are in absolute value less than $2\frac{1}{67}$.

3. Determine the numbers of the roots of the equation $x^{2p}+ax+b=0$, where a and b are real and p odd, which have their real parts positive and negative. Show that if $a>0$, $b>0$ the numbers are $p-1$ and $p+1$, if $a<0$, $b>0$ they are $p+1$ and $p-1$, and if $b<0$ they are p and p. Discuss the

particular cases in which $a=0$ or $b=0$. Verify the results when $p=1$. [Trace the variation of $\operatorname{am}(x^{2p}+ax+b)$ as x describes the contour formed by a large semicircle whose centre is the origin and whose radius is R, and the part of the imaginary axis intercepted by the semicircle.]

4. Consider similarly the equations

$$x^{4q}+ax+b=0, \quad x^{4q-1}+ax+b=0, \quad x^{4q+1}+ax+b=0.$$

5. Show that, if α and β are real, the numbers of the roots of the equation $x^{2n}+\alpha^2 x^{2n-1}+\beta^2=0$ which have their real parts positive and negative are $n-1$ and $n+1$ or n and n respectively, according as n is odd or even. (*Math. Trip.* 1891.)

6. Show that when z moves along the straight line joining the points $z=z_1$, $z=z_2$, from a point near z_1 to a point near z_2, the increment of

$$\operatorname{am}\left(\frac{1}{z-z_1}+\frac{1}{z-z_2}\right)$$

is nearly equal to π.

7. A contour enclosing the three points $z=z_1$, $z=z_2$, $z=z_3$ is defined by parts of the sides of the triangle formed by z_1, z_2, z_3, and the parts exterior to the triangle of three small circles with their centres at those points. Show that when z describes the contour the increment of

$$\operatorname{am}\left(\frac{1}{z-z_1}+\frac{1}{z-z_2}+\frac{1}{z-z_3}\right)$$

is equal to -2π.

8. Prove that a closed oval path which surrounds all the roots of a cubic equation $f(z)=0$ also surrounds those of the derived equation $f'(z)=0$. [Use the equation

$$f'(z)=f(z)\left(\frac{1}{z-z_1}+\frac{1}{z-z_2}+\frac{1}{z-z_3}\right),$$

where z_1, z_2, z_3 are the roots of $f(z)=0$, and the result of Ex. 7.]

9. Show that the roots of $f'(z)=0$ are the foci of the ellipse which touches the sides of the triangle (z_1, z_2, z_3) at their middle points. [For a proof see Cesàro's *Elementares Lehrbuch der Algebraischen Analysis*, p. 352.]

10. Extend the result of Ex. 8 to equations of any degree.

11. If $f(z)$ and $\phi(z)$ are two polynomials in z, and γ is a contour which does not pass through any root of either, and $|\phi(z)|<|f(z)|$ for all points on γ, then the numbers of the roots of the equations

$$f(z)=0, \quad f(z)+\phi(z)=0$$

which lie inside γ are the same.

APPENDIX II.

(To Chapters IX, X.)

A Note on Double Limit Problems.

In the course of Chapters IX and X we came on several occasions into contact with problems of a kind which invariably puzzle beginners and are indeed, when treated in their most general forms, problems of great difficulty and of the utmost interest and importance in higher mathematics.

Let us consider some special instances. In § 195 we proved that

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \dots$$

by integrating the equation

$$1/(1+t) = 1 - t + t^2 - \dots$$

between the limits 0 and x, and what we proved amounted to this, that

$$\int_0^x \frac{dt}{1+t} = \int_0^x dt - \int_0^x t\,dt + \int_0^x t^2\,dt - \dots;$$

or in other words that *the integral of the sum of the infinite series* $1 - t + t^2 - \dots$, *taken between the limits* 0 *and* x, *is equal to the sum of the integrals of its terms taken between the same limits.* Another way of expressing this fact is to say that the operations of summation from 0 to ∞, and of integration from 0 to x, are *commutative* when applied to the function $(-1)^n t^n$, i.e. that it does not matter in what order they are performed on the function.

Again in §§ 187, 194 we proved that the differential coefficient of the exponential function

$$\exp x = 1 + x + \frac{x^2}{2!} + \dots$$

is itself equal to $\exp x$, or that

$$D_x\left(1 + x + \frac{x^2}{2!} + \dots\right) = D_x(1) + D_x(x) + D_x\left(\frac{x^2}{2!}\right) + \dots;$$

that is to say that *the differential coefficient of the sum of the series is equal to the sum of the differential coefficients of its terms*, or that the operations of summation from 0 to ∞ and of differentiation with respect to x are commutative when applied to $x^n/n!$.

Finally, we proved incidentally in the same sections that the function $\exp x$ is a continuous function of x, or in other words that

$$\lim_{x \to \xi}\left(1 + x + \frac{x^2}{2!} + \dots\right) = 1 + \xi + \frac{\xi^2}{2!} + \dots = \lim_{x \to \xi}(1) + \lim_{x \to \xi}(x) + \lim_{x \to \xi}\left(\frac{x^2}{2!}\right) + \dots;$$

i.e. that the limit of the sum of the series is equal to the sum of the limits of the terms, or that the sum of the series is continuous for $x=\xi$, or that the operations of summation from 0 to ∞ and of making $x \to \xi$ are commutative when applied to $x^n/n\,!$.

In each of these cases we gave a special proof of the correctness of the result. We have not proved, and in this volume shall not prove, any general theorem from which the truth of any one of them could be immediately inferred. In Ex. XXXVIII. 1 we saw that the sum of a finite number of continuous terms is itself continuous: in § 94 that the differential coefficient of the sum of a finite number of terms is equal to the sum of their differential coefficients: and in § 144 we stated the corresponding theorem for integrals. Thus we have proved that in certain circumstances the operations symbolised by

$$\lim_{x \to \xi} \dots, \quad D_x \dots, \quad \int \dots dx$$

are commutative with respect to the operation of summation of a *finite* number of terms. And it is natural to suppose that, in certain circumstances which it should be possible to define precisely, they should be commutative also with respect to the operation of summation of an *infinite* number. It is natural to suppose so: but that is all that we have a right to say at present.

A few further instances of commutative and non-commutative operations may help to elucidate these points.

(1) Multiplication by 2 and multiplication by 3 are always commutative, for

$$2 \times 3 \times x = 3 \times 2 \times x$$

for all values of x.

(2) The operation of taking the real part of z is *never* commutative with that of multiplication by i, except when $z=0$. For

$$i \times R(x+iy) = ix, \quad R\{i \times (x+iy)\} = -y.$$

(3) The operations of proceeding to the limit zero with each of two variables x and y, applied to a function $f(x, y)$ may or may not be commutative. Thus

$$\lim_{x \to 0} \{\lim_{y \to 0} (x+y)\} = \lim_{x \to 0} x = 0, \quad \lim_{y \to 0} \{\lim_{x \to 0} (x+y)\} = \lim_{y \to 0} y = 0;$$

but on the other hand

$$\lim_{x \to 0} \left(\lim_{y \to 0} \frac{x-y}{x+y}\right) = \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} (1) = 1,$$

$$\lim_{y \to 0} \left(\lim_{x \to 0} \frac{x-y}{x+y}\right) = \lim_{y \to 0} \frac{-y}{y} = \lim_{y \to 0} (-1) = -1.$$

(4) The operations $\sum\limits_0^\infty \dots$, $\lim\limits_{x \to 1} \dots$ may or may not be commutative. Thus, if $x \to 1$ through values less than 1,

$$\lim_{x \to 1} \left\{\sum_1^\infty \frac{(-1)^n}{n} x^n\right\} = \lim_{x \to 1} \log(1+x) = \log 2,$$

$$\sum_1^\infty \left\{\lim_{x \to 1} \frac{(-1)^n}{n} x^n\right\} = \sum_1^\infty \frac{(-1)^n}{n} = \log 2;$$

but on the other hand

$$\lim_{x \to 1} \left\{ \sum_1^\infty (x^n - x^{n+1}) \right\} = \lim \{(1-x)+(x-x^2)+\dots\} = \lim_{x \to 1} (1) = 1,$$

$$\sum_1^\infty \left\{ \lim_{x \to 1} (x^n - x^{n+1}) \right\} = \sum_1^\infty (1-1) \qquad = \sum_1^\infty (0) \quad = 0.$$

The preceding examples suggest that there are three possibilities with respect to the commutation of two given operations, viz.: (1) the operations may *always* be commutative; (2) they may *never* be commutative, *except in very special circumstances*; (3) they may be commutative *in most of the ordinary cases which occur practically*.

The really important case (as is suggested by the instances which we gave from Ch. IX) is that in which each operation is one which involves a passage to the limit, such as a differentiation or the summation of an infinite series: such operations are called *limit operations*. The general question as to the circumstances in which two given limit operations are commutative is one of the most important in all mathematics. But to attempt to deal with general questions of this character by means of general theorems would carry us far beyond the scope of this volume.

We may however remark that the answer to the general question is on the lines suggested by the examples above. If L and L' are two limit operations the numbers $LL'z$ and $L'Lz$ are not *generally* equal, in the strict theoretical sense of the word 'general.' We can always, by the exercise of a little ingenuity, find z so that $LL'z \neq L'Lz$. But they *are* equal generally, if we use the word in a more practical sense, viz. as meaning 'in a great majority of such cases as are likely to occur naturally' or in *ordinary* cases.

Of course, in an exact science like pure mathematics, we cannot be satisfied with an answer of this kind; and in the higher branches of mathematics the detailed investigation of these questions is an absolute necessity. But for the present the reader may be content if he realises the point of the remarks which we have just made. In practice, a result obtained by assuming that two limit-operations are commutative is *probably* true: it at any rate affords a valuable *suggestion* as to the answer to the problem under consideration. But an answer thus obtained must, in default of a further study of the general question or a special investigation of the particular problem, such as we gave in the instances which occurred in Ch. IX, be regarded as suggested only and not proved. Detailed investigations of a large number of important double limit problems will be found in Mr Bromwich's book on *Infinite Series*.

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